

Research Article

Unbiased Minimum-Variance Filter for State and Fault Estimation of Linear Time-Varying Systems with Unknown Disturbances

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This paper presents a new recursive filter to joint fault and state estimation of a linear time-varying discrete systems in the presence of unknown disturbances. The method is based on the assumption that no prior knowledge about the dynamical evolution of the fault and the disturbance is available. As the fault affects both the state and the output, but the disturbance affects only the state system. Initially, we study the particular case when the direct feedthrough matrix of the fault has full rank. In the second case, we propose an extension of the previous case by considering the direct feedthrough matrix of the fault with an arbitrary rank. The resulting filter is optimal in the sense of the unbiased minimum-variance (UMV) criteria. A numerical example is given in order to illustrate the proposed method.

1. Introduction

This paper is concerned with the problem of joint fault and state estimation of linear time-varying discrete-time stochastic systems in the presence of unknown disturbances. In spite of the presence of the unknown inputs, the robust estimate of the state and the fault enables us to implement a Fault Tolerant Control (FTC). A simple idea consists of using an architecture FTC resting on the compensation of the effect of the fault; see, for example, [1].

Initially, we refer to the unknown input filtering problem largely treated in the literature by two different approaches. The first approach was based on the augmentation of the state vector with an unknown input vector. However, this approach assumes that the model for the dynamical evolution of the unknown inputs is available. When the statistical

properties of the unknown input are perfectly known, the augmented state Kalman filter (ASKF) is an optimal solution. To reduce computation costs of the ASKF, Friedland [2] developed the two-stage Kalman filter (TSKF). This latter is optimal only for a constant bias. Many authors have extended Friedland's idea to treat the stochastic bias, for example, [3–5]. Recently, Kim et al. [6, 7] have developed an adaptive two-stage Kalman filter (ATSKF). The second approach treats the case when we do not have a prior knowledge about the dynamical evolution of the unknown input. Kitanidis [8] was the first to solve this problem using the linear unbiased minimum-variance (UMV). Darouach et al. [9] extended Kitanidis's filter using a parameterizing technique to have an optimal estimator filter (OEF). Hsieh [10] has developed an equivalent to Kitanidis's filter noted robust two-stage Kalman filter (RTSKF). Later, Hsieh [11] developed an optimal minimum variance filter (OMVF) to solve the performance degradation problem encountered in OEF. Gillijns and Moor [12] have treated the problem of estimating the state in the presence of unknown inputs which affect the system model. They developed a recursive filter which is optimal in the sense of minimum-variance. This filter has been extended by the same authors [13] for joint input and state estimation to linear discrete-time systems with direct feedthrough where the state and the unknown input estimation are interconnected. This filter is called recursive three-step filter (RTSF) and is limited to direct feedthrough matrix with full rank. Recently, Cheng et al. [14] proposed a recursive optimal filter with global optimality in the sense of unbiased minimum-variance over all linear unbiased estimators, but this filter is limited to estimate the state (i.e., no estimate of the unknown input). In [15], the author has extended an RTSF-noted ERTSF, where he solved a general case when the direct feedthrough matrix has an arbitrary rank.

In this paper, we develop a new recursive filter to joint fault and state estimation for linear stochastic, discrete-time, and time-varying systems in the presence of unknown disturbances. We assume that the unknown disturbances affect only the state equation. While, the fault affects both the state and the output equations, as well, we consider that the direct feedthrough matrix has an arbitrary rank [15].

This paper is organized as follows. Section 2 states the problem of interest. Section 3 is dedicated to the design of the proposed filter. In Section 4, the obtained filter is summarized. An illustrative example is presented in Section 5. Finally, in Section 6 we conclude our obtained results.

2. Statement of the Problem

Assume the following linear stochastic discrete-time system:

$$\begin{aligned}x_{k+1} &= A_k x_k + B_k u_k + F_k^x f_k + E_k^x d_k + w_k, \\y_k &= H_k x_k + F_k^y f_k + v_k,\end{aligned}\tag{2.1}$$

where $x_k \in \mathfrak{R}^n$ is the state vector, $y_k \in \mathfrak{R}^m$ is the observation vector, $u_k \in \mathfrak{R}^r$ is the known control input, $f_k \in \mathfrak{R}^p$ is the additive fault vector, and $d_k \in \mathfrak{R}^q$ is the unknown disturbances. w_k and v_k are uncorrelated white noise sequences of zero-mean and covariance matrices are $Q_k \geq 0$ and $R_k > 0$, respectively. The disturbance d_k is assumed to have no stochastic description and must be decoupled. The initial state is uncorrelated with the white noises

processes w_k and v_k and x_0 is a Gaussian random variable with $\mathcal{E}[x_0] = \hat{x}_0$ and $\mathcal{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] = P_0^x$ where $\mathcal{E}[\cdot]$ denotes the expectation operator. The matrices A_k , B_k , F_k^x , E_k^x , H_k , and F_k^y are known and have appropriate dimensions. We consider the following assumptions:

- (i) A_1 : (H_k, A_k) is observable,
- (ii) A_2 : $n > m \geq p + q$,
- (iii) A_3 : $0 < \text{rank}(F_k^y) \leq p$,
- (iv) A_4 : $\text{rank}(H_k E_{k-1}^x) = \text{rank}(E_{k-1}^x) = q$.

The objective of this paper is to design an unbiased minimum-variance linear estimator of the state x_k and the fault f_k without any information concerning the fault f_k and the unknown disturbances d_k . We can consider that the filter has the following form:

$$\hat{x}_{k/k-1} = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} + F_{k-1}^x \hat{f}_{k-1}, \quad (2.2)$$

$$\hat{f}_k = K_k^f (y_k - H_k \hat{x}_{k/k-1}), \quad (2.3)$$

$$\hat{x}_k = \hat{x}_{k/k-1} + K_k^x (y_k - H_k \hat{x}_{k/k-1}), \quad (2.4)$$

where the gain matrices $K_k^f \in \mathfrak{R}^{p \times m}$ and $K_k^x \in \mathfrak{R}^{n \times m}$ are determined to satisfy the following criteria.

Unbiasedness

The estimator must satisfy

$$\mathcal{E}[\tilde{f}_k] = \mathcal{E}[f_k - \hat{f}_k] = 0, \quad (2.5)$$

$$\mathcal{E}[\tilde{x}_k] = \mathcal{E}[x_k - \hat{x}_k] = 0. \quad (2.6)$$

Minimum-Variance

The estimator is determined such that

- (i) the mean square errors $\mathcal{E}[\tilde{f}_k \tilde{f}_k^T]$ is minimized under the constraint (2.5);
- (ii) the trace $\{P_k^x = \mathcal{E}[\tilde{x}_k \tilde{x}_k^T]\}$ is minimized under the constraints (2.5) and (2.6).

3. Filter Design

In this section, the fault and the state estimation are considered in the presence of the unknown disturbance in two cases with respect to assumption A_3 . Section 3.1 is dedicated to deriving a UMV state and fault estimation filter if matrix F_k^y has full rank (i.e., $\text{rank}(F_k^y) = p$). A general case will be solved by an extension of the UMV state and fault estimation filter in Section 3.2.

3.1. UMV Fault and State Estimation

In this subsection, we will study a particular case when the $\text{rank}(F_k^y) = p$. The gain matrices K_k^f and K_k^x will be determined as that (2.3) and (2.4) can give an unbiased estimation of f_k and x_k . In the next, the UMV fault and state estimation are solved.

3.1.1. Unbiased Estimation

The innovation error has the following form

$$\tilde{y}_k := y_k - H_k \hat{x}_{k/k-1} = F_k^y f_k + H_k E_{k-1}^x d_{k-1} + e_k, \quad (3.1)$$

where

$$e_k = H_k \tilde{x}_{k/k-1} + v_k, \quad (3.2)$$

$$\tilde{x}_{k/k-1} = A_{k-1} \tilde{x}_{k-1} + F_{k-1}^x \tilde{f}_{k-1} + w_{k-1}. \quad (3.3)$$

The fault estimation error and the state estimation error are, respectively, given by

$$\begin{aligned} \tilde{f}_k &:= f_k - \hat{f}_k \\ &= (I - K_k^f F_k^y) f_k - K_k^f H_k E_{k-1}^x d_{k-1} - K_k^f e_k, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \tilde{x}_k &:= x_k - \hat{x}_k \\ &= (I - K_k^x H_k) \tilde{x}_{k/k-1} - K_k^x F_k^y f_k - (K_k^x H_k E_{k-1}^x - E_{k-1}^x) d_{k-1} - K_k^x v_k. \end{aligned} \quad (3.5)$$

The estimators \hat{x}_k and \hat{f}_k are unbiased if K_k^f and K_k^x satisfy the following constraints:

$$K_k^f G_k = \mathcal{F}_k, \quad (3.6)$$

$$K_k^x G_k = \Gamma_k, \quad (3.7)$$

where $G_k = [F_k^y \quad H_k E_{k-1}^x]$, $\mathcal{F}_k = [I_p \quad 0]$ and $\Gamma_k = [0 \quad E_{k-1}^x]$.

Lemma 3.1. *Let $\text{rank}(F_k^y) = p$; under the assumptions A_2 and A_4 , the necessary and sufficient condition so that the estimators (2.3) and (2.4) are unbiased as matrix G_k is full column rank, that is,*

$$\text{rank}(G_k) = \text{rank}\left(F_k^y \quad H_k E_{k-1}^x\right) = p + q. \quad (3.8)$$

Proof. Equations (3.6) and (3.7) can be written as

$$\begin{bmatrix} K_k^f \\ K_k^x \end{bmatrix} G_k = \begin{bmatrix} \mathcal{F}_k \\ \Gamma_k \end{bmatrix}. \quad (3.9)$$

The necessary and sufficient condition for the existence of the solution to (3.9) is

$$\text{rank} \begin{bmatrix} \mathcal{F}_k \\ \Gamma_k \\ G_k \end{bmatrix} = \text{rank}(G_k). \quad (3.10)$$

We clarify (3.10), and we obtain

$$\text{rank} \begin{bmatrix} I_p & 0 \\ 0 & E_{k-1}^x \\ F_k^y & H_k E_{k-1}^x \end{bmatrix} = \text{rank} \begin{pmatrix} F_k^y & H_k E_{k-1}^x \end{pmatrix}. \quad (3.11)$$

However, the matrix on the left of the equality has a rank equal to $p + q$. According to assumptions A_2 , A_4 and $\text{rank}(F_k^y) = p$, this can be easily justified by considering that the faults and the unknown disturbances have an independent influences. The condition to satisfy is thus given by (3.8). \square

3.1.2. UMV Estimation

In this subsection, we propose to determine the gain matrices K_k^f and K_k^x by satisfying the unbiasedness constraints (2.5) and (2.6).

(a) Fault Estimation

Equation (3.1) will be written as

$$\tilde{y}_k = G_k \begin{bmatrix} f_k \\ d_{k-1} \end{bmatrix} + e_k. \quad (3.12)$$

Since, e_k does not have unit variance and \tilde{y}_k does not satisfy the assumptions of the Gauss-Markov theorem [16], the least-square (LS) solutions do not have a minimum-variance. Nevertheless, the covariance matrix of e_k has the following form:

$$C_k = \mathcal{E}[e_k \ e_k^T] = H_k \bar{P}_{k/k-1}^x H_k^T + R_k, \quad (3.13)$$

where $\bar{P}_{k/k-1}^x = \mathcal{E}[\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T]$.

For that, f_k can be obtained by a weighted least-square (WLS) estimation with a weighting matrix C_k^{-1} .

Theorem 3.2. Let $\tilde{x}_{k/k-1}$ be unbiased; the matrix C_k is positive definite and the matrix G_k is full column rank; then to have a UMV fault estimation, the matrix gain K_k^f is given by

$$K_k^{f*} = \mathcal{F}_k G_k^*, \quad (3.14)$$

where $G_k^* = (G_k^T C_k^{-1} G_k)^{-1} G_k^T C_k^{-1}$.

Proof. Under that C_k is positive definite and an invertible matrix $S_k \in \mathfrak{R}^{m \times m}$ verifies $S_k S_k^T = C_k$, so we can rewrite (3.12) as follow.

$$S_k^{-1} \tilde{y}_k = S_k^{-1} G_k \begin{bmatrix} f_k \\ d_{k-1} \end{bmatrix} + S_k^{-1} e_k. \quad (3.15)$$

If the matrix G_k is full column rank, that is, $\text{rank}(G_k) = p + q$, then the matrix $G_k^T C_k^{-1} G_k$ is invertible. Solving (3.15) by an LS estimation is equivalent to solve (3.12) by WLS solution:

$$\hat{f}_k^* = \mathcal{F}_k (G_k^T C_k^{-1} G_k)^{-1} G_k^T C_k^{-1} \tilde{y}_k. \quad (3.16)$$

In this way, we can consider that $S_k^{-1} e_k$ has a unit variance and (3.15) can satisfy the assumptions of the Gauss-Markov theorem. Hence, (3.16) is the UMV estimate of f_k . \square

In this case, the fault estimation error is rewritten as follows:

$$\tilde{f}_k^* = -K_k^{f*} e_k. \quad (3.17)$$

Using (3.17), the covariance matrix P_k^f is given by

$$P_k^{f*} = \mathcal{E}[\tilde{f}_k^* \tilde{f}_k^{*T}] = K_k^{f*} C_k K_k^{f*T} = \mathcal{F}_k (G_k^T C_k^{-1} G_k)^{-1} \mathcal{F}_k^T. \quad (3.18)$$

(b) State Estimation

In this part, we propose to obtain an unbiased minimum variance state estimator to calculate the gain matrix K_k^x which will minimize the trace of covariance matrix P_k^x under the unbiasedness constraint (3.7).

Theorem 3.3. Let $G_k^T C_k^{-1} G_k$ be nonsingular; then the state gain matrix K_k^x is given by

$$K_k^{x*} = \bar{P}_{k/k-1}^x H_k^T C_k^{-1} (I - G_k G_k^*) + \Gamma_k G_k^*. \quad (3.19)$$

Proof. Considering (3.7) and (3.5), we determine P_k^x as follows:

$$\begin{aligned} P_k^x &= (I - K_k^x H_k) \bar{P}_{k/k-1}^x (I - K_k^x H_k)^T + K_k^x R_k K_k^{xT} \\ &= K_k^x C_k K_k^{xT} - 2 \bar{P}_{k/k-1}^x H_k^T K_k^{xT} + \bar{P}_{k/k-1}^x. \end{aligned} \quad (3.20)$$

So, the optimization problem can be solved using Lagrange multipliers:

$$\text{trace}\left\{K_k^x C_k K_k^{xT} - 2\bar{P}_{k/k-1}^x H_k^T K_k^{xT} + \bar{P}_{k/k-1}^x\right\} - 2 \text{trace}\left\{(K_k^x G_k - \Gamma_k) \Lambda_k^T\right\}, \quad (3.21)$$

where Λ_k is the matrix of Lagrange multipliers.

To derive (3.21) with respect to K_k^x , we obtain

$$C_k K_k^{x*T} - H_k \bar{P}_{k/k-1}^x - G_k \Lambda_k^T = 0. \quad (3.22)$$

Equations (3.7) and (3.22) form the linear system of equations:

$$\begin{bmatrix} C_k & -G_k \\ G_k^T & 0 \end{bmatrix} \begin{bmatrix} K_k^{x*T} \\ \Lambda_k^T \end{bmatrix} = \begin{bmatrix} H_k \bar{P}_{k/k-1}^x \\ \Gamma_k^T \end{bmatrix}. \quad (3.23)$$

If $G_k^T C_k^{-1} G_k$ is nonsingular, (3.23) will have a unique solution. \square

3.1.3. The Filter Time Update

From (3.3), the prior covariance matrix $\bar{P}_{k/k-1}^x = \mathcal{E}[\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T]$ has the following form:

$$\bar{P}_{k/k-1}^x = \begin{bmatrix} A_{k-1} & F_{k-1}^x \end{bmatrix} \begin{bmatrix} P_{k-1}^{x*} & P_{k-1}^{xf*} \\ P_{k-1}^{fx*} & P_{k-1}^{f*} \end{bmatrix} \begin{bmatrix} A_{k-1}^T \\ F_{k-1}^{xT} \end{bmatrix} + Q_{k-1}, \quad (3.24)$$

where $P_k^{xf*} := \mathcal{E}[\tilde{x}_k^* \tilde{f}_k^{*T}]$ is calculated by using (2.3) and (2.4):

$$P_k^{xf*} = -(I - K_k^{x*} H_k) \bar{P}_{k/k-1}^x H_k^T K_k^{f*T} + K_k^{x*} R_k K_k^{f*T}. \quad (3.25)$$

3.2. Extended UMV Fault and State Estimation

In this section, we consider that $0 < \text{rank}(F_k^y) \leq p$. To solve this interesting problem we will use the proposed approach by Hsieh in (2009) [15]. If we introduce (3.2) and (3.3) into (3.4), then we will be able to write the fault error estimation as follows:

$$\begin{aligned} \tilde{f}_k &:= \left(I - K_k^f F_k^y\right) f_k - K_k^f H_k E_{k-1}^x d_{k-1} - K_k^f \left(H_k \tilde{x}_{k/k-1} + v_k\right) \\ &= -K_k^f H_k F_{k-1}^x \tilde{f}_{k-1} - K_k^f H_k A_{k-1} \tilde{x}_{k-1} + \left(I - K_k^f F_k^y\right) f_k - K_k^f H_k E_{k-1}^x d_{k-1} \\ &\quad - K_k^f H_k w_{k-1} - K_k^f v_k. \end{aligned} \quad (3.26)$$

Assuming that $\mathcal{E}[\tilde{x}_{k-1}] = 0$ we define the following notations:

$$\begin{aligned}\Phi_k &= K_k^f F_k^y = I_p - \Sigma_k, \\ G_k^f &= K_k^f H_k F_{k-1}^x, \\ G_k^d &= K_k^f H_k E_{k-1}^x,\end{aligned}\tag{3.27}$$

where $\Sigma_k = I - (F_k^y)^+ F_k^y$.

Using the same technique presented in [15], the expectation value of the \tilde{f}_k is given by

$$\begin{aligned}\mathcal{E}[\tilde{f}_k] &= \Sigma_k f_k - G_k^f \Sigma_{k-1} f_{k-1} + G_k^f (G_{k-1}^f \Sigma_{k-2} - 2) f_{k-2} + \dots + (-1)^k G_k^f \times \dots \times G_2^f (G_1^f \Sigma_0) f_0 \\ &\quad - G_k^d d_{k-1} + G_k^f G_{k-1}^d d_{k-2} + \dots + (-1)^k G_k^f \times \dots \times G_1^f G_1^d d_0.\end{aligned}\tag{3.28}$$

When we assume that $G_i^f \Sigma_{i-1} = 0$ and $G_i^d = 0$ for $i = 1, \dots, k$, then we obtain

$$\mathcal{E}[\tilde{f}_k] = \Sigma_k f_k.\tag{3.29}$$

To obtain an unbiased estimation of the fault, the gain matrix K_k^f should respect the following constraints:

$$\begin{aligned}K_k^f F_k^y &= \Phi_k, \\ K_k^f H_k F_{k-1}^x \Sigma_{k-1} &= 0, \\ K_k^f H_k E_{k-1}^x &= 0.\end{aligned}\tag{3.30}$$

Equation (3.30) can be written as

$$K_k^f \bar{G}_k = \bar{\mathcal{F}}_k,\tag{3.31}$$

where

$$\begin{aligned}\bar{G}_k &= \begin{bmatrix} F_k^y & H_k F_{k-1}^x \Sigma_{k-1} & H_k E_{k-1}^x \end{bmatrix}, \\ \bar{\mathcal{F}}_k &= [\Phi_k \ 0 \ 0].\end{aligned}\tag{3.32}$$

Using (3.31), we can determine the gain matrix K_k^f as follows:

$$K_k^{f*} = \bar{\mathcal{F}}_k \bar{G}_k^*, \quad (3.33)$$

where $\bar{G}_k^* = (\bar{G}_k^T C_k^{-1} \bar{G}_k)^+ \bar{G}_k^T C_k^{-1}$ and X^+ denotes the Moore-Penrose pseudoinverse of X .
The state estimation error is given by

$$\begin{aligned} \tilde{x}_k &:= (I - K_k^x H_k) \tilde{x}_{k/k-1} - K_k^x F_k^y f_k - (K_k^x H_k E_{k-1}^x - E_{k-1}^x) d_{k-1} - K_k^x v_k \\ &= (I - K_k^x H_k) A_{k-1} \tilde{x}_{k-1} + (I - K_k^x H_k) F_{k-1}^x \tilde{f}_k - (K_k^x H_k E_{k-1}^x - E_{k-1}^x) d_{k-1} \\ &\quad - K_k^x F_k^y f_k + (I - K_k^x H_k) w_{k-1} - K_k^x v_k. \end{aligned} \quad (3.34)$$

To have an unbiased estimation of the state, the gain matrix K_k^x should satisfy the following constraints:

$$\begin{aligned} K_k^x F_k^y &= 0, \\ K_k^x H_k F_{k-1}^x \Sigma_{k-1} &= F_{k-1}^x \Sigma_{k-1}, \\ K_k^x H_k E_{k-1}^x &= E_{k-1}^x. \end{aligned} \quad (3.35)$$

From (3.35), we obtain

$$K_k^x \bar{G}_k = \bar{\Gamma}_k, \quad (3.36)$$

where

$$\bar{\Gamma}_k = [0 \quad F_{k-1}^x \Sigma_{k-1} \quad E_{k-1}^x]. \quad (3.37)$$

Referring to (3.34), we calculate the error state covariance matrix:

$$\begin{aligned} P_k^x &= (I - K_k^x H_k) \bar{P}_{k/k-1}^x (I - K_k^x H_k)^T + K_k^x R_k K_k^{xT} \\ &= K_k^x C_k K_k^{xT} - 2 \bar{P}_{k/k-1}^x H_k^T K_k^{xT} + \bar{P}_{k/k-1}^x. \end{aligned} \quad (3.38)$$

The gain matrix K_k^x is determined by minimizing the trace of the covariance matrix P_k^x such as (3.36). Using the Kitanidis method [8], we obtain

$$\begin{bmatrix} C_k & -\bar{G}_k \\ \bar{G}_k^T & 0 \end{bmatrix} \begin{bmatrix} K_k^{x*T} \\ \Lambda_k^T \end{bmatrix} = \begin{bmatrix} H_k \bar{P}_{k/k-1}^x \\ \bar{\Gamma}_k^T \end{bmatrix}, \quad (3.39)$$

where Λ_k is the matrix of Lagrange multipliers.

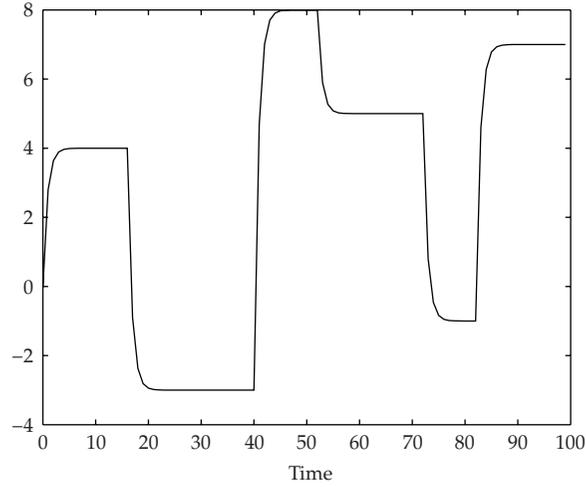


Figure 1: Known input u_k .

If $\bar{G}_k^T C_k^{-1} \bar{G}_k$ is nonsingular, (3.39) will have a unique solution. So, the gain matrix K_k^x is given by

$$K_k^{x*} = \bar{P}_{k/k-1}^x H_k^T C_k^{-1} (I - \bar{G}_k \bar{G}_k^*) + \bar{\Gamma}_k \bar{G}_k^*. \quad (3.40)$$

The filter time update is the same as that given by (3.24) and (3.25). The obtained filters will be tested by an illustrative example in Section 5.

4. Summary of Filter Equations

We suppose to know the following:

- (i) the known input u_k ,
- (ii) matrices A_k , B_k , H_k , F_k^x , F_k^y and E_k^x ,
- (iii) covariance matrices Q_k^x and R_k ,
- (iv) initial values \hat{x}_0 and P_0^x .

We assume that the estimate of the initial state is unbiased and we take the initial covariance matrix $\bar{P}_{0/0-1}^x = P_0^x$.

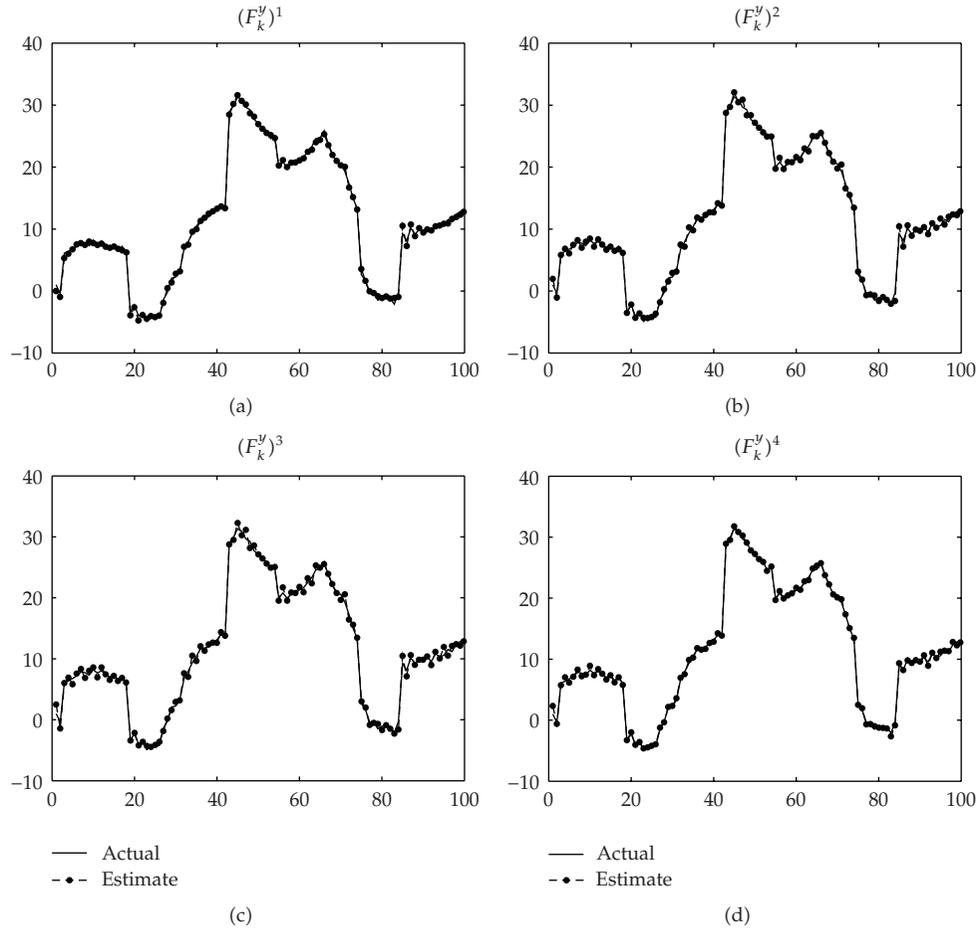


Figure 2: Actual state x_k^1 and estimated state \hat{x}_k^1 .

Step 1. Estimation of fault is

$$\begin{aligned}
 C_k &= H_k \bar{P}_{k/k-1}^x H_k^T + R_k, \\
 \bar{G}_k &= \begin{bmatrix} F_k^y & H_k F_{k-1}^x \Sigma_{k-1} & H_k E_{k-1}^x \end{bmatrix}, \\
 \bar{\mathcal{F}}_k &= [\Phi_k \ 0 \ 0], \\
 \bar{G}_k^* &= (\bar{G}_k^T C_k^{-1} \bar{G}_k)^+ \bar{G}_k^T C_k^{-1}, \\
 K_k^f &= \bar{\mathcal{F}}_k \bar{G}_k^*, \\
 \hat{f}_k &= K_k^f (y_k - H_k \hat{x}_{k/k-1}), \\
 P_k^f &= K_k^f C_k K_k^{fT}.
 \end{aligned} \tag{4.1}$$

Step 2. Measurement update is

$$\begin{aligned}\bar{\Gamma}_k &= [0 \quad F_{k-1}^x \Sigma_{k-1} \quad E_{k-1}^x], \\ K_k^x &= \bar{P}_{k/k-1}^x H_k^T C_k^{-1} (I - \bar{G}_k \bar{G}_k^*) + \bar{\Gamma}_k \bar{G}_k^*, \\ \hat{x}_k &= \hat{x}_{k/k-1} + K_k^x (y_k - H_k \hat{x}_{k/k-1}), \\ P_k^x &= (I - K_k^x H_k) \bar{P}_{k/k-1}^x (I - K_k^x H_k)^T + K_k^x R_k K_k^{xT}, \\ P_k^{xf} &= -(I - K_k^x H_k) \bar{P}_{k/k-1}^x H_k^T K_k^{fT} + K_k^x R_k K_k^{fT}.\end{aligned}\tag{4.2}$$

Step 3. Time update is

$$\begin{aligned}\hat{x}_{k+1/k} &= A_k \hat{x}_k + B_k u_k + F_k^x \hat{f}_k, \\ \bar{P}_{k+1/k}^x &= [A_k \quad F_k^x] \begin{bmatrix} P_k^x & P_k^{xf} \\ P_k^{fx} & P_k^f \end{bmatrix} \begin{bmatrix} A_k^T \\ F_k^{xT} \end{bmatrix} + Q_k.\end{aligned}\tag{4.3}$$

Remark 4.1. If $\text{rank}(F_k^y) = p$, then we have $\Sigma_k = 0$ for all $k \geq 0$ and it is easier to use the filter obtained in Section 3.1. In this case, the gain matrices K_k^f and K_k^x are given by (3.14) and (3.19), respectively.

Remark 4.2. These remarks give the relationships with the existing literature results.

- (i) If $E_k^x = 0$ and $0 < \text{rank}(F_k^y) \leq p$, the obtained filter is equivalent to ERTSF developed by [15].
- (ii) If $E_k^x = 0$ and $\text{rank}(F_k^y) = p$, then we have $\Sigma_k = 0$ for all $k \geq 0$ and the obtained filter is equivalent to RTSF proposed by [13].
- (iii) In the case where $F_k^x = 0$ and $F_k^y = 0$, the filter of [8] is obtained.
- (iv) In the case where $F_k^x = 0$, $F_k^y = 0$ and $E_k^x = 0$, we obtain the standard Kalman filter.

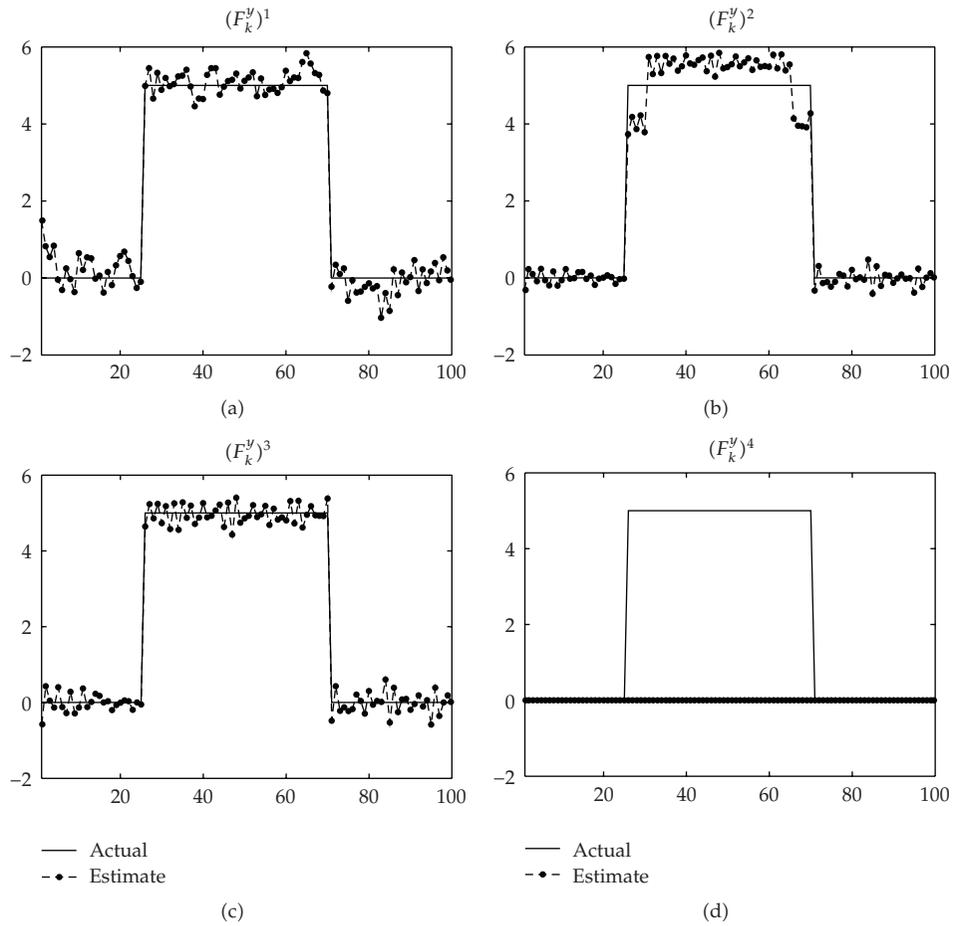


Figure 3: Actual fault f_k^1 and estimated fault \hat{f}_k^1 .

5. An Illustrative Example

To apply our proposed filters we will treat different cases to respect assumption A_3 . The parameters of the system (2.1) are given by

$$x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \end{bmatrix}, \quad A_k = \begin{bmatrix} a_k & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.5 & 0.1 & 0.25 \end{bmatrix}, \quad a_k = 0.4 + 0.3 \sin(0.2k), \quad B_k = \begin{bmatrix} 2 \\ -1.5 \\ 0.5 \end{bmatrix},$$

$$F_k^x = \begin{bmatrix} 0.5 & 0.7 \\ 1.5 & 1.1 \\ 0.8 & 0.9 \end{bmatrix}, \quad E_k^x = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad H_k = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad Q_k = 0.1I_{3 \times 3}, \quad R_k = 0.01I_{3 \times 3},$$

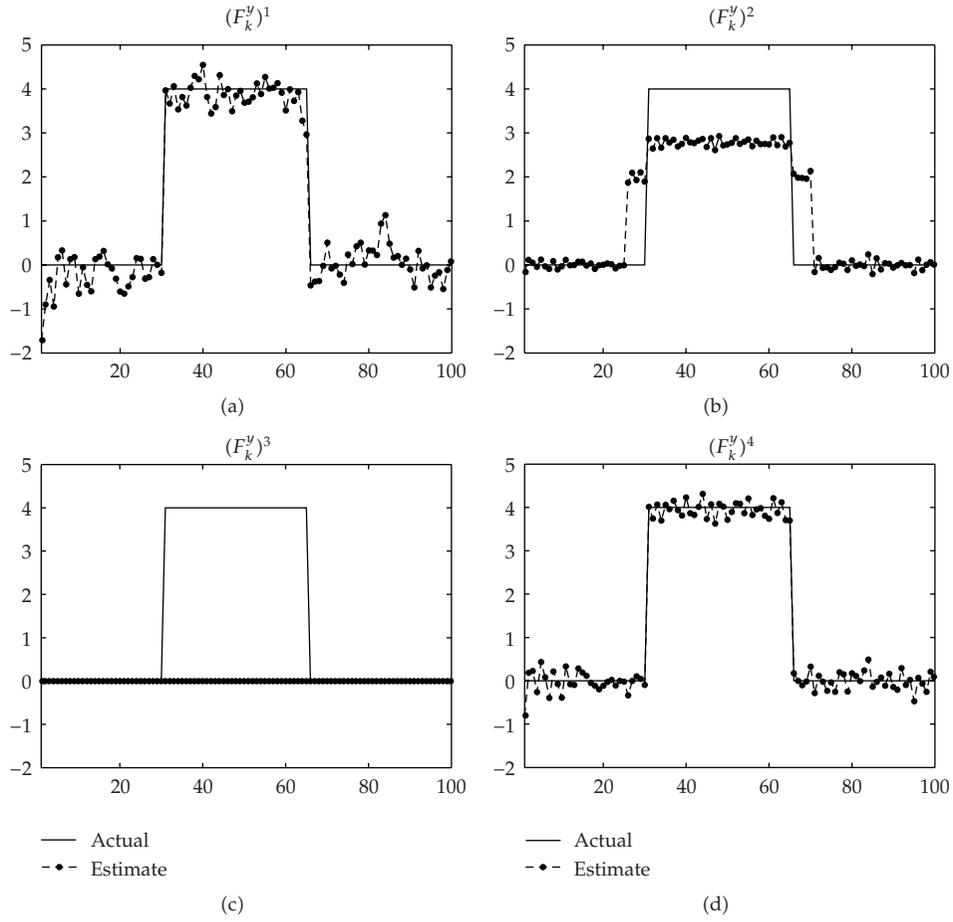


Figure 4: Actual fault f_k^2 and estimated fault \hat{f}_k^2 .

$$x_0 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \hat{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_0^x = I_{3 \times 3}. \quad (5.1)$$

In this simulation, four cases of F_k^y will be considered as follows:

$$\begin{aligned} (F_k^y)^1 &= \begin{bmatrix} 2 & 1.4 \\ 0.6 & 0.3 \\ 0.2 & 1.6 \end{bmatrix}, & (F_k^y)^2 &= \begin{bmatrix} 2 & 1 \\ 0.6 & 0.3 \\ 0.2 & 0.1 \end{bmatrix}, \\ (F_k^y)^3 &= \begin{bmatrix} 2 & 0 \\ 0.6 & 0 \\ 0.2 & 0 \end{bmatrix}, & (F_k^y)^4 &= \begin{bmatrix} 0 & 1.4 \\ 0 & 0.3 \\ 0 & 1.6 \end{bmatrix}. \end{aligned} \quad (5.2)$$

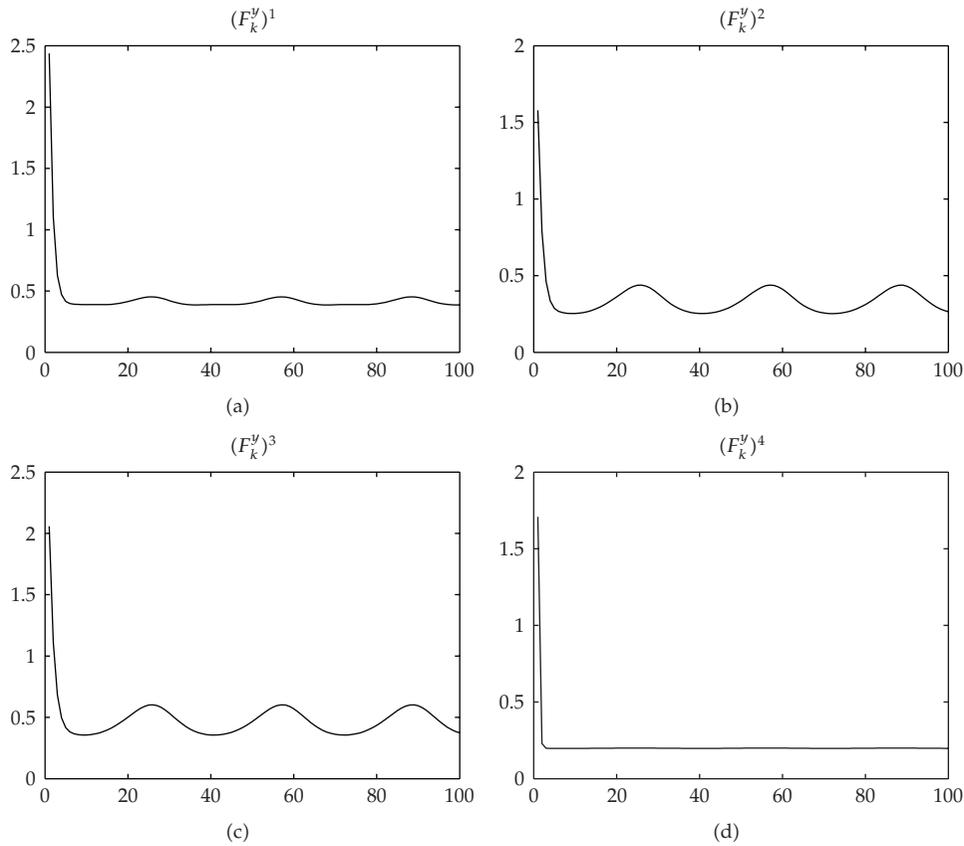


Figure 5: Trace of the covariance matrix P_k^x .

We assume that the fault and the disturbance are given by

$$\begin{bmatrix} f_{1,k} \\ f_{2,k} \end{bmatrix} = \begin{bmatrix} 5u_s(k-10) - 5u_s(k-70) \\ 4u_s(k-30) - 4u_s(k-65) \end{bmatrix}, \quad (5.3)$$

$$d_k = 4u_s(k-15) - 4u_s(k-55),$$

where $u_s(k)$ is the unit-step function.

Figure 1 presents the input sequence of the system (2.1). The simulation time is 100 time steps.

In Figure 2, we have plotted the actual and the estimated value of the first element of the state vector $x_k = [x_k^1 \ x_k^2 \ x_k^3]^T$. Figures 3 and 4 present the actual and the estimated value of the first element and the second element of the fault vector $f_k = [f_k^1 \ f_k^2]^T$, respectively. The convergence of the trace of covariance matrices P_k^x and P_k^f is shown, respectively, in Figures 5 and 6.

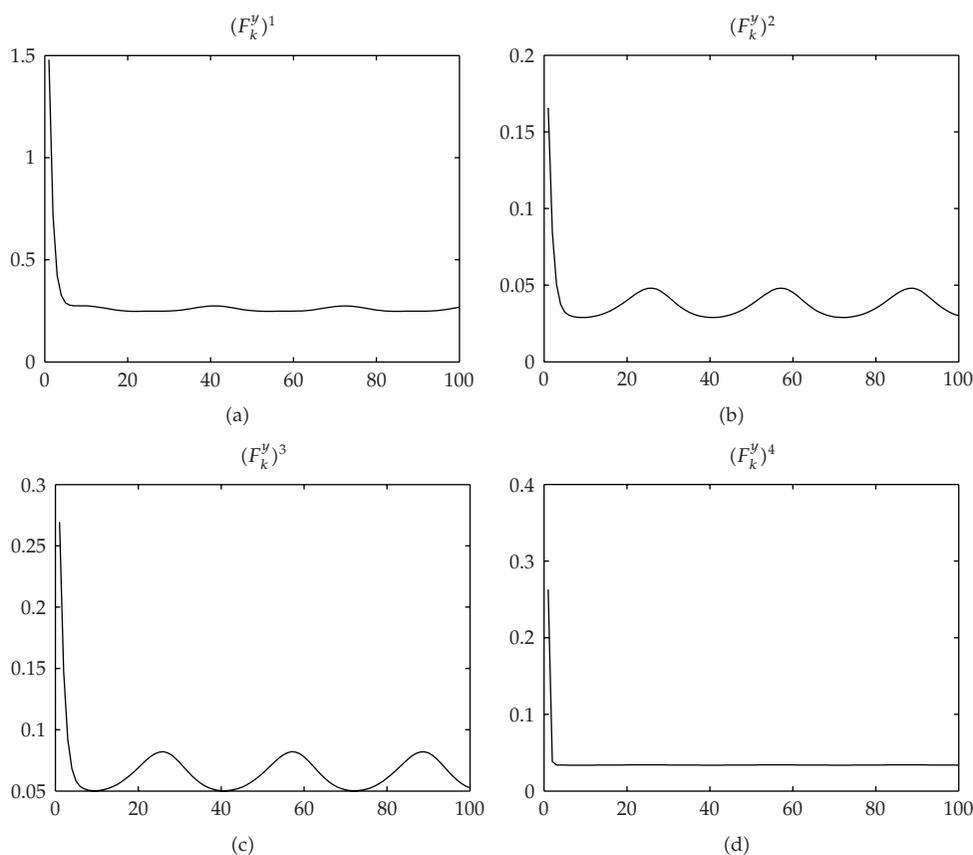


Figure 6: Trace of the covariance matrix P_k^f .

Table 1: RMSE values.

RMSE	$x_{1,k}$	$x_{2,k}$	$x_{3,k}$	$f_{1,k}$	$f_{2,k}$
$(F_k^y)^1$	0.4496	0.1531	0.5114	0.3875	0.4139
$(F_k^y)^2$	0.5248	0.1346	0.1863	0.4941	0.9612
$(F_k^y)^3$	0.6301	0.1542	0.2129	0.2565	2.3664
$(F_k^y)^4$	0.3378	0.0938	0.3647	3.3541	0.4139

The simulation results in Table 1 show the average root mean square errors (RMSEs) in the estimated states and faults.

According to Figures 2–6 and Table 1, we can conclude that if the matrix F_k^y has full rank, then we obtain a best estimate of the state and the fault (Figures 2(a), 3(a) and 4(a)). On the other hand, when the matrix F_k^y has not full rank, it is not possible to obtain a best estimate of the various components of the fault (Figures 3(b), 3(c), 3(d), 4(b), 4(c) and 4(d)), but the state estimation remains acceptable (Figures 2(b), 2(c), and 2(d)).

6. Conclusion

In this paper, the problem of the state and the fault estimation are solved in the case of stochastic linear discrete-time and varying-time systems. A recursive unbiased minimum-variance (UMV) filter is proposed when the direct feedthrough matrix of the fault has an arbitrary rank. The advantages of this filter are especially important in the case when we do not have any priory information about the unknown disturbances and the fault. An application of the proposed filter has been shown by an illustrative example. This recursive filter is able to obtain a robust and unbiased minimum-variance of the state and the fault estimation in spite of the presence of the unknown disturbances.

References

- [1] M. Blanke, M. Kinnaert, J. Lunze, and M. Staroswiecki, *Diagnosis and Fault-Tolerant Control*, Springer, Berlin, Germany, 2006.
- [2] B. Friedland, "Treatment of bias in recursive filtering," *IEEE Transactions on Automatic Control*, vol. 14, pp. 359–367, 1969.
- [3] A. T. Alouani, T. R. Rice, and W. D. A. Blair, "Two-stage filter for state estimation in the presence of dynamical stochastic bias," in *Proceedings of the American Control Conference*, vol. 2, pp. 1784–1788, Chicago, Ill, USA, 1992.
- [4] J. Y. Keller and M. Darouach, "Two-stage Kalman estimator with unknown exogenous inputs," *Automatica*, vol. 35, no. 2, pp. 339–342, 1999.
- [5] C. S. Hsieh and F. C. Chen, "Optimal solution of the two-stage Kalman estimator," *IEEE Transactions on Automatic Control*, vol. 44, no. 1, pp. 194–199, 1999.
- [6] K. H. Kim, J. G. Lee, and C. G. Park, "Adaptive two-stage Kalman filter in the presence of unknown random bias," *International Journal of Adaptive Control and Signal Processing*, vol. 20, no. 7, pp. 305–319, 2006.
- [7] K. H. Kim, J. G. Lee, and C. G. Park, "The stability analysis of the adaptive two-stage Kalman filter," *International Journal of Adaptive Control and Signal Processing*, vol. 21, no. 10, pp. 856–870, 2007.
- [8] P. K. Kitanidis, "Unbiased minimum-variance linear state estimation," *Automatica*, vol. 23, no. 6, pp. 775–778, 1987.
- [9] M. Darouach, M. Zasadzinski, and M. Boutayeb, "Extension of minimum variance estimation for systems with unknown inputs," *Automatica*, vol. 39, no. 5, pp. 867–876, 2003.
- [10] C. S. Hsieh, "Robust two-stage Kalman filters for systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 45, no. 12, pp. 2374–2378, 2000.
- [11] C. S. Hsieh, "Optimal minimum-variance filtering for systems with unknown inputs," in *Proceedings of the 6th World Congress on Intelligent Control and Automation (WCICA '06)*, vol. 1, pp. 1870–1874, Dalian, China, 2006.
- [12] S. Gillijns and B. Moor, "Unbiased minimum-variance input and state estimation for linear discrete-time systems with direct feedthrough," *Automatica*, vol. 43, no. 5, pp. 934–937, 2007.
- [13] S. Gillijns and B. Moor, "Unbiased minimum-variance input and state estimation for linear discrete-time systems," *Automatica*, vol. 43, no. 1, pp. 111–116, 2007.
- [14] Y. Cheng, H. Ye, Y. Wang, and D. Zhou, "Unbiased minimum-variance state estimation for linear systems with unknown input," *Automatica*, vol. 45, no. 2, pp. 485–491, 2009.
- [15] C. S. Hsieh, "Extension of unbiased minimum-variance input and state estimation for systems with unknown inputs," *Automatica*, vol. 45, no. 9, pp. 2149–2153, 2009.
- [16] T. Kailath, A. H. Sayed, and B. Hassibi, *Linear Estimation*, Prentice-Hall, Englewood Cliffs, NJ, USA, 2000.