Research Article

On the Nonlinear Theory of Micromorphic Thermoelastic Solids

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This work aims to study some dynamical problems in the framework of nonlinear theory of micromorphic thermoelastic solids. First, the continuous dependence of smooth admissible thermodynamic processes upon initial state and supply terms is investigated in the region of state space where the internal energy is a convex function and the elastic material behaves as a definite conductor of heat. Then, a uniqueness result is demonstrated.

1. Introduction

Motivated by experimental studies, various continuous models of deformable bodies have been proposed in literature in order to describe the thermomechanical behavior of media with microstructure. In the micromorphic theory introduced by Eringen and Şuhubi [1] and Eringen [2], a material body is envisioned as a collection of a large number of deformable particles (subcontinua or microcontinua). Each particle possesses finite size and directions representing its microstructure. The microdeformation gives rise to extra degrees of freedom. Thus, the particle has nine independent degrees of freedom describing both rotations and stretches, in addition to the three classical translational degrees of freedom of its center.

Many deformable solids point to the necessity for the incorporation of micromotions into mechanics. Porous solids with deformable grains and pores, composites, polymers with deformable molecules, crystals, solids with microcracks, dislocation and disclinations, and biological tissues (bones and muscles) are just a few examples of deformable solids which require the degrees of freedom given by the micromorphic theory. As a consequence, the micromorphic mechanics is the subject of detailed studies both from theoretical and practical reasons. In the linear context, uniqueness theorems have been proved by Soós [3] and Ieşan [4], variational principles have been established by Maugin [5] and Nappa [6], applications

to earthquake problems have been suggested by Teisseyre [7], Dresen et al. [8], and Teisseyre et al. [9], plane harmonic waves have been studied by Eringen [2], reciprocal and existence theorems have been demonstrated by Ieşan [4], and material constants for isotropic materials have been determined by Chen and Lee [10].

On the other hand, the theory has been generalized to mixtures of micromorphic materials by Twiss and Eringen [11, 12], to higher-grade materials by Eringen [13], and to electromagnetic micromorphic thermoelastic solids by Eringen [14]. Moreover, the constitutive theory of micromorphic thermoplasticity has been formulated by Lee and Chen [15], the concept of material forces was extended to micromorphic theory by Lee and Chen [16], the problem of heat flow in a micromorphic continua with microtemperatures has been investigated by Ieşan and Nappa [17].

This paper deals with the nonlinear micromorphic thermoelasticity. The main purpose is to investigate the stability of smooth thermodynamic processes. In this sense, we use the method developed for nonlinear thermoelastic solids which are nonconductor of heat by Dafermos [18] and updated later by Chiriţă [19] to heat-conducting thermoelastic materials. In the general context of heat-conducting nonlinear micromorphic solids, we prove the continuous dependence of smooth admissible thermodynamic processes upon initial state and supply terms. We present also a uniqueness theorem. Both results are obtained in the region of state space where the internal energy is a convex function and the elastic material behaves as a definite conductor of heat.

We recall that the Dafermos method has been utilized recently to prove continuous dependence results for nonconductor-of-heat mixtures [20–22] and for materials with voids [23].

The paper is organized as follows. In the next section, we recall the basic equations of the nonlinear theory of micromorphic thermoelasticity. Then, in Section 3, we use the consequences of the second law of thermodynamics to prove a uniqueness theorem and the continuous dependence of smooth thermodynamic processes upon initial state and body loads.

2. Basic Formulation

We consider a micromorphic continuum, and we assume that at time t_0 the body occupies the region B of the Euclidean three-dimensional space and is bounded by a piecewise smooth surface ∂B. Following [2], a point C in the body is characterized by its rectangular coordinates X_1, X_2, X_3, in a fixed system of rectangular Cartesian axes OX_K, $K = 1, 2, 3$, and a deformable vector Ξ. One writes $C(\mathbf{X}, \Xi)$. Deformation carries $C(\mathbf{X}, \Xi)$ to $c(\mathbf{x}, \boldsymbol{\xi})$ in the spatial configuration at time t, where x_1, x_2, and x_3 are the rectangular coordinates with respect to a new coordinate Cartesian system ox_i, $i = 1, 2, 3$, and $\boldsymbol{\xi}$ is the vector attached to c. The motion can be expressed as

$$x_i = x_i(\mathbf{X}, t), \quad \xi_i = \chi_{iK}(\mathbf{X}, t)\Xi_K, \quad \mathbf{X} \in B, \ t \in I, \tag{2.1}$$

where, and henceforth, the summation convention on repeated indices is understood and $I = [t_0, t_1)$, $t_1 > t_0$.

The inverse motion can be written as

$$X_K = X_K(\mathbf{x}, t), \qquad \Xi_K = \mathfrak{X}_{Ki}(\mathbf{x}, t)\xi_i, \tag{2.2}$$

where

$$\chi_{iK}\mathfrak{X}_{Kj} = \delta_{ij}, \qquad \chi_{iK}\mathfrak{X}_{Li} = \delta_{KL}. \tag{2.3}$$

In what follows $\dot{f}$ denotes the material time rate of f. Moreover, we use the notations $f_K = \partial f/\partial X_K$ and $f_i = \partial f/\partial x_i$.

The balance laws of micromorphic continua can be expressed as follows [1, 2]:

(i) the conservation of mass:

$$\rho_0 = J\rho \quad \text{or} \quad \dot{\rho} + \rho v_{i,i} = 0, \tag{2.4}$$

where $\rho_0(\mathbf{X})$ and $\rho(\mathbf{x}, t)$ are the mass densities at time t_0 and at present time, respectively, $J = \det(\partial x_i/\partial X_K)$, and v_i are the components of velocity vector;

(ii) the conservation of micromorphic inertia:

$$i_{kl} = I_{KL}\chi_{kK}\chi_{lL}, \quad I_{KL} = i_{kl}\mathfrak{X}_{Kk}\mathfrak{X}_{Ll} \quad \text{or} \quad \frac{di_{kl}}{dt} = i_{km}v_{lm} + i_{lm}v_{km}, \tag{2.5}$$

where v_{kl} is the microgyration tensor defined by

$$v_{ij} = \dot{\chi}_{iK}\mathfrak{X}_{Kj}, \tag{2.6}$$

and $I_{KL}(\mathbf{X})$ are the components of the inertia stress tensor at time t_0 and $i_{kl}(\mathbf{x}, t)$ are the components of the microinertia tensor at time t. Clearly, $I_{KL}(\mathbf{X})$ and $i_{kl}(\mathbf{x}, t)$ are symmetric and positive definite;

(iii) the balance of momentum:

$$t_{ji,j} + \rho f_i = \rho\dot{v}_i, \tag{2.7}$$

(iv) the balance of moment of momentum:

$$m_{kij,k} + t_{ji} - s_{ji} + \rho l_{ij} = \rho\sigma_{ij}, \tag{2.8}$$

(v) the balance of energy:

$$\rho\dot{e} = t_{ji}v_{i,j} + m_{kij}v_{ij,k} + (s_{ji} - t_{ji})v_{ij} + q_{i,i} + \rho r, \tag{2.9}$$

where **t**, **m**, **s**, σ, e, **q** are the stress tensor, moment stress tensor, microstress average, spin inertia, internal energy, and heat input, respectively, and **f**, **l**, r are the body force, external body moment, and heat source, respectively. The tensor **s** is symmetric, that is, $s_{ij} = s_{ji}$. The spin inertia σ_{ij} may be expressed as

$$\sigma_{ij} = I_{KL}\ddot{\chi}_{iK}\chi_{jL} = i_{mj}(\dot{v}_{im} + v_{in}v_{nm}). \tag{2.10}$$

The second law of thermodynamics is written as

$$\rho\theta\dot{\eta} - q_{i,i} - \rho r + \frac{1}{\theta}q_i\theta_{,i} \geq 0, \tag{2.11}$$

where η is the entropy density and θ is the absolute temperature.

The above formulation is described in detail in the book by Eringen [2]. Since we deal with micromorphic solids, we reformulate the basic equations in Lagrangean description. Thus, introducing the Piola-Kirchhoff tensors

$$T_{Ki} = JX_{K,j}t_{ji}, \qquad M_{KiL} = JX_{K,j}\mathfrak{X}_{Lk}m_{jik}, \qquad Q_K = JX_{K,i}q_i,$$

$$S_{Ki} = JX_{K,j}s_{ji}, \qquad L_{iK} = \mathfrak{X}_{Kj}l_{ij}, \tag{2.12}$$

and making the notation

$$G_{iL} = \mathfrak{X}_{Lj}[x_{j,K}(S_{Ki} - T_{Ki}) - \chi_{jM,K}M_{KiM}], \tag{2.13}$$

then, with the help of the relations (2.4), (2.6), and (2.10) and the identity

$$\left(\frac{x_{i,K}}{J}\right)_{,i} = 0, \tag{2.14}$$

we obtain the equations of motion

$$T_{Ki,K} + \rho_0 f_i = \rho_0\ddot{x}_i,$$

$$M_{KiL,K} - G_{iL} + \rho_0 L_{iL} = \rho_0 I_{KL}\ddot{\chi}_{iK}, \tag{2.15}$$

the energy equation

$$\rho_0\dot{e} = T_{Ki}v_{i,K} + M_{KiL}\dot{\chi}_{iL,K} + G_{iK}\dot{\chi}_{iK} + Q_{K,K} + \rho_0 r, \tag{2.16}$$

and the inequality

$$\rho_0 \theta \dot{\eta} - Q_{K,K} - \rho_0 r + \frac{1}{\theta} Q_K \theta_{,K} \geq 0. \tag{2.17}$$

Introducing Helmholtz's free energy ψ as

$$\psi = e - \theta \eta, \tag{2.18}$$

then, with the help of (2.15), the energy equation can be expressed as

$$\frac{d}{dt} \left[\rho_0 \left(\psi + \theta \eta + \frac{1}{2} v_i v_i + \frac{1}{2} I_{KL} \dot{\chi}_{iK} \dot{\chi}_{iL} \right) \right] = (T_{Ki} v_i + M_{KiL} \dot{\chi}_{iL} + Q_K)_{,K} + \rho_0 (f_i v_i + L_{iL} \dot{\chi}_{iL} + r). \tag{2.19}$$

Moreover, from (2.16), (2.17), and (2.18), we deduce

$$-\rho_0 (\dot{\psi} + \eta \dot{\theta}) + T_{Ki} v_{i,K} + M_{KiL} \dot{\chi}_{iL,K} + G_{iK} \dot{\chi}_{iK} + \frac{1}{\theta} Q_K \theta_{,K} \geq 0. \tag{2.20}$$

The response of a micromorphic thermoelastic solid is characterized by the following constitutive equations:

$$\psi = \widehat{\psi}(\mathfrak{L}), \qquad T_{Lj} = \widehat{T}_{Lj}(\mathfrak{L}), \qquad G_{jL} = \widehat{G}_{jL}(\mathfrak{L}), \qquad M_{LjP} = \widehat{M}_{LjP}(\mathfrak{L}),$$
$$\eta = \widehat{\eta}(\mathfrak{L}), \qquad Q_L = \widehat{Q}_L(\mathfrak{L}), \tag{2.21}$$

where

$$\mathfrak{L} = \{x_{i,K}, \chi_{iK}, \chi_{iK,M}, \theta, \theta_{,K}; X_K\} \tag{2.22}$$

and $\widehat{\psi}$, $\widehat{T}_{Lj}$, $\widehat{G}_{jL}$, $\widehat{M}_{LjP}$, $\widehat{\eta}$, and $\widehat{Q}_L$ are sufficient smooth functions.

We assume that there is no kinematical constraint. Then, it follows from the inequality (2.20) that

$$\psi = \widehat{\psi}(x_{i,K}, \chi_{iK}, \chi_{iK,M}, \theta; X_K), \qquad Q_L = \widehat{Q}_L(\mathfrak{L}),$$
$$T_{Lj} = \rho_0 \frac{\partial \widehat{\psi}}{\partial x_{j,L}}, \qquad G_{jL} = \rho_0 \frac{\partial \widehat{\psi}}{\partial \chi_{jL}}, \qquad M_{KjL} = \rho_0 \frac{\partial \widehat{\psi}}{\partial \chi_{jL,K}}, \qquad \eta = -\frac{\partial \widehat{\psi}}{\partial \theta}, \tag{2.23}$$
$$\widehat{Q}_L(x_{i,K}, \chi_{iK}, \chi_{iK,M}, \theta, 0; X_K) = 0. \tag{2.24}$$

The inequality (2.20) reduces to

$$\widehat{Q}_K \theta_{,K} \geq 0. \tag{2.25}$$

3. Uniqueness and Continuous Dependence

In this section, we establish a uniqueness result and the continuous dependence of smooth thermodynamic processes upon initial state and supply terms.

We assume that B is a bounded region and that ∂B is sufficiently regular to assure the common laws of transformation of surface integrals. We will employ the following notations: N_K are the components of the unit outward normal vector to the surface ∂B; I denotes the time interval $[t_0, t_1)$, where t_1 may be infinity; the symbol $|\cdot|$ denotes a norm, either in Euclidean vector space or in a tensor space, while $\|\cdot\|_{L^2(B)}$ denotes the L^2-norm.

Definition 3.1. By a thermodynamic process on $B \times I$ one means an ordered array of functions $\{x_i, \chi_{iL}, \theta, \psi, T_{Ki}, G_{iL}, M_{KiL}, \eta, Q_K, f_i, L_{iL}, r\}$ which satisfy the equations of motion (2.15), the energy equation (2.19), and the constitutive relations (2.21).

Definition 3.2. A thermodynamic process will be called admissible if it also satisfies the Clausius-Duhem inequality (2.17). From the previous section it follows that the Clausius-Duhem inequality (2.17) holds for all admissible processes in $B \times I$ if and only if (2.23), (2.24), and (2.25) are satisfied.

For admissible thermodynamic processes, one may write the energy equation in reduced form

$$\rho_0 \theta \dot{\eta} = Q_{K,K} + \rho_0 r. \tag{3.1}$$

Definition 3.3. One will say that $U = \{x_i, \chi_{iL}, \theta\}$ is an admissible state corresponding to the load (f_i, L_{iL}, r) if $\{x_i, \chi_{iL}, \theta, \psi, T_{Ki}, G_{iL}, M_{KiL}, \eta, Q_K, f_i, L_{iL}, r\}$ is an admissible thermodynamic process. The admissible state $U = \{x_i, \chi_{iL}, \theta\}$ is smooth if x_i, χ_{iL}, and θ are functions of class C^2.

Let $U = \{x_i, \chi_{iL}, \theta\}$ and $\overline{U} = \{\overline{x}_i, \overline{\chi}_{iL}, \overline{\theta}\}$ be two smooth admissible states on $B \times I$ corresponding to the loads (f_i, L_{iL}, r) and $(\overline{f}_i, \overline{L}_{iL}, \overline{r})$, respectively. We define the function $D(\cdot)$ on I by

$$\begin{aligned}
D = \int_B \Bigg[& \frac{1}{2}\rho_0 (v_i - \overline{v}_i)(v_i - \overline{v}_i) + \frac{1}{2}\rho_0 I_{KL}\left(\dot{\chi}_{iK} - \dot{\overline{\chi}}_{iK}\right)\left(\dot{\chi}_{iL} - \dot{\overline{\chi}}_{iL}\right) + \rho_0 \psi - \rho_0 \overline{\psi} \\
& -\overline{T}_{Ki}(x_{i,K} - \overline{x}_{i,K}) - \overline{G}_{iK}\left(\chi_{iK} - \overline{\chi}_{iK}\right) - \overline{M}_{KiL}\left(\chi_{iL,K} - \overline{\chi}_{iL,K}\right) + \rho_0 \eta \left(\theta - \overline{\theta}\right) \Bigg] dV,
\end{aligned} \tag{3.2}$$

where

$$\overline{v}_i = \dot{\overline{x}}_i, \quad \overline{\psi} = \hat{\psi}\left(\overline{x}_{i,K}, \overline{\chi}_{iK}, \overline{\chi}_{iK,M}, \overline{\theta}; X_K\right), \quad \overline{Q}_L = \hat{Q}_L\left(\overline{x}_{i,K}, \overline{\chi}_{iK}, \overline{\chi}_{iK,M}, \overline{\theta}, \overline{\theta}_{,K}; X_K\right),$$

$$\overline{T}_{Lj} = \rho_0 \frac{\partial \overline{\psi}}{\partial \overline{x}_{j,L}}, \quad \overline{G}_{jL} = \rho_0 \frac{\partial \overline{\psi}}{\partial \overline{\chi}_{jL}}, \quad \overline{M}_{KjL} = \rho_0 \frac{\partial \overline{\psi}}{\partial \overline{\chi}_{jL,K}}, \quad \overline{\eta} = -\frac{\partial \overline{\psi}}{\partial \overline{\theta}}. \tag{3.3}$$

On account of (2.23) and (3.3) it is easy to see that D is of quadratic order in

$$\left\|\mathbf{v} - \overline{\mathbf{v}}, \dot{\chi} - \dot{\overline{\chi}}, \mathbf{F} - \overline{\mathbf{F}}, \chi - \overline{\chi}, \gamma - \overline{\gamma}, \theta - \overline{\theta}\right\|_{L^2(B)}, \tag{3.4}$$

where $\mathbf{F} = (x_{i,K})$, $\chi = (\chi_{iK})$, and $\gamma = (\chi_{iL,K})$. The evolution in time of this function is described by the following.

Theorem 3.4. *If $U = \{x_i, \chi_{iL}, \theta\}$ and $\overline{U} = \{\overline{x}_i, \overline{\chi}_{iL}, \overline{\theta}\}$ are two smooth admissible states on $B \times I$ corresponding to the loads (f_i, L_{iL}, r) and $(\overline{f}_i, \overline{L}_{iL}, \overline{r})$ in $L^\infty(B \times I)$, then*

$$\dot{D} = \int_{\partial B} \Gamma dA + \int_B \left[\Lambda + Z + \rho_0\left(f_i - \overline{f}_i\right)(v_i - \overline{v}_i) + \rho_0\left(L_{iL} - \overline{L}_{iL}\right)\left(\dot{\chi}_{iL} - \dot{\overline{\chi}}_{iL}\right) \right.$$
$$\left. + \frac{\rho_0}{\theta}(r - \overline{r})\left(\theta - \overline{\theta}\right) - \frac{1}{\theta\overline{\theta}}\left(\overline{Q}_{K,K} + \rho_0\overline{r}\right)\left(\theta - \overline{\theta}\right)^2\right] dV, \tag{3.5}$$

where

$$\Gamma = \left[\left(T_{Ki} - \overline{T}_{Ki}\right)(v_i - \overline{v}_i) + \left(M_{KiL} - \overline{M}_{KiL}\right)\left(\dot{\chi}_{iL} - \dot{\overline{\chi}}_{iL}\right) + \frac{1}{\theta}\left(Q_K - \overline{Q}_K\right)\left(\theta - \overline{\theta}\right)\right] N_K, \tag{3.6}$$

$$\Lambda = \dot{\overline{x}}_{i,K}\left[T_{Ki} - \overline{T}_{Ki} - \frac{\partial\overline{T}_{Ki}}{\partial\overline{x}_{j,L}}(x_{j,L} - \overline{x}_{j,L}) - \frac{\partial\overline{T}_{Ki}}{\partial\overline{\chi}_{jL}}\left(\chi_{jL} - \overline{\chi}_{jL}\right)\right.$$
$$\left. - \frac{\partial\overline{T}_{Ki}}{\partial\overline{\chi}_{jL,M}}\left(\chi_{jL,M} - \overline{\chi}_{jL,M}\right) - \frac{\partial\overline{T}_{Ki}}{\partial\overline{\theta}}\left(\theta - \overline{\theta}\right)\right]$$

$$+ \dot{\overline{\chi}}_{iK}\left[G_{iK} - \overline{G}_{iK} - \frac{\partial\overline{G}_{iK}}{\partial\overline{x}_{j,L}}(x_{j,L} - \overline{x}_{j,L}) - \frac{\partial\overline{G}_{iK}}{\partial\overline{\chi}_{jL}}\left(\chi_{jL} - \overline{\chi}_{jL}\right) - \frac{\partial\overline{G}_{iK}}{\partial\overline{\chi}_{jL,M}}\left(\chi_{jL,M} - \overline{\chi}_{jL,M}\right)\right.$$
$$\left. - \frac{\partial\overline{G}_{iK}}{\partial\overline{\theta}}\left(\theta - \overline{\theta}\right)\right]$$

$$+ \dot{\overline{\chi}}_{iL,K}\left[M_{KiL} - \overline{M}_{KiL} - \frac{\partial\overline{M}_{KiL}}{\partial\overline{x}_{j,M}}(x_{j,M} - \overline{x}_{j,M})\right.$$
$$\left. - \frac{\partial\overline{M}_{KiL}}{\partial\overline{\chi}_{jM}}\left(\chi_{jM} - \overline{\chi}_{jM}\right) - \frac{\partial\overline{M}_{KiL}}{\partial\overline{\chi}_{jM,N}}\left(\chi_{jM,N} - \overline{\chi}_{jM,N}\right) - \frac{\partial\overline{M}_{KiL}}{\partial\overline{\theta}}\left(\theta - \overline{\theta}\right)\right]$$

$$- \rho_0\dot{\overline{\theta}}\left[\eta - \overline{\eta} - \frac{\partial\overline{\eta}}{\partial\overline{x}_{i,K}}(x_{i,K} - \overline{x}_{i,K}) - \frac{\partial\overline{\eta}}{\partial\overline{\chi}_{iK}}\left(\chi_{iK} - \overline{\chi}_{iK}\right)\right.$$
$$\left. - \frac{\partial\overline{\eta}}{\partial\overline{\chi}_{iL,K}}\left(\chi_{iL,K} - \overline{\chi}_{iL,K}\right) - \frac{\partial\overline{\eta}}{\partial\overline{\theta}}\left(\theta - \overline{\theta}\right)\right], \tag{3.7}$$

$$Z = -\left(Q_K - \overline{Q}_K\right)\left(\frac{\theta - \overline{\theta}}{\theta}\right)_{,K}. \tag{3.8}$$

Proof. From (3.2) we obtain

$$
\begin{aligned}
\dot{D} = \int_B \Bigg\{ & \frac{d}{dt}\left[\rho_0\left(\psi + \theta\eta + \frac{1}{2}v_iv_i + \frac{1}{2}I_{KL}\dot{\chi}_{iK}\dot{\chi}_{iL}\right)\right] \\
& - \frac{d}{dt}\left[\rho_0\left(\overline{\psi} + \overline{\theta}\overline{\eta} + \frac{1}{2}\overline{v}_i\overline{v}_i + \frac{1}{2}I_{KL}\dot{\overline{\chi}}_{iK}\dot{\overline{\chi}}_{iL}\right)\right] \\
& - \rho_0\left(v_i\dot{\overline{v}}_i + \dot{v}_i\overline{v}_i - 2\overline{v}_i\dot{\overline{v}}_i\right) - \rho_0 I_{KL}\left(\dot{\chi}_{iK}\ddot{\overline{\chi}}_{iL} + \ddot{\chi}_{iK}\dot{\overline{\chi}}_{iL} - 2\dot{\overline{\chi}}_{iK}\ddot{\overline{\chi}}_{iL}\right) \\
& - \dot{\overline{T}}_{Ki}(x_{i,K} - \overline{x}_{i,K}) - \dot{\overline{G}}_{iK}(\chi_{iK} - \overline{\chi}_{iK}) - \dot{\overline{M}}_{KiL}\left(\chi_{iL,K} - \overline{\chi}_{iL,K}\right) \\
& - \overline{T}_{Ki}\left(\dot{x}_{i,K} - \dot{\overline{x}}_{i,K}\right) - \overline{G}_{iK}\left(\dot{\chi}_{iK} - \dot{\overline{\chi}}_{iK}\right) - \overline{M}_{KiL}\left(\dot{\chi}_{iL,K} - \dot{\overline{\chi}}_{iL,K}\right) \\
& - \rho_0\dot{\overline{\theta}}(\eta - \overline{\eta}) - \rho_0\overline{\theta}\left(\dot{\eta} - \dot{\overline{\eta}}\right) \Bigg\}dV.
\end{aligned}
\tag{3.9}
$$

Using the balance laws (2.15) and (2.19), we may write (3.9) in the form

$$
\begin{aligned}
\dot{D} = \int_B \Bigg\{ & \left[\left(T_{Ki} - \overline{T}_{Ki}\right)(v_i - \overline{v}_i) + \left(M_{KiL} - \overline{M}_{KiL}\right)\left(\dot{\chi}_{iL} - \dot{\overline{\chi}}_{iL}\right)\right]_{,k} \\
& + \rho_0\left(f_i - \overline{f}_i\right)(v_i - \overline{v}_i) + \rho_0\left(L_{iL} - \overline{L}_{iL}\right)\left(\dot{\chi}_{iL} - \dot{\overline{\chi}}_{iL}\right) + R + P \Bigg\}dV,
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
R = & -\dot{\overline{T}}_{Ki}(x_{i,K} - \overline{x}_{i,K}) + \dot{\overline{x}}_{i,K}\left(T_{Ki} - \overline{T}_{Ki}\right) - \dot{\overline{M}}_{KiL}\left(\chi_{iL,K} - \overline{\chi}_{iL,K}\right) \\
& + \dot{\overline{\chi}}_{iL,K}\left(M_{KiL} - \overline{M}_{KiL}\right) - \dot{\overline{G}}_{iK}(\chi_{iK} - \overline{\chi}_{iK}) + \dot{\overline{\chi}}_{iK}\left(G_{iK} - \overline{G}_{iK}\right) \\
& - \rho_0\dot{\overline{\theta}}(\eta - \overline{\eta}) + \rho_0\dot{\overline{\eta}}\left(\theta - \overline{\theta}\right),
\end{aligned}
\tag{3.11}
$$

$$
P = \rho_0 r + Q_{K,K} - \rho_0\overline{r} - \overline{Q}_{K,K} - \rho_0\left(\dot{\overline{\eta}}\theta + \dot{\eta}\overline{\theta} - 2\dot{\overline{\eta}}\overline{\theta}\right).
$$

It follows from (3.3) that

$$
\begin{aligned}
\dot{\overline{T}}_{Ki} &= \frac{\partial\overline{T}_{Ki}}{\partial\overline{x}_{j,L}}\dot{\overline{x}}_{j,L} + \frac{\partial\overline{T}_{Ki}}{\partial\overline{\chi}_{jL}}\dot{\overline{\chi}}_{jL} + \frac{\partial\overline{T}_{Ki}}{\partial\overline{\chi}_{jL,M}}\dot{\overline{\chi}}_{jL,M} + \frac{\partial\overline{T}_{Ki}}{\partial\overline{\theta}}\dot{\overline{\theta}}, \\
\dot{\overline{G}}_{iK} &= \frac{\partial\overline{G}_{iK}}{\partial\overline{x}_{j,L}}\dot{\overline{x}}_{j,L} + \frac{\partial\overline{G}_{iK}}{\partial\overline{\chi}_{jL}}\dot{\overline{\chi}}_{jL} + \frac{\partial\overline{G}_{iK}}{\partial\overline{\chi}_{jL,M}}\dot{\overline{\chi}}_{jL,M} + \frac{\partial\overline{G}_{iK}}{\partial\overline{\theta}}\dot{\overline{\theta}}, \\
\dot{\overline{M}}_{KiL} &= \frac{\partial\overline{M}_{KiL}}{\partial\overline{x}_{j,M}}\dot{\overline{x}}_{j,M} + \frac{\partial\overline{M}_{KiL}}{\partial\overline{\chi}_{jM}}\dot{\overline{\chi}}_{jM} + \frac{\partial\overline{M}_{KiL}}{\partial\overline{\chi}_{jM,N}}\dot{\overline{\chi}}_{jM,N} + \frac{\partial\overline{M}_{KiL}}{\partial\overline{\theta}}\dot{\overline{\theta}}, \\
\dot{\overline{\eta}} &= \frac{\partial\overline{\eta}}{\partial\overline{x}_{i,K}}\dot{\overline{x}}_{i,K} + \frac{\partial\overline{\eta}}{\partial\overline{\chi}_{iK}}\dot{\overline{\chi}}_{iK} + \frac{\partial\overline{\eta}}{\partial\overline{\chi}_{iK,L}}\dot{\overline{\chi}}_{iK,L} + \frac{\partial\overline{\eta}}{\partial\overline{\theta}}\dot{\overline{\theta}}.
\end{aligned}
\tag{3.12}
$$

With the help of (3.12) and (3.3), we find that

$$R = \Lambda. \tag{3.13}$$

On the other hand, using (3.1), we have

$$P = \rho_0 r + Q_{K,K} - \rho_0 \overline{r} - \overline{Q}_{K,K} + \frac{\overline{\theta}}{\theta} \left(\overline{Q}_{K,K} + \rho_0 \overline{r} \right) - \frac{\overline{\theta}}{\theta} \left(Q_{K,K} + \rho_0 r \right)$$

$$- \left(\overline{Q}_{K,K} + \rho_0 \overline{r} \right) \left(\frac{\overline{\theta}}{\theta} - 2 + \frac{\theta}{\overline{\theta}} \right) = \frac{\rho_0}{\theta} (r - \overline{r}) \left(\theta - \overline{\theta} \right) \tag{3.14}$$

$$+ \left[\left(Q_K - \overline{Q}_K \right) \frac{\theta - \overline{\theta}}{\theta} \right]_{,K} - \left(Q_K - \overline{Q}_K \right) \left(\frac{\theta - \overline{\theta}}{\theta} \right)_{,K} - \frac{1}{\theta\overline{\theta}} \left(\overline{Q}_{K,K} + \rho_0 \overline{r} \right) (\theta - \overline{\theta})^2.$$

Collecting (3.10), (3.13), and (3.14) and using (3.6), (3.7), and (3.8) and the divergence theorem we conclude that (3.5) holds. The proof is complete. $\square$

From (2.21), it follows that

$$Q_K = \overline{Q}_K + \overline{K}_{KL} \left(\theta_{,L} - \overline{\theta}_{,L} \right) + \mathfrak{Q}_K + Q_K^0, \tag{3.15}$$

where

$$\overline{K}_{KL} = \frac{\partial \widehat{Q}_K}{\partial \theta_{,L}} \left(\overline{\mathfrak{L}} \right), \quad \overline{\mathfrak{L}} = \left(\overline{x}_{i,M}, \overline{\chi}_{iM}, \overline{\chi}_{iM,N}, \overline{\theta}, \overline{\theta}_{,M}; X_M \right), \tag{3.16}$$

and $\mathfrak{Q}_K$ is given by

$$\mathfrak{Q}_K = \overline{B}_{KiL} (x_{i,L} - \overline{x}_{i,L}) + \overline{C}_{KiL} (\chi_{iL} - \overline{\chi}_{iL}) + \overline{\mathcal{E}}_{KiLM} \left(\chi_{iL,M} - \overline{\chi}_{iL,M} \right) + \overline{\alpha}_K \left(\theta - \overline{\theta} \right), \tag{3.17}$$

$$\overline{B}_{KiL} = \frac{\partial \widehat{Q}_K}{\partial x_{i,L}} \left(\overline{\mathfrak{L}} \right), \quad \overline{C}_{KiL} = \frac{\partial \widehat{Q}_K}{\partial \chi_{iL}} \left(\overline{\mathfrak{L}} \right), \quad \overline{\mathcal{E}}_{KiLM} = \frac{\partial \widehat{Q}_K}{\partial \chi_{iL,M}} \left(\overline{\mathfrak{L}} \right), \quad \overline{\alpha}_K = \frac{\partial \widehat{Q}_K}{\partial \theta} \left(\overline{\mathfrak{L}} \right). \tag{3.18}$$

In (3.15), Q_K^0 is a function of order $o(\varrho)$, ϱ being defined by

$$\varrho = \left| \mathbf{F} - \overline{\mathbf{F}} \right| + |\chi - \overline{\chi}| + |\gamma - \overline{\gamma}| + \left| \theta - \overline{\theta} \right| + |\mathbf{g} - \overline{\mathbf{g}}|, \tag{3.19}$$

where $\mathbf{F} = (x_{i,K})$, $\chi = (\chi_{iK})$, $\gamma = (\chi_{iL,K})$, and $\mathbf{g} = (\theta_{,K})$.

Let us introduce the following definition for a definite heat conductor material (see Chiriță [19]).

Definition 3.5. One says that the admissible state $\overline{U}$ resides in the region where the material is a definite heat conductor if

$$\overline{k}_{KL}\zeta_K\zeta_L > 0, \tag{3.20}$$

for any nonzero ζ_K, where $\overline{k}_{KL}$ is given by

$$\overline{k}_{KL} = \frac{1}{2}\left(\overline{K}_{KL} + \overline{K}_{LK}\right). \tag{3.21}$$

We introduce the following notation:

$$y(t) = \left\|\left(\mathbf{F} - \overline{\mathbf{F}}, \chi - \overline{\chi}, \gamma - \overline{\gamma}, \theta - \overline{\theta}\right)(\cdot, t)\right\|_{L^2(B)}, \quad t \in I. \tag{3.22}$$

Theorem 3.6. *Let* $\overline{U} = \{\overline{x}_i, \overline{\chi}_{iL}, \overline{\theta}\}$ *be a smooth admissible state residing in the region where the material is a definite conductor of heat. Then there exist the positive constants* δ, m_1, *and* m_2 *with the following property: if* $U = \{x_i, \chi_{iL}, \theta\}$ *is any smooth admissible process defined on* $B \times I$, *such that*

$$\varrho = \left|\mathbf{F} - \overline{\mathbf{F}}\right| + |\chi - \overline{\chi}| + |\gamma - \overline{\gamma}| + \left|\theta - \overline{\theta}\right| + |\mathbf{g} - \overline{\mathbf{g}}| < \delta, \tag{3.23}$$

on $B \times I$, *then*

$$\int_B Z dV \leq -m_1 \|(\mathbf{g} - \overline{\mathbf{g}})(\cdot, t)\|_{L^2(B)}^2 + m_2 y^2(t), \quad t \in I. \tag{3.24}$$

Proof. In view of (3.8), (3.15), and (3.21), it follows that

$$
\begin{aligned}
Z &= -\frac{1}{\theta}\left(Q_K - \overline{Q}_K\right)\left(\theta_{,K} - \overline{\theta}_{,K}\right) + \frac{1}{\theta^2}\theta_{,K}\left(Q_K - \overline{Q}_K\right)\left(\theta - \overline{\theta}\right) \\
&= -\frac{1}{\theta}\overline{k}_{KL}\left(\theta_{,K} - \overline{\theta}_{,K}\right)\left(\theta_{,L} - \overline{\theta}_{,L}\right) + \mathfrak{K} + o\left(\varrho^2\right),
\end{aligned}
\tag{3.25}
$$

where

$$\mathfrak{K} = -\frac{1}{\theta}\mathfrak{Q}_K\left(\theta_{,K} - \overline{\theta}_{,K}\right) + \frac{1}{\theta^2}\theta_{,K}\left[\overline{K}_{KL}\left(\theta_{,L} - \overline{\theta}_{,L}\right) + \mathfrak{Q}_K\right]\left(\theta - \overline{\theta}\right). \tag{3.26}$$

Using the arithmetic-geometric inequality

$$2ab \leq a^2\beta^2 + b^2\beta^{-2}, \quad \text{for all nonzero constant } \beta, \tag{3.27}$$

Schwarz's inequality, and (3.17), we deduce

$$
\begin{aligned}
2 \int_B \Re dV \leq &\left(\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2\right) \|(\mathbf{g} - \overline{\mathbf{g}})(\cdot, t)\|_{L^2(B)}^2 \\
&+ \left(M_1^2 \beta_1^{-2} + \gamma_1^2\right) \|\left(\mathbf{F} - \overline{\mathbf{F}}\right)(\cdot, t)\|_{L^2(B)}^2 + \left(M_2^2 \beta_2^{-2} + \gamma_2^2\right) \|(\chi - \overline{\chi})(\cdot, t)\|_{L^2(B)}^2 \\
&+ \left(M_3^2 \beta_3^{-2} + \gamma_3^2\right) \|(\gamma - \overline{\gamma})(\cdot, t)\|_{L^2(B)}^2 \\
&+ \left(M_4^2 \beta_4^{-2} + N_1^2 \gamma_1^{-2} + N_2^2 \gamma_2^{-2} + N_3^2 \gamma_3^{-2} + N^2\right) \|\left(\theta - \overline{\theta}\right)(\cdot, t)\|_{L^2(B)}^2,
\end{aligned}
\tag{3.28}
$$

where β_p, $(p = 1, 2, 3, 4)$ and γ_s, $(s = 1, 2, 3)$ are arbitrary nonzero constants and

$$
M_1 = \max\left|\theta^{-1}\overline{\mathcal{B}}\right|, \qquad M_2 = \max\left|\theta^{-1}\overline{C}\right|, \qquad M_3 = \max\left|\theta^{-1}\overline{\mathcal{E}}\right|,
$$

$$
M_4 = \max\left|\theta^{-1}\overline{\alpha} + \theta^{-2}\overline{K}^T \mathbf{g}\right|, \qquad N_1 = \max\left|\theta^{-2}\overline{\mathcal{B}}^T \mathbf{g}\right|, \qquad N_2 = \max\left|\theta^{-2}\overline{C}^T \mathbf{g}\right|, \tag{3.29}
$$

$$
N_3 = \max\left|\theta^{-2}\overline{\mathcal{E}}^T \mathbf{g}\right|, \qquad N^2 = 2 \max\left|\theta^{-2}\overline{\alpha}\mathbf{g}\right|.
$$

In view of (3.20), we conclude that there exists a positive constant λ such that

$$
-\int_B \frac{1}{\theta} \overline{k}_{KL}\left(\theta_{,K} - \overline{\theta}_{,K}\right)\left(\theta_{,L} - \overline{\theta}_{,L}\right) dV \leq -\lambda \|(\mathbf{g} - \overline{\mathbf{g}})(\cdot, t)\|_{L^2(B)}^2, \qquad t \in I. \tag{3.30}
$$

Collecting (3.25), (3.28), and (3.30), we conclude that there exists a positive constant δ with the property that whenever (3.23) holds, we have

$$
\begin{aligned}
\int_B Z \, dV \leq &-m_1 \|(\mathbf{g} - \overline{\mathbf{g}})(\cdot, t)\|_{L^2(B)}^2 + c_1 \|\left(\mathbf{F} - \overline{\mathbf{F}}\right)(\cdot, t)\|_{L^2(B)}^2 \\
&+ c_2 \|(\chi - \overline{\chi})(\cdot, t)\|_{L^2(B)}^2 + c_3 \|(\gamma - \overline{\gamma})(\cdot, t)\|_{L^2(B)}^2 + c_4 \|\left(\theta - \overline{\theta}\right)(\cdot, t)\|_{L^2(B)}^2,
\end{aligned}
\tag{3.31}
$$

where

$$
m_1 = \lambda - \frac{1}{2}\left(\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2\right), \qquad 2c_1 = M_1^2 \beta_1^{-2} + \gamma_1^2, \qquad 2c_2 = M_2^2 \beta_2^{-2} + \gamma_2^2,
$$

$$
2c_3 = M_3^2 \beta_3^{-2} + \gamma_3^2, \qquad 2c_4 = M_4^2 \beta_4^{-2} + N_1^2 \gamma_1^{-2} + N_2^2 \gamma_2^{-2} + N_3^2 \gamma_3^{-2} + N^2. \tag{3.32}
$$

Now, taking the constants β_p, $(p = 1, 2, 3, 4)$ such that $m_1 > 0$ and setting

$$
m_2 = \max\{c_1, c_2, c_3, c_4\}, \tag{3.33}
$$

from (3.22) and (3.31), we obtain (3.24), and the proof is complete. $\qquad \square$

Following [18], we introduce the following.

Definition 3.7. One says that the admissible state $\overline{U} = \{\overline{x}_i, \overline{\chi}_{iL}, \overline{\theta}\}$ resides in the convexity region of the internal energy if the following two conditions are satisfied:

(i) for each $(\mathbf{X}, t) \in \overline{B} \times I$, there exists a positive constant μ such that

$$
\frac{\partial \overline{\psi}}{\partial \overline{x}_{i,K} \overline{x}_{j,L}} \Upsilon_{iK} \Upsilon_{jL} + \frac{\partial \overline{\psi}}{\partial \overline{\chi}_{iK} \overline{\chi}_{jL}} \Omega_{iK} \Omega_{jL} + \frac{\partial \overline{\psi}}{\partial \overline{\chi}_{iK,M} \overline{\chi}_{jL,N}} \Phi_{iKM} \Phi_{jLN} + 2 \frac{\partial \overline{\psi}}{\partial \overline{x}_{i,K} \overline{\chi}_{jL}} \Upsilon_{iK} \Omega_{jL}
$$

$$
+ 2 \frac{\partial \overline{\psi}}{\partial \overline{x}_{i,K} \overline{\chi}_{jL,M}} \Upsilon_{iK} \Phi_{jLM} + 2 \frac{\partial \overline{\psi}}{\partial \overline{\chi}_{iK} \overline{\chi}_{jL,M}} \Omega_{iK} \Phi_{jLM} \geq \mu (\Upsilon_{iK} \Upsilon_{iK} + \Omega_{iK} \Omega_{iK} + \Phi_{iKL} \Phi_{iKL}),
$$

$$(3.34)$$

for all Υ_{iK}, Ω_{iK}, and Φ_{iKL}

(ii) and

$$
\frac{\partial \overline{\eta}}{\partial \overline{\theta}} > 0. \tag{3.35}
$$

Our study on stability and uniqueness is based on the following Gronwall-type inequality [18].

Lemma 3.8. *Assume that the nonnegative functions $z(t) \in L^{\infty}[0,s]$ and $g(t) \in L^1[0,s]$ satisfy the inequality*

$$
z^2(\tau) \leq M^2 z^2(0) + 2 \int_0^{\tau} \left[(\alpha + 2\beta\tau) z^2(t) + N g(t) z(t) \right] dt, \quad \tau \in [0,s] \tag{3.36}
$$

with α, β, M, and N being nonnegative constants. Then

$$
z(s) \leq \left[M z(0) + N \int_0^s g(t) dt \right] \exp\left(\sigma s + \beta s^2 \right), \tag{3.37}
$$

where $\sigma = \alpha + \beta/\alpha$.

Now, we are ready to state the following stability result.

Theorem 3.9. *Let $\overline{U} = \{\overline{x}_i, \overline{\chi}_{iK}, \overline{\theta}\}$ be a smooth admissible state on $B \times I$ corresponding to the loading $(\overline{f}_i, \overline{L}_{iL}, \overline{r}) \in L^{\infty}(B \times I)$ and residing in the region where the internal energy is a convex function and the material is a definite conductor of heat. Then there exist the positive constants $\tilde{\delta}_0$, α_0, M_0, and N_0 with the following property: if $U = \{x_i, \chi_{iK}, \theta\}$ is any smooth admissible state on $B \times I$ corresponding to the loading $(f_i, L_{iK}, r) \in L^{\infty}(B \times I)$, such that*

$$
\varrho = \left| \mathbf{F} - \overline{\mathbf{F}} \right| + |\chi - \overline{\chi}| + |\gamma - \overline{\gamma}| + \left| \theta - \overline{\theta} \right| + |\mathbf{g} - \overline{\mathbf{g}}| < \tilde{\delta}_0, \tag{3.38}
$$

on $B \times I$, and

$$\left[\left(T_{Ki} - \overline{T}_{Ki}\right)(v_i - \overline{v}_i) + \left(M_{KiL} - \overline{M}_{KiL}\right)\left(\dot{\chi}_{iL} - \dot{\overline{\chi}}_{iL}\right) + \frac{1}{\theta}\left(Q_K - \overline{Q}_K\right)\left(\theta - \overline{\theta}\right)\right]N_K = 0, \tag{3.39}$$

$$\text{on } \partial B \times I,$$

then for any $s \in I$, one has

$$z_0(s) \leq \left[M_0 z_0(0) + N_0 \int_0^s g_0(t)dt\right]\exp(\alpha_0 s), \tag{3.40}$$

where

$$z_0(s) = \left\|\left(\mathbf{v} - \overline{\mathbf{v}}, \mathbf{F} - \overline{\mathbf{F}}, \chi - \overline{\chi}, \dot{\chi} - \dot{\overline{\chi}}, \gamma - \overline{\gamma}, \theta - \overline{\theta}\right)(\cdot, s)\right\|_{L^2(B)},$$

$$g_0(s) = \left\|\left(\mathbf{f} - \overline{\mathbf{f}}, \mathbf{L} - \overline{\mathbf{L}}, r - \overline{r}\right)(\cdot, s)\right\|_{L^2(B)}. \tag{3.41}$$

Proof. From (3.6) and (3.39), we have $\Gamma = 0$. In view of (3.5), (3.7), (3.24), and Schwarz inequality, it follows that there exist the positive constants δ, ν_1, and ν_2 such that whenever (3.23) holds, we have

$$\dot{D}(t) \leq \nu_1 y^2(t) + \nu_2 g_0(t)w(t), \quad t \in I, \tag{3.42}$$

where $y(t)$ is defined in (3.22) and

$$w(t) = \left\|\left(\mathbf{v} - \overline{\mathbf{v}}, \dot{\chi} - \dot{\overline{\chi}}, \theta - \overline{\theta}\right)(\cdot, t)\right\|_{L^2(B)}. \tag{3.43}$$

Let us fix $s \in I$ and integrate (3.42) over $[0, \tau]$, with $\tau \in [0, s]$. Then, we have

$$D(\tau) \leq D(0) + \nu_1 \int_0^\tau z_0^2(t)dt + \nu_2 \int_0^\tau g_0(t)z_0(t)dt. \tag{3.44}$$

Here we used the inequalities $y(t) \leq z_0(t)$ and $w(t) \leq z_0(t)$, $t \in I$.

On the other hand, in view of (2.23) and (3.3), we obtain

$$
\rho_0 \psi - \rho_0 \overline{\psi} - \overline{T}_{Ki}(x_{i,K} - \overline{x}_{i,K}) - \overline{G}_{iK}(\chi_{iK} - \overline{\chi}_{iK}) - \overline{M}_{KiL}\left(\chi_{iL,K} - \overline{\chi}_{iL,K}\right) + \rho_0 \eta\left(\theta - \overline{\theta}\right)
$$

$$
= \frac{\rho_0}{2}\left[\frac{\partial \overline{\psi}}{\partial \overline{x}_{i,K} \partial \overline{x}_{j,L}}(x_{i,K} - \overline{x}_{i,K})(x_{j,L} - \overline{x}_{j,L}) \right.
$$

$$
+ \frac{\partial \overline{\psi}}{\partial \overline{\chi}_{iK} \partial \overline{\chi}_{jL}}(\chi_{iK} - \overline{\chi}_{iK})\left(\chi_{jL} - \overline{\chi}_{jL}\right)
$$

$$
+ \frac{\partial \overline{\psi}}{\partial \overline{\chi}_{iK,M} \partial \overline{\chi}_{jL,N}}\left(\chi_{iK,M} - \overline{\chi}_{iK,M}\right)\left(\chi_{jL,N} - \overline{\chi}_{jL,N}\right)
$$

$$
+ 2\frac{\partial \overline{\psi}}{\partial \overline{x}_{i,K} \partial \overline{\chi}_{jL}}(x_{i,K} - \overline{x}_{i,K})\left(\chi_{jL} - \overline{\chi}_{jL}\right)
$$

$$
+ 2\frac{\partial \overline{\psi}}{\partial \overline{x}_{i,K} \partial \overline{\chi}_{jL,M}}(x_{i,K} - \overline{x}_{i,K})\left(\chi_{jL,M} - \overline{\chi}_{jL,M}\right)
$$

$$
\left. + 2\frac{\partial \overline{\psi}}{\partial \overline{\chi}_{iK} \partial \overline{\chi}_{jL,M}}(\chi_{iK} - \overline{\chi}_{iK})\left(\chi_{jL,M} - \overline{\chi}_{jL,M}\right)\right]
$$

$$
+ \frac{1}{2}\frac{\partial \overline{\eta}}{\partial \overline{\theta}}\left(\theta - \overline{\theta}\right)^2 + o\left(\left|\mathbf{F} - \overline{\mathbf{F}}\right|^2 + |\chi - \overline{\chi}|^2 + |\gamma - \overline{\gamma}|^2 + \left|\theta - \overline{\theta}\right|^2\right).
$$

(3.45)

It follows from (3.2), (3.34), (3.35), and (3.45) that there exist the positive constants δ_0 and ν_3 such that whenever

$$
\left|\mathbf{F} - \overline{\mathbf{F}}\right| + |\chi - \overline{\chi}| + |\gamma - \overline{\gamma}| + \left|\theta - \overline{\theta}\right| < \delta_0,
$$

(3.46)

we have

$$
\nu_3 z_0(t) \leq 2D(t), \quad t \in I.
$$

(3.47)

Setting $\widetilde{\delta}_0 = \min(\delta, \delta_0)$ in (3.38), from (3.44) and (3.47), we obtain

$$
\nu_3 z_0^2(\tau) \leq 2D(0) + 2\nu_1 \int_0^\tau z_0^2(t)dt + 2\nu_2 \int_0^\tau g_0(t)z_0(t)dt.
$$

(3.48)

Using the estimate

$$
D(0) \leq \nu_4 z_0(0), \quad \nu_4 > 0,
$$

(3.49)

and the notations

$$
M_0^2 = \frac{2\nu_4}{\nu_3}, \qquad \alpha_0 = \frac{\nu_1}{\nu_3}, \qquad N_0 = \frac{\nu_2}{\nu_3},
$$

(3.50)

then (3.48) implies that

$$z_0^2(\tau) \le M_0^2 z_0(0) + 2 \int_0^\tau \left[\alpha_0 z_0^2(t) dt + N_0 g_0(t) z_0(t) \right] dt. \tag{3.51}$$

An application of Lemma 3.8 completes the proof. □

A direct consequence of the above theorem is the following uniqueness result.

Theorem 3.10. *Let U and $\overline{U}$ be as in Theorem 3.9. Assume that the corresponding body loads coincide on $B \times I$ and U and $\overline{U}$ originate from the same state, namely,*

$$
\begin{aligned}
x_i(\mathbf{X}, 0) &= \overline{x}_i(\mathbf{X}, 0), \quad v_i(\mathbf{X}, 0) = \overline{v}_i(\mathbf{X}, 0), \quad \chi_{iL}(\mathbf{X}, 0) = \overline{\chi}_{iL}(\mathbf{X}, 0), \\
\dot{\chi}_{iL}(\mathbf{X}, 0) &= \dot{\overline{\chi}}_{iL}(\mathbf{X}, 0), \quad \theta(\mathbf{X}, 0) = \overline{\theta}(\mathbf{X}, 0),
\end{aligned}
\qquad \mathbf{X} \in B. \tag{3.52}
$$

Then, U and $\overline{U}$ coincide on $B \times I$.

Acknowledgment

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