Research Article

Modified Variational Iteration Method for Free-Convective Boundary-Layer Equation Using Padé Approximation

Syed Tauseef Mohyud-Din,[1] Ahmet Yildirim,[2] Sefa Anıl Sezer,[2] and Muhammad Usman[3]

[1] HITEC University, Taxila Cantt, Pakistan

[2] Department of Mathematics, Ege University, Bornova 35100, Izmir, Turkey

[3] Department of Mathematics, University of Dayton, 300 College Park, Dayton, OH 45469-2316, USA

Correspondence should be addressed to Syed Tauseef Mohyud-Din, syedtauseefs@hotmail.com

This paper is devoted to the study of a free-convective boundary-layer flow modeled by a system of nonlinear ordinary differential equations. We apply a modified variational iteration method (MVIM) coupled with He's polynomials and Padé approximation to solve free-convective boundary-layer equation. It is observed that the combination of MVIM and the Padé approximation improves the accuracy and enlarges the convergence domain.

1. Introduction

The boundary-layer flows of viscous fluids are of utmost importance for industry and applied sciences. These flows can be modeled by systems of nonlinear ordinary differential equations on an unbounded domain, see [1–4] and the references therein. Keeping in view the physical importance of such problems, there is a dire need of extension of some reliable and efficient technique for the solution of such problems. He [1, 2, 5–15] developed the variational iteration (VIM) and homotopy perturbation (HPM) methods which are very efficient and accurate and are [1, 2, 4–42] being used very frequently for finding the appropriate solutions of nonlinear problems of physical nature. In a later work, Ghorbani and Nadjfi [24] introduced He's polynomials which are calculated for He's homotopy perturbation method. It is also established [24] that He's polynomials are compatible with Adomian's polynomials but are easier to implement and are more user friendly. Recently, Mohyud-Din, Noor and Noor [4, 33–36] made the elegant coupling of He's polynomials and the correction functional of

variational iteration method (VIM) and found the solutions of number of nonlinear singular and nonsingular problems. It is observed that [4, 33–36] the modified version of VIM is very efficient in solving nonlinear problems. The basic motivation of this paper is the extension of the modified variational iteration method (MVIM) coupled with Padé approximation to solve a free-convective boundary-layer flow modeled by a system of nonlinear ordinary differential equations. Numerical and figurative illustrations show that it is a promising tool to solve nonlinear problems. It needs to be highlighted that Herisanu and Marinca [41] suggested an optimal variational iteration algorithm. It needs to be highlighted that He in his latest article "The variational iteration method which should be followed" [42] presented a very comprehensive and detailed study on various aspects of variational iteration method in connection with partial differential equations, ordinary differential equations, fractional differential equations, fractal-differential equations, and difference-differential equations.

2. Modified Variational Iteration Method (MVIM)

To illustrate the basic concept of the modified variational iteration method (MVIM), we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is the forcing term. According to variational iteration method [1, 2, 4, 10–23, 28, 33–39, 41, 42], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\big(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\big)d\xi, \tag{2.2}$$

where λ is a Lagrange multiplier [1, 2, 10–15, 42], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation; $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n = 0$; (2.2) is called a correction functional. Now, we apply He's polynomials [24]

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p\int_0^x \lambda(\xi)\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\right)d\xi - \int_0^x \lambda(\xi)g(\xi)d\xi, \tag{2.3}$$

which is the coupling of variational iteration method and He's polynomials and is called the modified variational iteration method (MVIM) [4, 33–36]. The comparison of like powers of p gives solutions of various orders.

3. Mathematical Model

Let us consider the problem of cooling of a low-heat-resistance sheet that moves downwards in a viscous fluid [3]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_0),$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2},$$

(3.1)

subject to

$$u = 0, \qquad v = 0 \quad \text{at } y = 0,$$

$$u \longrightarrow 0, \qquad T \longrightarrow T_0 \quad \text{as } y \longrightarrow \infty,$$

(3.2)

where u and v are the velocity components in the x- and y-directions, respectively. ψT is the temperature, T_0 is the temperature of the surrounding fluid, v is the kinematic viscosity, κ is the thermal diffusivity, g is the acceleration due to gravity, and β is the coefficient of thermal expansion. Using the similarity variables

$$\psi = \left[g\beta(T_1 - T_0)v^2 x_0^3\right]^{1/4} f(\eta),$$

$$T = T_0 + (T_1 - T_0)\left[\frac{x_0}{(x_0 - x)}\right]^3 \theta(\eta),$$

$$\eta = \left[\frac{g\beta(T_1 - T_0)x_0^3}{v^2}\right]^{1/4} \frac{y}{(x_0 - x)},$$

(3.3)

where ψ is the stream function defined by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, f and θ are the similarity functions dependent on η, $T(0,0) = T_1$ and $\theta(0) = 1$, (3.1) is transformed to

$$f'''(\eta) + \theta(\eta) - (f'(\eta))^2 = 0,$$

$$\theta''(\eta) - 3\sigma f'(\eta)\theta(\eta) = 0,$$

(3.4)

subject to the boundary conditions

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(+\infty) = 0,$$

$$\theta(0) = 1, \qquad \theta(+\infty) = 0,$$

(3.5)

where the primes denote differentiation with respect to η, and σ is the Prandtl number.

Table 1: Numerical values of $\alpha_1 = f''(0)$.

σ	[4,4]	[5,5]	[6,6]	α_1
0.001	1.1135529418	1.1272760416	1.1252849854	1.1231381347
0.01	1.0631737963	1.0741895683	1.0638385351	1.0633808585
0.1	0.9128082210	0.9238226280	0.9242158493	0.9240830397
1	0.6941230861	0.6929598014	0.6932195158	0.6932116298
10	0.4511240728	0.4502429544	0.4476712316	0.4471165250
100	0.2679197151	0.2681474363	0.2641295627	0.2645235434
1000	0.2204061432	0.1524783266	0.1500456755	0.1512901971
10000	0.0858587180	0.0858519249	0.0844775473	0.0855408524

Table 2: Numerical values of $\alpha_2 = \theta'(0)$.

σ	[4,4]	[5,5]	[6,6]	α_2 of [3]
0.001	−0.0371141028	−0.0415417739	−0.0436188230	−0.0468074648
0.01	−0.1274922800	−0.1221616907	−0.1351353865	−0.1357607439
0.1	−0.3621215470	−0.3505589981	−0.3499273453	−0.3500596733
1	−0.7694165843	−0.7695971295	−0.7698955992	−0.7698611967
10	−1.5028543431	−1.5007437650	−1.4985484075	−1.4970992078
100	−2.7627624234	−2.7637067330	−2.7445541894	−2.7468855016
1000	−5.7787858408	−4.9468469883	−4.9104728566	−4.9349476252
10000	−8.8057265644	−8.8032691004	−8.7384279086	−8.8044492660

4. The Padé Approximation

We denote L, M Padé approximants to $f(z)$ by

$$\left[\frac{L}{M}\right] = \frac{P_L(z)}{Q_M(z)}, \tag{4.1}$$

where $P_L(z)$ is polynomial of degree at most L and $Q_M(z)(Q_M(z) \neq 0)$ is a polynomial of degree at most M. The former power series is

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot z^k. \tag{4.2}$$

And we write the $P_L(z)$ and $Q_M(z)$ as

$$P_L(z) = p_0 + p_1 \cdot z + p_2 \cdot z^2 + p_3 \cdot z^3 + \cdots + p_L \cdot z^L,$$
$$Q_M(z) = q_0 + q_1 \cdot z + q_2 \cdot z^2 + q_3 \cdot z^3 + \cdots + q_M \cdot z^M, \tag{4.3}$$

so

$$f(z) - \frac{P_L(z)}{Q_M(z)} = O\left(z^{L+M+1}\right) \quad \text{as } z \longrightarrow 0, \tag{4.4}$$
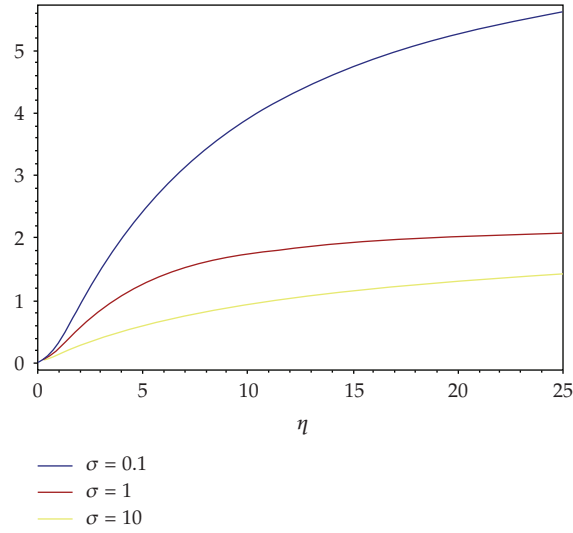
Figure 1: Variation of $f(\eta)$ using $\phi_{6[6,6]}$ for $\sigma = 0.1$, $\phi_{6[5,5]}$ for $\sigma = 1$, and $\phi_{6[4,4]}$ for $\sigma = 10$.
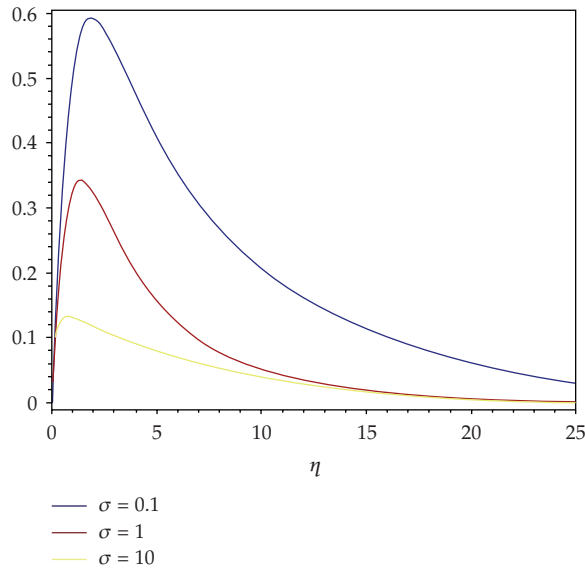

Figure 2: Variation of $f'(\eta)$ using $\phi'_{6[6,6]}$ for $\sigma = 0.1$, $\phi'_{6[5,5]}$ for $\sigma = 1$, and $\phi'_{6[4,4]}$ for $\sigma = 10$.

and the coefficients of $P_L(z)$ and $Q_M(z)$ are determined by the equation. From (4.4), we have

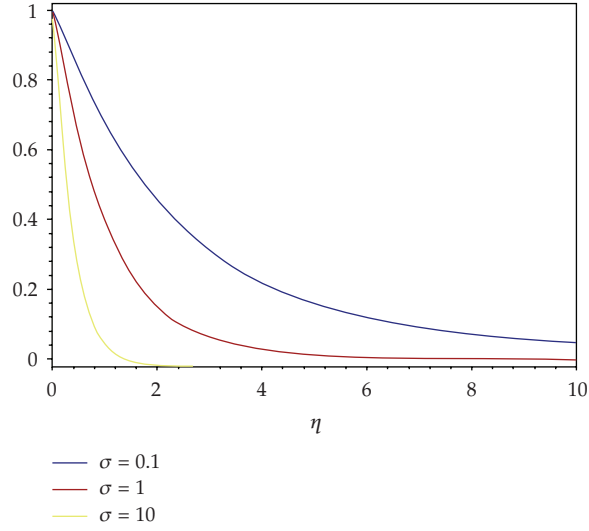$$f(z) \cdot Q_M(z) - P_L(z) = O\left(z^{L+M+1}\right), \tag{4.5}$$

Figure 3: Variation of $\theta(\eta)$ using $\varpi_{7[6,6]}$ for $\sigma = 0.1$, $\varpi_{7[5,5]}$ for $\sigma = 1$ and $\phi_{7[4,4]}$ for $\sigma = 10$.

which system of $L + M + 1$ homogeneous equations with $L + M + 2$ unknown quantities. We impose the normalization condition

$$Q_M(0) = 1. \tag{4.6}$$

We can write out (4.5) as

$$c_{L+1} + c_L \cdot q_1 + \cdots + c_{L-M+1} \cdot q_M = 0,$$

$$c_{L+2} + c_{L+1} \cdot q_1 + \cdots + c_{L-M+2} \cdot q_M = 0,$$

$$\vdots,\tag{4.7}$$

$$c_{L+M} + c_{L+M-1} \cdot q_1 + \cdots + c_L \cdot q_M = 0,$$

$$c_0 = p_0,$$

$$c_1 + c_0 \cdot q_1 = p_1,$$

$$c_2 + c_1 \cdot q_1 + c_0 \cdot q_2 = p_2,\tag{4.8}$$

$$\vdots,$$

$$c_L + c_{L-1} \cdot q_1 + \cdots + c_0 \cdot q_L = p_L.$$

From (4.7), we can obtain $q_i (1 \leq i \leq M)$. Once the values of $q_1, q_2, \ldots, q_M$ are all known (4.8) gives an explicit formula for the unknown quantities $p_1, p_2, \ldots, p_L$. For the diagonal approximants like $[2/2], [3/3], [4/4], [5/5],$ or $[6/6]$ have the most accurate approximants by built-in utilities of Maple.

5. Solution Procedure

Consider problems (3.4)–(3.5) formulated in Section 3 and is related to the free-convective boundary-layer flow.

The correction functional is given by

$$f_{n+1}(\eta) = f_n(\eta) + \int_0^x \lambda_1(s) \left(\frac{d^3 f_n}{ds^3} + \widetilde{\theta}_n(\eta) - \left(\frac{d\widetilde{f}_n}{d\eta} \right)^2 \right) ds,$$

$$\theta_{n+1}(\eta) = \theta_n(\eta) + \int_0^x \lambda_2(s) \left(\frac{d^2 \theta_n}{ds^2} - 3\sigma \left(\frac{d\widetilde{f}_n}{d\eta} \right) \right) \widetilde{\theta}_n(\eta) ds.$$

$$(5.1)$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified

$$\lambda_1(s) = -\frac{1}{2!} (s - \eta)^2, \qquad \lambda_2(s) = (s - \eta).$$

$$(5.2)$$

Consequently,

$$f_{n+1}(\eta) = f_n(\eta) - \int_0^x \frac{1}{2!} (s - \eta)^2 \left(\frac{d^3 f_n}{ds^3} + \theta_n(\eta) - \left(\frac{df_n}{d\eta} \right)^2 \right) ds,$$

$$\theta_{n+1}(\eta) = \theta_n(\eta) + \int_0^x (s - \eta) \left(\frac{d^2 \theta_n}{ds^2} - 3\sigma \left(\frac{df_n}{d\eta} \right) \right) \theta_n(\eta) ds.$$

$$(5.3)$$

Applying the modified variational iteration method (MVIM), we get

$$f_0 + pf_1 + \cdots = f_0(\eta) - p \int_0^x \frac{1}{2!} (s - \eta)^2 \left(\left(\frac{d^3 f_0}{ds^3} + p \frac{d^3 f_1}{ds^3} + \cdots \right) + (\theta_0 + p\theta_1 + \cdots) \right.$$

$$\left. - \left(\frac{df_0}{d\eta} + p \frac{df_1}{d\eta} + \cdots \right)^2 \right) ds,$$

$$\theta_0 + p\theta_1 + \cdots = \theta_0(\eta) + \int_0^x (s - \eta) \left(\left(\frac{d^2 \theta_0}{ds^2} + p \frac{d^2 \theta_1}{ds^2} + \cdots \right) - 3\sigma (\theta_0 + p\theta_1 + \cdots) \right.$$

$$\left. - \left(\frac{df_0}{d\eta} + p \frac{df_1}{d\eta} + \cdots \right)^2 \right) ds.$$

$$(5.4)$$

Comparing the coefficient of like powers of p, we get

$$p^0 : f_0(\eta) = \left(\frac{\alpha_1}{2}\right)\eta^2,$$

$$p^1 : f_1(\eta) = \left(\frac{\alpha_1}{2}\right)\eta^2 - \left(\frac{1}{6}\right)\eta^3 + \left(\frac{\alpha_2}{24}\right)\eta^4 + \left(\frac{\alpha_1^2}{60}\right)\eta^5,$$

$$p^2 : f_2(\eta) = \left(\frac{\alpha_1}{2}\right)\eta^2 - \left(\frac{1}{6}\right)\eta^3 + \left(\frac{\alpha_2}{24}\right)\eta^4 + \left(\frac{\alpha_1^2}{60}\right)\eta^5 - \left(\frac{\sigma\alpha_1}{240} + \frac{\alpha_1}{120}\right)\eta^6$$

$$+ \left(\frac{\alpha_1\alpha_2}{630} + \frac{\sigma\alpha_1\alpha_2}{120}\right)\eta^7 + \left(\frac{\alpha_1^3}{2016}\right)\eta^8,$$

$$p^3 : f_3(\eta) = \left(\frac{\alpha_1}{2}\right)\eta^2 - \left(\frac{1}{6}\right)\eta^3 + \left(\frac{\alpha_2}{24}\right)\eta^4 + \left(\frac{\alpha_1^2}{60}\right)\eta^5 - \left(\frac{\sigma\alpha_1}{240} + \frac{\alpha_1}{120}\right)\eta^6$$

$$+ \left(\frac{\alpha_1\alpha_2}{630} + \frac{\sigma\alpha_1\alpha_2}{120} + \frac{\sigma}{1680} + \frac{1}{840}\right)\eta^7$$

$$+ \left(\frac{\alpha_1^3}{2016} + \frac{\sigma\alpha_2}{3360} + \frac{\alpha_2}{2016}\right)\eta^8 + \left(-\frac{\alpha_1^2\sigma^2}{10080} + \frac{\alpha_1^2\sigma}{8640} + \frac{\sigma\alpha_2^2}{130240} - \frac{11\alpha_1^2}{30240} + \frac{\alpha_2^2}{18144}\right)\eta^9$$

$$+ \left(-\frac{\alpha_1^2\alpha_2}{14400} - \frac{19\alpha_1^2\sigma\alpha_2}{604800} - \frac{\alpha_1^2\sigma^2\alpha_2}{40320}\right)\eta^{10},$$

$$\vdots$$

$$p^0 : \theta_0(\eta) = 1 + (\alpha_2)\eta,$$

$$p^1 : \theta_1(\eta) = 1 + \alpha_2\eta + \left(\frac{\sigma\alpha_1\alpha_2}{2}\right)\eta^3 + \left(\frac{\sigma\alpha_1\alpha_2}{4}\right)\eta^4,$$

$$p^2 : \theta_2(\eta) = 1 + \alpha_2\eta + \left(\frac{\sigma\alpha_1\alpha_2}{2}\right)\eta^3 + \left(\frac{\sigma\alpha_1\alpha_2}{4}\right)\eta^4 - \left(\frac{\alpha_1\alpha_2}{10}\right)\eta^5 + \left(\frac{\alpha_1^2\sigma^2}{20} + \frac{\alpha_1^2\sigma}{120} - \frac{\alpha_2^2\sigma}{60}\right)\eta^6$$

$$\times \left(\frac{\alpha_1^2\sigma\alpha_2}{168} + \frac{\alpha_1^2\alpha_2\sigma^2}{56}\right)\eta^7,$$

$$p^3 : \theta_3(\eta) = 1 + \alpha_2\eta + \left(\frac{\sigma\alpha_1\alpha_2}{2}\right)\eta^3 + \left(\frac{\sigma\alpha_1\alpha_2}{4}\right)\eta^4 - \left(\frac{\alpha_1\alpha_2}{10}\right)\eta^5 + \left(\frac{\alpha_1^2\sigma^2}{20} + \frac{\alpha_1^2\sigma}{120} - \frac{\alpha_2^2\sigma}{60}\right)\eta^6$$

$$+ \left(\frac{\alpha_1^2\sigma\alpha_2}{168} + \frac{\alpha_1^2\alpha_2\sigma^2}{56} - \frac{\sigma\alpha_1}{280} - \frac{\alpha_1\sigma^2}{35}\right)\eta^7 + \left(-\frac{11\alpha_1\sigma\alpha_2}{3360} - \frac{41\alpha_1\sigma^2\alpha_2}{2240}\right)\eta^8$$

$$+ \left(\frac{\alpha_1^2\sigma^2\alpha_2^2}{360} + \frac{\alpha_1^2\sigma}{6048} + \frac{\sigma^3\alpha_1^3}{480} - \frac{\alpha_1\alpha_2^2\sigma}{2160}\right)\eta^9$$

$$+ \left(\frac{\alpha_1^3 \alpha_2 \sigma \alpha_2^2}{7560} + \frac{\alpha_1^2 \sigma^2 \alpha_2}{1120} + \frac{\sigma^3 \alpha_2 \alpha_1^3}{1680} \right) \eta^{10},$$

$$\vdots \tag{5.5}$$

The series solution is given by

$$f(\eta) = \left(\frac{\alpha_1}{2} \right) \eta^2 - \left(\frac{1}{6} \right) \eta^3 + \left(\frac{\alpha_2}{24} \right) \eta^4 + \left(\frac{\alpha_1^2}{60} \right) \eta^5 - \left(\frac{\sigma \alpha_1}{240} + \frac{\alpha_1}{120} \right) \eta^6$$

$$+ \left(\frac{\alpha_1 \alpha_2}{630} + \frac{\sigma \alpha_1 \alpha_2}{120} + \frac{\sigma}{1680} + \frac{1}{840} \right) \eta^7$$

$$+ \left(\frac{\alpha_1^3}{2016} + \frac{\sigma \alpha_2}{3360} + \frac{\alpha_2}{2016} \right) \eta^8 + \left(-\frac{\alpha_1^2 \sigma^2}{10080} + \frac{\alpha_1^2 \sigma}{8640} + \frac{\sigma \alpha_2^2}{130240} - \frac{11 \alpha_1^2}{30240} + \frac{\alpha_2^2}{18144} \right) \eta^9$$

$$+ \left(-\frac{\alpha_1^2 \alpha_2}{14400} - \frac{19 \alpha_1^2 \sigma \alpha_2}{604800} - \frac{\alpha_1^2 \sigma^2 \alpha_2}{40320} \right) \eta^{10} + \cdots,$$

$$\theta(\eta) = 1 + \alpha_2 \eta + \left(\frac{\sigma \alpha_1 \alpha_2}{2} \right) \eta^3 + \left(\frac{\sigma \alpha_1 \alpha_2}{4} \right) \eta^4 - \left(\frac{\alpha_1 \alpha_2}{10} \right) \eta^5 + \left(\frac{\alpha_1^2 \sigma^2}{20} + \frac{\alpha_1^2 \sigma}{120} - \frac{\alpha_2^2 \sigma}{60} \right) \eta^6$$

$$+ \left(\frac{\alpha_1^2 \sigma \alpha_2}{168} + \frac{\alpha_1^2 \alpha_2 \sigma^2}{56} - \frac{\sigma \alpha_1}{280} - \frac{\alpha_1 \sigma^2}{35} \right) \eta^7 + \left(-\frac{11 \alpha_1 \sigma \alpha_2}{3360} - \frac{41 \alpha_1 \sigma^2 \alpha_2}{2240} \right) \eta^8$$

$$+ \left(\frac{\alpha_1^2 \sigma^2 \alpha_2^2}{360} + \frac{\alpha_1^2 \sigma}{6048} + \frac{\sigma^3 \alpha_1^3}{480} - \frac{\alpha_1 \alpha_2^2 \sigma}{2160} \right) \eta^9$$

$$+ \left(\frac{\alpha_1^3 \alpha_2 \sigma \alpha_2^2}{7560} + \frac{\alpha_1^2 \sigma^2 \alpha_2}{1120} + \frac{\sigma^3 \alpha_2 \alpha_1^3}{1680} \right) \eta^{10} + \cdots. \tag{5.6}$$

It is observed in Figures 1 and 2 that the flow has a boundary-layer structure and the thickness of this boundary-layer decreases with increase in the Prandtl number, σ as expected. This is due to the inhibiting influence of the viscous forces.

Figure 3 shows the increase of the Prandtl number, σ, that results in the decrease, as expected, of temperature distribution at a particular point of the flow region, that is, there would be a decrease of the thermal boundary-layer thickness with the increase of values of σ implying a slow rate of thermal diffusion. Thus higher Prandtl number σ leads to faster cooling of the plane sheet.

6. Conclusions

In this study, we employed modified variational iteration method (MVIM) coupled with Padé approximation to solve a system of two nonlinear ordinary differential equations that describes a free-convective boundary-layer in glass-fiber production process. The results

show strong effects of the Prandtl number on the velocity and temperature profiles since the two model equations are coupled.

Acknowledgment

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