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Research Article

The Solution of Mitchell's Problem for the Elastic Infinite Cone with a Spherical Crack

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The new problem about the stress concentration around a spherical crack inside of an elastic cone is solved for the point tensile force enclosed to a cone's edge. The constructed discontinuous solutions of the equilibrium equations have allowed to express the displacements and stress in a cone through their jumps and the jumps of their normal derivatives across the crack's surface. The application of the integral transformation method under the generalized scheme has reduced the problem solving to the solving of the integrodifferential equation system with regard to the displacements' jumps. This system was solved approximately by the orthogonal polynomial method. The use of this method has allowed to take into consideration the order of the solution's singularities at the ends of an integral interval. The correlation between the crack's geometrical parameters, its distance from an edge, and the SIF values is established after the numerical analysis. The limit of the proposed method applicability is specified.

1. Introduction

The problems of the fracture mechanics and the non-destructive material testing demand the estimation of the stress intensity factor (SIF) around a crack located in an elastic body is important, as it is well known, because of it various applicatious in such engineering sciences as the fracture mechanics and the non-destructive material testing. The cracks' researching in the unbounded elastic matrix can not reflect all the complicity of the crack's phenomena in the real elastic body with boundaries. The mathematical complexity of the problems is caused by the necessity of the satisfaction to boundary conditions not only on the crack's branches, but also on the boundaries of an elastic body. The topological form of a crack is also concerned with a number of factors complicating the problem's solving. The big number of works, both in static and in dynamic statements, is devoted to the researching of the plane cracks with the different configurations of the contours [1–5]. The influence of the surface curvature, the variable curvature of a crack's contour, the interference of the applied loading and geometrical parameters of a crack are investigated in these papers. In three-dimensional statement the nonplanar cracks in the unlimited elastic bodies were considered in [6–14]. In comparison with this, the number of the works investigating the nonplanar cracks, located in an elastic body with boundaries, is limited.

The problems on the cracks' investigation in a finite elastic body are more often considered at the coincidence of a crack's and bodies topology, because that allows to choose the same coordinate system for their description. So, the behavior of the penny-shaped cracks in the finite elastic cylinders is investigated in [15–17]. The dynamic SIF around the spherical crack in a finite elastic shaft of the variable section is analyzed in [18]. The influence of an elastic cone boundaries on the SIF values around the spherical crack is shown in [19] in the assumption that at an cone's edge the compressing force is applied. In the proposed paper the loading at the cone's edge is the point tensile force, that essentially complicates the problem's solving and allows to establish more general laws of the SIF correlation with the crack's topology and the edge's influence on its values.

2. Formulation of the Boundary Value Problem and the Discontinuous Solution Method

Let us consider the infinite elastic cone $0 < r < \infty$, $0 \leq \theta \leq \omega$, $-\pi \leq \varphi \leq \pi$ (Poison's coefficient is μ, the shear module is G, r, θ, φ is the spherical coordinate system) at the vertex of which the concentrated force P is applied (Figure 1).

On the cone surface the stress is given:

$$\tau_{r\theta}(r,\theta)\big|_{\theta=\omega} = 0, \qquad \sigma_{\theta}(r,\theta)\big|_{\theta=\omega} = 0. \tag{2.1}$$

The spherical crack is situated inside the cone at the distance from the vertex, its surface is described by the following relations:

$$r = R, \quad 0 \leq \theta \leq \omega_0, \quad -\pi \leq \varphi \leq \pi(\omega_0 < \omega). \tag{2.2}$$

The crack's branches are free from the stress

$$\tau_{r\theta}(r,\theta)\big|_{r=R\pm 0} = 0, \qquad \sigma_{r}(r,\theta)\big|_{r=R\pm 0} = 0. \tag{2.3}$$

It is necessary to determine the SIF around the crack and to investigate the correlation between the SIF and the crack's location and geometrical parameters.

The searched solution is constructed as the superposition of the continuous solution (in the assumption of the crack's absence in the cone) and the discontinuous one (that one takes into consideration the existence of the crack). The first solution is marked by zero in the upper index and the second one by prim in the upper index:

$$\begin{aligned} u_r(r,\theta) &= u_r^0(r,\theta) + u_r^1(r,\theta), \qquad & \sigma_\theta(r,\theta) &= \sigma_\theta^0(r,\theta) + \sigma_\theta^1(r,\theta), \\ u_\theta(r,\theta) &= u_\theta^0(r,\theta) + u_\theta^1(r,\theta), \qquad & \tau_{r\theta}(r,\theta) &= \tau_{r\theta}^0(r,\theta) + \tau_{r\theta}^1(r,\theta). \end{aligned} \tag{2.4}$$

The continuous components are obtained in [20].

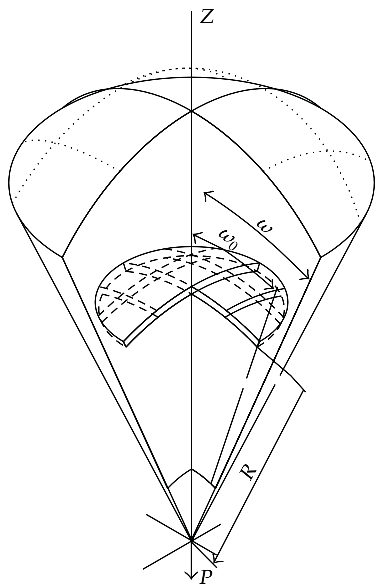

Figure 1

The method of the discontinuous solutions has been proved in Popov's works, and developed in the further in the papers [2, 12, 19]. The kernel of it lays in the construction of such solutions of Lame's equations [21]

$$\left(r^2u'\right)' - 2u - \frac{\mu_{**}}{\mu_*}\frac{(\nu\,\sin\theta)^{\bullet}}{\sin\theta} + \frac{(\sin\theta u^{\bullet})^{\bullet}}{\sin\theta}\frac{1}{\mu_*} + \frac{r(\nu'\sin\theta)^{\bullet}}{\sin\theta}\frac{\mu_0}{\mu_*} = 0,$$
$$\left(r^2\nu'\right)' + \mu_*\left(\frac{(\nu^{\bullet}\sin\theta)^{\bullet}}{\sin\theta} - \frac{\nu}{\sin^2\theta}\right) + \mu_0 r u'^{\bullet} + 2\mu_* u^{\bullet} = 0,$$
$$\mu_* = 1 + \mu_0,\quad \mu_{**} = \mu_0 + 2,\quad \mu_0 = \left(1 - 2\mu\right)^{-1},\quad u = u(r,\theta) = u_r^1(r,\theta),\quad \nu = \nu(r,\theta) = u_\theta^1(r,\theta). \tag{2.5}$$

(here the prime marks derivative with regard to the variable r, the point represents derivative with regard to the variable θ), which satisfy to these equations everywhere in the medium, except for the points of a defect. As a defect it can be understood both a crack, and an inclusion. At the transition across the defect's surface the mechanical characteristics have the discontinuities of a continuity of the first kind. The jumps of the displacements and stress are assumed set. They are determined further in the problem's statement and from the satisfaction of the boundary conditions. The constructed solutions allow to calculate the displacements and stress in any point of the medium with taking into consideration the discontinuity inside it.

We construct such solutions of (2.5) for a case of the crack defect of the spherical form. We will set jumps of the displacements and stress $\chi(\theta) = \langle u(R,\theta)\rangle$, $\psi(\theta) = \langle \nu(R,\theta)\rangle$, $\langle f(R,\theta)\rangle = f(R-0,\theta) - f(R+0,\theta)$. To (2.5) the Mellin's integral transformation is applied under the generalized scheme [2]

$$f_s(\theta) = \left(\int_0^{R-0} + \int_{R+0}^{\infty}\right) f(r,\theta)\ r^{s-1}dr, \qquad f(r,\theta) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f_s(\theta) r^{-s} ds, \tag{2.6}$$

(see Appendix A), and then the integral transformation with respect to the variable θ is used

$$u_{ks} = \int_0^{\omega_0} u_s(\theta) P^0_{\nu_k}(\cos\theta)\sin\theta\, d\theta, \qquad u_s(\theta) = \sum_{k=0}^{\infty} \frac{u_{sk} P^0_{\nu_k}(\cos\theta)}{\left\| P^0_{\nu_k}(\cos\theta)\right\|^2}, \tag{2.7a}$$

$$\nu_{ks} = \int_0^{\omega_0} \nu_s(\theta) P^1_{\nu_k}(\cos\theta)\sin\theta\, d\theta, \qquad \nu_s(\theta) = \sum_{k=0}^{\infty} \frac{\nu_{sk} P^1_{\nu_k}(\cos\theta)}{\left\| P^1_{\nu_k}(\cos\theta)\right\|^2} \tag{2.7b}$$

ν_k are the roots of the transcendental equation $P^1_{\nu_k}(\cos\omega) = 0, k = 0, 1, 2, \ldots$.

The equation system (2.5) is reduced by all of these transformations to the system of the linear algebraic equations in regard to the transformations of the displacements' jumps.

$$\begin{aligned}
&(s(s-1)-2)u_{sk} + \frac{1}{\mu_*}\nu_k(\nu_k+1)u_{sk} + \frac{\mu_{**}}{\mu_*}\nu_{sk} + \frac{\mu_0}{\mu_*}s\nu_{sk} \\
&\quad = R^s(s-1)\chi_k - R^{s+1}\langle u'_k(R)\rangle - \frac{\sin\omega}{\mu_*}u^\bullet_s(\omega)P^0_{\nu_k}(\cos\omega) \\
&\qquad + \frac{\mu_{**}}{\mu_*}\nu_s(\omega)\sin\omega P^0_{\nu_k}(\cos\omega) + \frac{\mu_0}{\mu_*}s\ \sin\omega P^0_{\nu_k}(\cos\omega)\nu_s(\omega) + \frac{\mu_0}{\mu_{**}}R^s\psi_k, \\
&s(s-1)\nu_{sk} - \mu_*\nu_k(\nu_k+1)\nu_{sk} - \mu_* s\nu_k(\nu_k+1)u_{sk} + 2\mu_* u_{sk} \\
&\quad = R^s(s-1)\psi_k - R^{s+1}\langle \nu'_k(R)\rangle + \mu_*\sin\omega\left(P^1_{\nu_k}(\cos\omega)\right)^\bullet \nu_s(\omega) - \mu_0 R^s\nu_k(\nu_k+1)\chi_k.
\end{aligned} \tag{2.8}$$

Let us resolve (2.8) and receive the transformations of the equilibrium equations' discontinuous solutions which allow to express the displacements in any point of a medium through the jumps of the displacements and their normal derivatives across the crack's surface.

To reduce the quantity of the unknown functions in the right parts (2.8) χ_k, ψ_k, $\nu_s(\omega)$, $u^\bullet_s(\omega)$, $\langle u'_k(R)\rangle$, $\langle \nu'_k(R)\rangle$, we use conditions on the crack, having written them down in terms of the displacements:

$$\frac{1}{2r}\left[r^2\left(\frac{\nu}{r}\right)' + u^\bullet\right]\Big|_{r=R\pm 0} = 0, \qquad [\mu\mu_0\theta + u']\big|_{r=R\pm 0} = 0, \tag{2.9}$$

where

$$\theta = \frac{(r^2u)'}{r^2} + \frac{(v\sin\theta)^\bullet}{r\sin\theta}. \tag{2.10}$$

One must satisfy to each condition on each crack's branch. The integral transformation (2.7b) is applied to the first condition, and the integral transformation (2.7a) to the second one. After deducting the equalities when $r = R \pm 0$, one must obtain the following relations:

$$\begin{aligned} \langle u_k'(R) \rangle &= \frac{\mu}{R(1-\mu)}\psi_k - \frac{3\mu}{R(1-\mu)}\chi_k, \\ \langle v_k'(R) \rangle &= \frac{\psi_k}{R} - \frac{\nu_k(\nu_k+1)}{R}\chi_k. \end{aligned} \tag{2.11}$$

For the further shortening of the unknown functions in the right-hand parts of the equation system, one must satisfy to the first condition on the conical surface (2.1). They must write them in the displacements and apply Mellin's integral transformation to it. After all these conversions the boundary condition will be as follows:

$$u_s^{\bullet}(\omega) = (s+1)v_s(\omega). \tag{2.12}$$

Now one must substitute the equalities (2.11), (2.12) to the equation system (2.8), and solve it in regard to the unknown displacements' transforms u_{sk} and v_{sk}:

$$\begin{aligned} u_{sk} &= \frac{\chi_k R^s \alpha(s,k) + \psi_k R^s \beta(s,k) + v_s(\omega)\gamma(s,k)}{\Delta_{sk}}, \\ v_{sk} &= \frac{\chi_k R^s q(s,k) + \psi_k R^s l(s,k) + v_s(\omega) p(s,k)}{\Delta_{sk}}. \end{aligned} \tag{2.13}$$

All coefficients are given in Appendix B. For obtaining of the discontinuous solution originals we apply the inverse Mellin's transformation to (2.13).

One must use the residue theorem for the integral calculation with the following notes:

(1) The roots of equation $\Delta_{sk} = 0$, $s = s_j$, and $j = \overline{1,4}$ are simple.

(2) For the Jordan lemma satisfaction it is necessary to close the contour or if on the left (then one must take into consideration the simple roots $s = s_1$ and $s = s_2$)—the case $r < R$, or if on the right (then one must take into consideration the simple roots $s = s_3, s_4$) the case $r > R$.

After the calculations we obtain the transformations of the displacements existing in the cone because of the crack's presence (see Appendix C). To get the displacements' originals, the inverse transformations (2.7a), (2.7b) should be applied to the solutions (2.13) with the equalities:

$$\chi_k = \int_0^{\omega_0} \chi(\eta) P_{\nu_k}^0(\cos\eta)\sin\eta\, d\eta, \qquad \psi_k = \int_0^{\omega_0} \psi(\eta) P_{\nu_k}^1(\cos\eta)\sin\eta\, d\eta. \tag{2.14}$$

Finally, the discontinuous solutions of the equilibrium equations are obtained

$$
\begin{aligned}
u(r,\theta) &= \int_0^{\omega_0} \chi(\eta) \sum_{k=0}^{\infty} \left\{ \sum_{j=1}^{2} \left[\frac{\alpha(s_{j,k})}{\Delta_{j,k}} \right] \left(\frac{R}{r} \right)^{s_j} \frac{P_{\nu_k}^0(\cos\theta)}{\left\| P_{\nu_k}^0(\cos\theta) \right\|^2} P_{\nu_k}^0(\cos\eta) \right\} d\eta \\
&\quad + \int_0^{\omega_0} \psi(\eta) \sum_{k=0}^{\infty} \left\{ \sum_{j=1}^{2} \left[\frac{\beta(s_{j,k})}{\Delta_{j,k}} \right] \left(\frac{R}{r} \right)^{s_j} \frac{P_{\nu_k}^0(\cos\theta)}{\left\| P_{\nu_k}^0(\cos\theta) \right\|} P_{\nu_k}^1(\cos\eta) \right\} d\eta \\
&\quad + \sum_{k=0}^{\infty} \frac{F_k(r) P_{\nu_k}^0(\cos\theta)}{\left\| P_{\nu_k}^0(\cos\theta) \right\|^2}, \quad r < R, \\
u(r,\theta) &= \int_{\text{o}}^{\omega_0} \chi(\eta) \sum_{k=0}^{\infty} \left\{ \sum_{j=3}^{4} \left[\frac{\alpha(s_{j,k})}{\Delta_{j,k}} \right] \left(\frac{R}{r} \right)^{s_j} \frac{P_{\nu_k}^0(\cos\theta)}{\left\| P_{\nu_k}^0(\cos\theta) \right\|^2} P_{\nu_k}^0(\cos\eta) \right\} d\eta \\
&\quad + \int_0^{\omega_0} \psi(\eta) \sum_{k=0}^{\infty} \left\{ \sum_{j=3}^{4} \left[\frac{\beta(s_{j,k})}{\Delta_{j,k}} \right] \left(\frac{R}{r} \right)^{s_j} \frac{P_{\nu_k}^0(\cos\theta)}{\left\| P_{\nu_k}^0(\cos\theta) \right\|^2} P_{\nu_k}^1(\cos\eta) \right\} d\eta \\
&\quad + \sum_{k=0}^{\infty} \frac{F_k(r) P_{\nu_k}^0(\cos\theta)}{\left\| P_{\nu_k}^0(\cos\theta) \right\|^2}, \quad r > R, \\
\nu(r,\theta) &= \int_0^{\omega_0} \chi(\eta) \sum_{k=0}^{\infty} \left\{ \sum_{j=1}^{2} \left[\frac{g(s_{j,k})}{\Delta_{j,k}} \right] \left(\frac{R}{r} \right)^{s_j} \frac{P_{\nu_k}^1(\cos\theta)}{\left\| P_{\nu_k}^1(\cos\theta) \right\|^2} P_{\nu_k}^0(\cos\eta) \right\} d\eta \\
&\quad + \int_0^{\omega_0} \psi(\eta) \sum_{k=0}^{\infty} \left\{ \sum_{j=1}^{2} \left[\frac{l(s_{j,k})}{\Delta_{j,k}} \right] \left(\frac{R}{r} \right)^{s_j} \frac{P_{\nu_k}^1(\cos\theta)}{\left\| P_{\nu_k}^1(\cos\theta) \right\|^2} P_{\nu_k}^1(\cos\eta) \right\} d\eta \\
&\quad + \sum_{k=0}^{\infty} \frac{G_k(r) P_{\nu_k}^1(\cos\theta)}{\left\| P_{\nu_k}^1(\cos\theta) \right\|^2}, \quad r < R, \\
\nu(r,\theta) &= \int_0^{\omega_0} \chi(\eta) \sum_{k=0}^{\infty} \left\{ \sum_{j=3}^{4} \left[\frac{g(s_{j,k})}{\Delta_{j,k}} \right] \left(\frac{R}{r} \right)^{s_j} \frac{P_{\nu_k}^1(\cos\theta)}{\left\| P_{\nu_k}^1(\cos\theta) \right\|^2} P_{\nu_k}^0(\cos\eta) \right\} d\eta \\
&\quad + \int_0^{\omega_0} \psi(\eta) \sum_{k=0}^{\infty} \left\{ \sum_{j=3}^{4} \left[\frac{l(s_{j,k})}{\Delta_{j,k}} \right] \left(\frac{R}{r} \right)^{s_j} \frac{P_{\nu_k}^1(\cos\theta)}{\left\| P_{\nu_k}^1(\cos\theta) \right\|^2} P_{\nu_k}^1(\cos\eta) \right\} d\eta \\
&\quad + \sum_{k=0}^{\infty} \frac{G_k(r) P_{\nu_k}^1(\cos\theta)}{\left\| P_{\nu_k}^1(\cos\theta) \right\|^2}, \quad r > R, \\
F_k(r) &= \int_0^{\infty} \nu(\psi,\omega) g_k\left(\frac{r}{\xi}\right) \frac{d\xi}{\xi}, \\
G_k(r) &= \int_0^{\infty} \nu(\xi,\omega) y_k\left(\frac{r}{\xi}\right) \frac{d\xi}{\xi}.
\end{aligned}
\tag{2.15}
$$

Let us note, that the specified procedure allows to receive the discontinuous solutions as for a crack case, so for a case of an inclusion, and under various conditions on a defect's surface.

3. The Problem Reducing to the System of the Integral-Differential Equations

With the aim to satisfy to the conditions on the crack, one must finally demand the stress on the any of crack's branches, for example, on the branch $r = R - 0$, will equal to zero. After substitution of the found discontinuous displacements in the boundary conditions when $r = R - 0$ with the expression of solution in the form (2.4), the system of the integral equations with regard to the unknown displacement jumps is obtained:

$$\begin{aligned}
&\int_0^{\omega_0} \chi(\eta) F_1(\theta,\eta)\, d\eta + \int_0^{\omega_0} \psi(\eta) F_2(\theta,\eta) + \int_0^{\omega_0} \nu(\xi,\omega)\alpha_1(\xi,\theta)\, d\xi + \int_0^{\infty} \nu'(\xi,\omega)\alpha_2(\xi,\theta)\, d\xi \\
&\quad = -\tau^{\circ}(\theta,R), \\
&\int_0^{\omega_0} \chi(\eta) F_3(\theta,\eta)\, d\eta + \int_0^{\omega_0} \psi(\eta) F_4(\theta,\eta)\, d\eta + \int_0^{\epsilon} \nu(\xi,\omega)\alpha_3(\xi,\theta)\, d\xi + \int_0^{\infty} \nu'(\xi,\omega)\alpha_4(\xi,\theta)\, d\xi \\
&\quad = -\sigma_2^0(\theta,R).
\end{aligned} \tag{3.1}$$

All taken notifications are written in Appendix D.

We must estimate the singularity order of the unknown functions in (3.1). As it is known, on the ends of the integration interval, $\eta = \omega_0$ the stress has the singularity of order: $-(1/2)$. In the integral equations (3.1) the unknown functions $\chi(\eta)$, $\psi(\eta)$ are the displacements' jumps, and hence, with the formulas of the displacements and stress correlation, one can make the conclusion that these functions have on the ends of the integration interval the singularity of order (1/2). For the estimation of function $\nu(r,\omega)$ singularity let us use the Williams's method [22]. As it shown in Appendix E the searched order of the displacement's singularity is $\nu(r,\omega) \sim r^{\lambda_* - 1}$. The further researching of the integral equations' kernels $F_j(\theta,\eta)$, $j = \overline{1,4}$ consists in the obtaining of their asymptotical expressions for $k \to \infty$. Therefore, they need to know the asymptotics of the functions $\alpha(s_{j,k}), \beta(s_{j,k}), l(s_{j,k}), q(s_{j,k}), y_k(\xi), g_k(\xi)$, and of the pairwise products, like $P^0_{\nu k}(\cos\theta)P^0_{\nu k}(\cos\eta)$, $P^0_{\nu k}(\cos\theta)P^1_{\nu k}(\cos\eta)$, $P^1_{\nu k}(\cos\theta)P^1_{\nu k}(\cos\eta)$; the last is possible because of well known formula [23], describing the asymptotic of Legendre's functions with the large values of the order. Also one needs the formulas that were obtained in [24] for the asymptotics of the eigenvalues and of the functions' norms:

$$\nu_k \sim \frac{k\pi}{\omega} \quad \text{with } k \longrightarrow \infty, \qquad \left\| P^{\{q\}}_{\nu_k}(\cos\theta) \right\|^2 \sim k, \quad q = 0,1 \quad \text{with } k \longrightarrow \infty. \tag{3.2}$$

After using of all these relations, the following asymptotics of the pairwise products were derived and substituted in the kernels $F_j(\theta,\eta)$, $j = \overline{1,4}$. The changing of the functions $\alpha(s_{j,k})$, $\beta(s_{j,k})$, $l(s_{j,k})$, $q(s_{j,k})$, $y_k(\xi), g_k(\xi)$, was done with the asymptotical relations. It is necessary to sum the series in the kernels $F_j(\theta,\eta)$, $j = \overline{1,4}$. The general scheme of this procedure is the following: the series $\sum_{k=1}^{\infty} a_k(\theta,\eta)$ is divided on the sum $\sum_{k=1}^{N} a_k(\theta,\eta) + \sum_{k=N+1}^{\infty} a_k(\theta,\eta)$, in the second addend the general series term is changed by its asymptotical expression $\tilde{a}_k(\theta,\eta)$ with the large values of k. The next step is the adding and deduction of the sum $\sum_{k=1}^{N} \tilde{a}_k(\theta,\eta)$. The initial series is written as the sum of the two addends $\sum_{k=1}^{\infty} a_k(\theta,\eta) = \sum_{k=1}^{\infty} \tilde{a}_k(\theta,\eta) + \sum_{k=1}^{N}(a_k(\theta,\eta) - \tilde{a}_k(\theta,\eta))$. The series in this formula is the well-known one and could be found at the tables of the series.

After all these operations with the kernels $F_j(\theta,\eta)$, $j = \overline{1,4}$, the table series where obtained [25]. The system of the integral equations (3.1) is reduced to the system of the two integro-differential equations (the derivation operator is exported from the integral sign with the aim of avoiding the divergent integrals in the kernels $F_j(\theta,\eta)$, $j = \overline{1,4}$. $\widehat{F}_j(\theta,\eta)$ are the regular kernels that were obtained by the scheme described earlier). In the last addends of both equations, the integration in parts was done. Finally the expression of the equation system (3.1) is

$$\begin{gathered}
\frac{d^2}{d\theta^2}\left[\int_0^{\omega_0}(\chi(\eta)+\psi(\eta))\ln\frac{1}{|\eta-\theta|}d\eta\right]+\int_0^{\omega_0}\chi(\eta)\widetilde{F}_1(\theta,\eta)d\eta \\
+\int_0^{\omega_0}\psi(\eta)\widetilde{F}_2(\theta,\eta)d\eta+\int_0^{\infty}\nu(\xi,\omega)B_1(\xi,\theta)d\xi=-\tau^{\circ}(\theta), \\
\frac{d^2}{d\theta^2}\left[\int_0^{\omega_0}(\chi(\eta)+\psi(\eta))\ln\frac{1}{|\eta-\theta|}d\eta\right]+\int_0^{\omega_0}\chi(\eta)\widehat{F}_3(\theta,\eta)d\eta+\int_0^{\omega_0}\psi(\eta)\widetilde{F}_3(\theta,\eta)d\eta \\
+\int_0^{\infty}\nu(\xi,\omega)B_2(\xi,\theta)d\xi=-\sigma_r^{\circ}(\theta).
\end{gathered} \tag{3.3}$$

4. The Solving of the Integro-Differential Equation System

One must realize the standard scheme of the orthogonal polynomial method [2]. The spectral relation

$$\frac{d^2}{dx^2}\int_{-1}^{1}\ln\frac{1}{|x-y|}\sqrt{1-y^2}U_n(y)dy=-\pi(n+1)U_n(x) \tag{4.1}$$

is needed for it (here $U_n(x)$—Chebyshev's polynomial of the second order). The variable changing is done for the passing to the interval $(0,1)$—$x = 2\eta - 1$, $y = 2\xi - 1$,

$$\frac{d^2}{d\eta^2}\int_0^1\ln\frac{1}{|\xi-\eta|}\sqrt{1-\xi^2}U_n(2\xi-1)d\xi=-\frac{\pi}{4}(n+1)U_n(2\eta-1). \tag{4.2}$$

In accordance with the singularity orders of the searched function $\chi(\eta)$, $\psi(\eta)$, and the spectral relation (4.2), the unknown functions will be searched as the following expansions:

$$\chi(\eta)=\sum_{k=0}^{\infty}\chi_k\sqrt{\eta-\eta^2}U_n(2\eta-1),\qquad \psi(\eta)=\sum_{k=0}^{\infty}\psi_k\sqrt{\eta-\eta^2}U_n(2\eta-1). \tag{4.3}$$

The infinite integral in the both integro-differential equations is changed by the finite one, and then the quadrature Sympson formula is applied with regard to the exponential character of function $B_j(\xi,\theta)$ decreasing

$$\int_0^T V(\xi)B_j(\xi,\theta)d\xi = \sum_{n=1}^{N} V_n A_n^j - B_n^j(\theta), \quad j = 1,2, \tag{4.4}$$

where A_n^j are the quadrature Simpson formula coefficients, and V_n are the unknown coefficients of the expansion.

The realization of the orthogonal polynomial method standard scheme leads to the system of the two-linear algebraic infinite equation system with regard to the following expansion coefficients:

$$\begin{gathered}\chi_l + \psi_l + \sum_{k=1}^{\infty} \chi_k \widetilde{F}_{kl}^1 + \sum_{k=1}^{\infty} \psi_k \widetilde{F}_{kl}^2 = f_l^1 + \sum_{n=1}^{N} V_n A_n^1 B_{nl}^1, \\ \chi_l + \psi_l + \sum_{k=1}^{\infty} \chi_k \widetilde{F}_{kl}^3 + \sum_{k=1}^{\infty} \psi_k \widetilde{F}_{kl}^4 = f_l^2 + \sum_{n=1}^{N} V_n A_n^2 B_{nl}^2.\end{gathered} \tag{4.5}$$

(In Appendix F one could see the linear algebraic equation system coefficients.) Taking into consideration the linearity of the SLAE solution, one must perform the unknown coefficients χ_k, ψ_k $(k = \overline{1,\infty})$ as the superposition of the $N+1$ unknown set of the constants:

$$\chi_k = \sum_{l=1}^{N+1} \chi_k^l, \quad \psi_k = \sum_{l=1}^{N+1} \psi_k^l, \qquad k = \overline{1,\infty}. \tag{4.6}$$

Thus, it is necessary to solve the $N+1$ systems of the sort (4.5), differentiating one from another only by their right-hand parts:

$$\left(f_l^1, f_l^2\right), \left(B_{1l}^1 A_1^1, B_{1l}^2 A_1^2\right), \ldots, \left(B_{Nl}^1 A_N^1, B_{Nl}^2 A_N^2\right). \tag{4.7}$$

Each of these systems is solved by the reduction method. The argumentation of its availability could be done by the method proposed in [26].

After the solving of the equation system (4.5), the coefficients of the expansion (4.3) are obtained, and this would be the final step of the displacement jumps searching. For the estimation of the cone's stress state, all that is left is to define the displacements along the conical surface.

5. The Calculation of the Displacements on the Conical Surface $v = (r, \omega)$ and SIF Values

It is needed to use the second condition (2.1) to find the displacement $v(r, \omega)$. For the condition's satisfaction, one must demand that

$$\sigma_\theta(r,\theta)|_{\theta=\omega} = 0 \quad \text{when } r < R, \qquad \sigma_\theta(r,\theta)|_{\theta=\omega} = 0 \quad r > R. \tag{5.1}$$

These conditions are written with the displacements' expression

$$\begin{aligned}
&\nu(r,\omega) - \int_0^\infty \nu(\xi,\omega)G(\xi,r)d\xi = -\int_0^{\sigma_0} \chi(\eta)\varphi_1(\eta,r)d\eta - \int_0^{\omega_0} \psi(\eta)\varphi_2(\eta,r)d\eta, \quad r < R,\\
&\nu(r,\omega) - \int_0^\infty \nu(\xi,\omega)G(\xi,r)d\xi = -\int_0^{\omega_0} \chi(\eta)\varphi_3(\eta,r)d\eta - \int_0^{\omega_0} \psi(\eta)\varphi_4(\eta,r)d\eta, \quad r > R,\\
&G(\xi,r) = \sum_{k=0}^{\infty}\left[g_k\left(\frac{r}{\xi}\right) + g_k'\left(\frac{r}{\xi}\right)\right]\frac{P_{\nu_k}^1(\cos\omega)}{\left\|P_{\nu_k}^1(\cos\omega)\right\|^2} + \sum_{k=0}^{\infty} y_k\left(\frac{r}{\xi}\right)\frac{\left(P_{\nu_k}^1(\cos\omega)\right)^\bullet}{\left\|P_{\nu_k}^1(\cos\omega)\right\|^2}.
\end{aligned} \tag{5.2}$$

The functions $\varphi_j(\eta, r)$ are defined by the discontinuous solutions' kernels. Mellin's transformation is applied to the relations (5.2)

$$\nu_s(\omega) = -\int_0^{\omega_0}\int_0^\infty \frac{\varphi_1(\eta,r)r^{s-1}dr}{1-G(s)}\frac{\chi(\eta)d\eta}{\omega_0} - \int_0^{\omega_0}\psi(\eta)\int_0^\infty \varphi_2(\eta,r)r^{s-1}dr\,d\eta, \tag{5.3a}$$

$$\nu_s(\omega) = \frac{-\int_0^{\omega_0}\chi(\eta)\int_0^\infty \varphi_3(\eta,r)r^{s-1}dr - \int_0^{\omega_0}\psi(\eta)\int_0^\infty \varphi_4(\eta,r)r^{s-1}dr\,d\eta}{1-G(s)}. \tag{5.3b}$$

One must show that the transformation $\nu_s(\omega)$, which is searched by the formula (5.3a) and by the formula (5.3b) is the same one. Really after the deduction from the right-hand part of (5.3a) of the right-hand part of (5.3b), zero will be the answer, so the coinciding of the left-hand parts is also proved. The fact of Mellin's transformation $\nu_s(\omega)$ equality in the both parts gives the result, that the originals of this transformations are also equal on the intervals $0 < r < R$ and $R < r < +\infty$. That is why it is enough to solve the integral equation (5.2) on any of these intervals, for example, when $r < R$. The solving is done by the method which was first used in [21]. It is important to use the fact that the singularity of the function $\nu(r, \omega)$ in the vicinity of zero is equal to $-\lambda^*$, as it is shown earlier. On the base of this fact the solution of the equation is constructed as the following expansion:

$$\nu(r,\omega) = \widetilde{\chi}(r) = \sum_{n=0}^{\infty}\overline{\chi}_n \overline{e}^r r^{-\lambda_*} L_n^{(-\lambda_*)}(2r), \tag{5.4}$$

where $L_n^{(-\alpha)}(r)$ is Chebyshev-Lager polynomials. The series (5.4) are substituted to the equation. The obtained expression is multiplied by $r^{2-\lambda} L_m^{(2-\lambda)}(2r)\overline{e}^r$ $(m = 0, 1, 2, \ldots)$, and each member of it is integrated on the interval $(0;+\infty)$. As a result, the infinite system of linear equations is obtained:

$$X_m + \sum_{n=0}^{\infty} A_{mn} X_n = C_m(\lambda), \quad m, n = 0, 1, 2, \ldots, \tag{5.5}$$

where

$$\begin{gathered} A_{mn} = \int_0^{\infty} r^{2-\lambda} \overline{e}^{r} L_m^{(2-\lambda)}(2r) K(r,\lambda) dr, \qquad K(r,\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{e}^{\xi} L_n^{(-\lambda)}(2\xi) \overline{e}^{\xi} G(r,\xi) d\xi, \\ C_m(\lambda) = \int_0^{\infty} r^{2-\lambda} e^{-r} L_m^{(2-\lambda)}(2r) F(r) dr. \end{gathered} \tag{5.6}$$

This system is solved approximately by the reduction method. For the proof of the method's convergence, the method proposed in [2] could be used.

The destruction criterion for the space case is Cherepanov's formula [1] $K_I + K_{II} + K_{III} = C$, where C is the material's constant.

In the stated problem, we have $K_{III} \equiv 0$, and K_I, K_{II} are the coefficients at the stress σ_r and $\tau_{r\theta}$ singularities correspondently:

$$\begin{aligned} K_I &= \lim_{\theta \to \omega_0 + 0} \sqrt{2\pi} \sqrt{\theta - \omega_0}\, \tau_{r\theta}(R,\theta), \\ K_{II} &= \lim_{\theta \to \sigma_0 + 0} \sqrt{2\pi} \sqrt{\theta - \omega_0}\, \sigma_r(R,\theta). \end{aligned} \tag{5.7}$$

The stress in the formulas (5.7) is defined by the equalities

$$\begin{gathered} \tau_{r\theta}(R,\theta) = \frac{d^2}{d\theta^2} \int_0^{\omega_0} \chi(\eta) \ln \frac{1}{|\eta - \theta|} d\eta + \frac{d^2}{d\theta^2} \int_0^{\omega_0} \psi(\eta) \ln \frac{1}{|\eta - \theta|} d\eta + R_1(\theta) + \tau^0(\theta, R), \\ \sigma_r(R,\theta) = (\mu\mu_0 + 1) \frac{d^2}{d\theta^2} \int_0^{\omega_0} \ln \frac{1}{|\eta - \theta|} d\eta + (\mu\mu_0 + 1) \int_0^{\omega_0} \psi(\eta) \ln \frac{1}{|\eta - \theta|} d\eta + R_2(\theta) + \sigma_r^0(\theta, R). \end{gathered} \tag{5.8}$$

For the limit calculation in (5.7), it is necessary to use the continuation of the spectral relation (4.2) on the interval $|\eta| > 1$. The following equation is used [25]

$$\begin{aligned} &\frac{d^2}{dx^2} \frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|x-s|} \sqrt{1-s^2} V_m(s) ds = \frac{(m+1)^2 2^{m+2}}{(x-1)^{m+2}} \\ &\quad \times \left[F\left(\frac{3}{2} + m, m+2; \frac{3}{2}; \frac{x+1}{x-1}\right) - \frac{m+1}{2} \sqrt{\frac{1-x}{-1-x}}\, \Gamma\left(\frac{3}{2} + m, m+1; \frac{1}{2}; \frac{x+1}{x-1}\right) \right]. \end{aligned} \tag{5.9}$$

Taking into consideration the variable change (4.2) and expressions (4.3), the result is obtained:

$$K_I = \sqrt{\frac{\pi}{2}}\left(\sum_{m=1}^{\infty}(-1)^{m+1}\sqrt{m}X_m + \sum_{m=1}^{\infty}(-1)^{m+1}\sqrt{m}\psi_m\right),$$
$$K_{II} = \sqrt{\frac{\pi}{2}}\left(\frac{\mu\mu_0+1}{\mu_*}\right)\left(\sum_{m=1}^{\infty}(-1)^{m+1}\sqrt{m}X_m + \sum_{m=1}^{\infty}(-1)^{m+1}\sqrt{m}\psi_m\right). \tag{5.10}$$

6. Numerical Results and Discussion

The dependence of the mode I SIF values K_I and of the mode II SIF K_{II} from the distance of a crack up to an edge is investigated at various values of a crack's angle ω_0.

On Figure 2 values of $\widetilde{K}_I = (K_I/P\sqrt{R})$, $\widetilde{K}_{II} = (K_{II}/P\sqrt{R})$

, calculated for a steel cone, which angle is $\omega = 75°$, are resulted. The dotted curves correspond to the crack's radius $R = R_1$, and continuous to the radius $R = 2R_1$. The analysis has shown that the angle of a crack, at which the mode I SIF reaches a maximum, almost twice is less than the value of the crack's angle at which the mode II SIF reaches its one. The distance to the cone's edge, as it is appreciable, influences the SIF absolute values, which are noticeably larger for the normal stress. The change of the crack's distance up to an edge insignificantly influences the value of a crack's angle at which maximum of SIF is reached.

The increase in loading essentially increases absolute SIF values though the cracks' angle at which SIF reach the maxima vary insignificantly. The maximum of mode I SIF is reached by the smaller values of the crack's angle than the values of the crack's angle at which the mode II SIF reaches its peak.

Comparison has been lead and gave enough good concurrence with numerical results of SIF values K_I^{∞}, K_{II}^{∞} calculation for the case of a spherical crack, located in the unlimited elastic medium at its stretching [13]. It is necessary to specify, that at ratio of the cone's angle to the crack's angle $l = \omega/\omega_0$, smaller than 1, 2, calculations lost stability that testifies that the proposed approach to the problem solving in this case is inapplicable, and it is necessary to use, for example, a method of a small parameter.

On Figures 3 and 4 the curves show the dependences of $K_I^* = K_I/K_I^{\infty}$ and $K_{II}^* = K_{II}/K_{II}^{\infty}$ on the ratio $l = \omega/\omega_0$ correspondently.

From the analysis of the curves, notice that the crack's distance from the cone's surface essentially influences the SIF absolute values. Noticeably, already at values $l \geq 8$ for the mode I SIF results coincide with a case of the infinite medium with a spherical crack, that is, the influence of the boundaries becomes insignificant. For the mode II SIF, the edge ceases to influence at $l \geq 6$.

7. Conclusions

(1) The approach proposed in the paper allows to solve the new axisymmetrical problem about stress concentration near the spherical crack located in an elastic cone at conditions of the first main elasticity problem on the cone's surface at the point tensile force enclosed to edge.

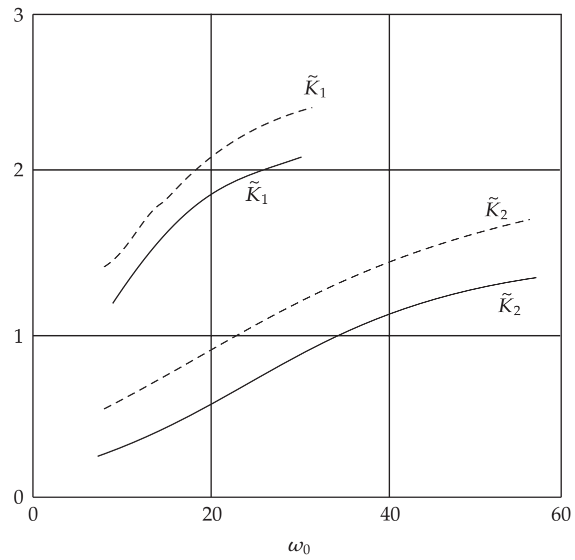

Figure 2

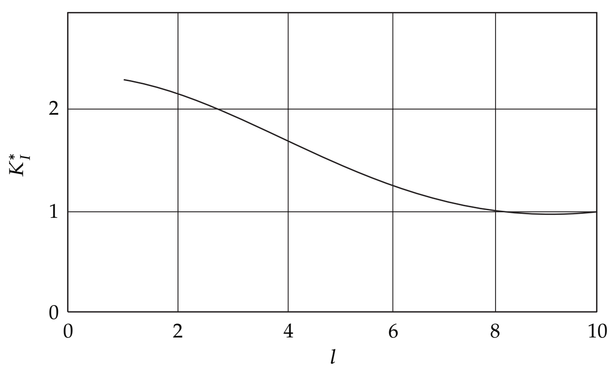

Figure 3

(2) It is established that values of the mode I SIF are larger on absolute values than the mode II SIF. The maximal crack's angle, at which the mode I SIF reaches the peak, almost is twice less than the crack's angle at which reaches the maximum the mode II SIF.

(3) The crack's distance to a cone's surface renders more essential influence on size of absolute values of SIF than the distance on which the crack is located from an edge.

(4) The proposed approach to the problem solving is available when the ratio of a cone's angle to the crack's angle is not less than 1, 2.

(5) The method that was used in the paper allows to solve a similar problem for a defect of the inclusion type, and also to solve the more complicate problem for a compound elastic cone in which the crack settles down on a surface of an elastic constants changing, that is, interphase crack. Moreover, the proposed approach will allow to solve a problem for a case of the arbitrary oriented force enclosed to a cone's edge for which it is necessary to construct the discontinuous solutions of the equilibrium equations for such case.

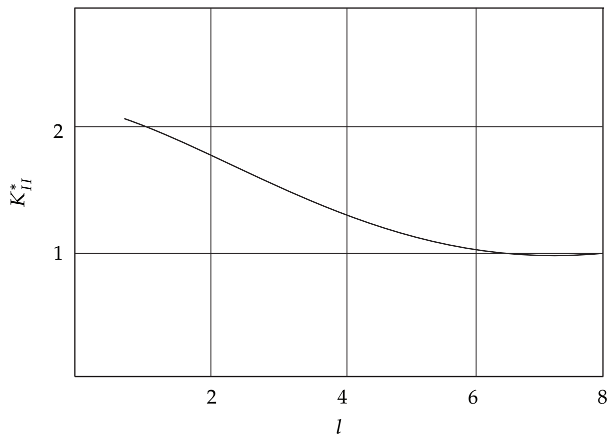

Figure 4

Appendices

A. The Equilibrium Equations' Representation in the Space of Mellin's Integral Transformation Which was Applied by the Generalized Scheme

$$\begin{aligned}
&R^{s+1}\langle u'(R,\theta)\rangle + R^s(s-1)\chi(\theta) + (s(s-1)-2)u_s(\theta) + \frac{1}{\mu_*}\frac{(\sin\theta u_s^\bullet(\theta))^\bullet}{\sin\theta} \\
&\quad - \frac{\mu_{**}}{\mu_*}\frac{(\nu_s\sin\theta)^\bullet}{\sin\theta} + \frac{\mu_0}{\mu_*}\frac{\left(\sin\theta\left(R^s\psi(\theta)-s\nu_s(\theta)\right)\right)^\bullet}{\sin\theta} = 0, \\
&R^{s+1}\langle \nu'(R,\theta)\rangle - R^s(s-1)\psi(\theta) + s(s-1)\nu_s(\theta) + \mu_*\left(\frac{(\nu_s^\bullet(\theta)\sin\theta)^\bullet}{\sin\theta} - \frac{\nu_s(\theta)}{\sin^2\theta}\right) \\
&\quad + \mu_0\left(R^s\chi^\bullet(\theta) - su_s^\bullet(\theta)\right) + 2\mu_* u_s^\bullet(\theta) = 0.
\end{aligned} \tag{A.1}$$

B. The Representation of the Transformations' Coefficients of the Equilibrium Equations' Discontinuous Solution

$$\Delta_{sk} = s^4 - 2s^3 + (-2N_k - 1)s^2 + (2N_k + 2)s + N_k^2 - 2N_k = \prod_{j=1}^{4}(s - s_j),$$

$$s_1 = -\nu_k - 2, \qquad s_2 = -\nu_k, \qquad s_3 = \nu_k - 1, \qquad s_4 = \nu_k + 1, \qquad N_k = \nu_k(\nu_k + 1),$$

$$\alpha(s,k) = s^3 + a_{1k}s^2 + a_{2k}s + a_{3k}, \qquad q(s,k) = s^2 b_{1k} + s b_{1k} + b_{3k},$$

$$\beta(s,k) = s^2 a_{4k} + s a_{5k} + a_{6k}, \qquad l(s,k) = s^3 - 3s^2 + s b_{4k} + b_{5k},$$

$$\gamma(s,k) = s^3 a_{7k} + s^2 a_{8k} + s a_{9k} + a_{10k}, \qquad p(s,k) = s^2 b_{6k} + s b_{7k} + b_{8k},$$

$$a_{1k}=\frac{5\mu-2}{1-\mu},\quad a_{2k}=\left(\frac{1-4\mu}{1-\mu}+N_k\left(2\mu\mu_0^2-\mu_*\right)\right),\quad a_{3k}=\left(2\mu\mu_0N_{**}-\frac{\mu_*(4\mu-1)}{1-\mu}\right)N_k,$$
$$a_{4k}=\frac{1}{2\mu_0(1-\mu)}-\mu_0,\quad a_{5k}=\mu_0-2-\frac{1}{2\mu_0(1-\mu)},\quad a_{6k}=2\mu_{**}-\frac{\mu_*N_k}{2\mu_0(1-\mu)},$$
$$a_{7k}=\frac{\sin\omega P_{\nu_k}^1(\cos\omega)}{\mu_*}2\mu\mu_0,\qquad a_{8k}=\sin\omega P_{\nu_k}^1(\cos\omega)-\sin\omega\frac{P_{\nu_k}^1(\cos\omega)}{\mu_*}2\mu\mu_0,$$
$$a_{9k}=\sin\omega P_{\nu_k}^1(\cos\omega)(-1-N_k2\mu\mu_0)-\sin\omega\left(P_{\nu_k}^1(\cos\omega)\right)^{\bullet}\mu_*\mu_0,$$
$$a_{10k}=-\sin\omega P_{\nu_k}^1(\cos\omega)\mu_*N_k-\sin\omega\left(P_{\nu_k}^1(\cos\omega)\right)^{\bullet}\mu_*\mu_{**},$$
$$b_{1k}=(1-2\eta\mu_0)N_k,\quad b_{2k}=2N_k(\mu\mu_0-1)+\frac{4\mu-1}{1-\mu},\quad b_{3k}=N_k\left(2\mu\mu_0\left(2+\frac{N_k}{\mu_*}\right)-\frac{8\mu-2}{1-\mu}\right),$$
$$b_{4k}=\frac{N_k}{2\mu_0(1-\mu)}-\left(2+\frac{N_k}{\mu_*}\right),\qquad b_{5k}=2\left(2+\frac{N_k}{\mu_*}\right)-\frac{N_k}{\mu_0(1-\mu)},$$
$$b_{6k}=\sin\omega\left(\mu_*\left(P_{\nu_k}^1(\cos\omega)\right)^{\bullet}+\frac{2\mu\mu_0}{\mu_*}N_kP_{\nu_k}^1(\cos\omega)\right),$$
$$b_{7k}=-\mu_*\sin\omega\left(P_{\nu_k}^1(\cos\omega)\right)^{\bullet}+P_{\nu_k}^1(\cos\omega)\sin\omega\left(1-\frac{4\mu\mu_0}{\mu_*}\right)N_k,$$
$$b_{8k}=-\mu_*\left(2+\frac{N_k}{\mu_*}\right)\sin\omega\left(P_{\nu_k}^1(\cos\omega)\right)^{\bullet}-2N_k\sin\omega P_{\nu_k}^1(\cos\omega). \tag{B.1}$$

C. The Discontinuous Solutions in the Space of the Integral Transformation with Regard to the Variable θ

$$u_k(r)=\begin{cases}\displaystyle\sum_{j=1}^{2}\left(\frac{R}{r}\right)^{s_j}\left[\frac{\alpha(s_{j,k})}{\Delta_{j,k}}\chi_k+\frac{\beta(s_{j,k})}{\Delta_{j,k}}\psi_k\right]+\int_0^\infty\nu(\xi,\omega)g_k\left(\frac{r}{\xi}\right)\frac{d\xi}{\xi}, & r<R,\\[2ex] \displaystyle\sum_{j=3}^{4}\left(\frac{R}{r}\right)^{s_j}\left[\frac{\alpha(s_{j,k})}{\Delta_{j,k}}\chi_k+\frac{\beta(s_{j,k})}{\Delta_{j,k}}\psi_k\right]+\int_0^\infty\nu(\xi,\omega)g_k\left(\frac{r}{\xi}\right)\frac{d\xi}{\xi}, & r>R,\end{cases}$$

$$\Delta_{j,k}=\prod_{\substack{i=1\\ i\neq j}}^{4}(s_j-s_i),\qquad g_k(z)=\begin{cases}\displaystyle\sum_{j=1}^{2}\left(\frac{R}{z}\right)^{s_j}\left[\frac{\gamma(s_{j,k})}{\Delta_{j,k}}\right], & z<R,\\[2ex] \displaystyle\sum_{j=3}^{4}\left(\frac{R}{z}\right)^{s_j}\left[\frac{\gamma(s_{j,k})}{\Delta_{j,k}}\right], & z>R,\end{cases}$$

$$\nu_k(r) = \begin{cases} \sum_{j=1}^{2}\left(\frac{R}{r}\right)^{s_j}\left[\frac{g(s_{j,k})}{\Delta_{j,k}}\chi_k + \frac{l(s_{j,k})}{\Delta_{j,k}}\psi_k\right] + \int_0^\infty \nu(\xi,\omega)h_k\left(\frac{r}{\xi}\right)\frac{d\xi}{\xi}, & r < R, \\ \sum_{j=3}^{4}\left(\frac{R}{r}\right)^{s_j}\left[\frac{g(s_{j,k})}{\Delta_{j,k}}\chi_k + \frac{l(s_{j,k})}{\Delta_{j,k}}\psi_k\right] + \int_0^\infty \nu(\xi,\omega)h_k\left(\frac{r}{\xi}\right)\frac{d\xi}{\xi}, & r > R, \end{cases}$$

$$\Delta_{j,k} = \prod_{\substack{i=1 \\ i\neq j}}^{4}(s_j - s_i), \qquad h_k(z) = \begin{cases} \sum_{j=1}^{2}\left(\frac{R}{z}\right)^{s_j}\left[\frac{p(s_{j,k})}{\Delta_{j,k}}\right], & z < R, \\ \sum_{j=3}^{4}\left(\frac{R}{z}\right)^{s_j}\left[\frac{p(s_{j,k})}{\Delta_{j,k}}\right], & z > R. \end{cases} \tag{C.1}$$

D. The Kernels and the Right-Hand Parts of the Integral Equations for the Unknown Jumps Searching

$$\begin{aligned}
F_1(\theta,\eta) &= \sum_{k=0}^{\infty}\sum_{j=1}^{2}\frac{(-1-s_j)q(s_{j,k})}{\Delta_{j,k}}\frac{P^1_{\nu_k}(\cos\theta)}{\left\|P^1_{\nu_k}(\cos\theta)\right\|^2}P^0_{\nu_k}(\cos\eta)\sin\eta \\
&\quad + \sum_{k=0}^{\infty}\sum_{j=1}^{2}\frac{\alpha(s_{j,k})}{\Delta_{j,k}}\frac{P^1_{\nu_k}(\cos\theta)}{\left\|P^0_{\nu_k}(\cos\theta)\right\|^2}P^0_{\nu_k}(\cos\eta)\sin\eta, \\
F_2(\theta,\eta) &= \sum_{k=0}^{\infty}\sum_{j=1}^{2}\frac{(-1-s_j)l(s_{j,k})}{\Delta_{j,k}}\frac{P^1_{vk}(\cos\theta)}{\left\|P^1_{vk}(\cos\theta)\right\|^2}P^1_{vk}(\cos\eta)\sin\eta \\
&\quad + \sum_{k=0}^{\infty}\sum_{j=1}^{2}\frac{\beta(s_{j,k})}{\Delta_{j,k}}\frac{P^1_{vk}(\cos\theta)}{\left\|P^0_{vk}(\cos\theta)\right\|^2}P^1_{vk}(\cos\eta)\sin\eta, \\
\alpha_1(\xi,\theta) &= -\sum_{k=0}^{\infty}\frac{P^1_{vk}(\cos\theta)}{\left\|P^1_{vk}(\cos\theta)\right\|^2}y_k(\xi) + \sum_{k=0}^{\infty}\frac{P^1_{vk}(\cos\theta)}{\left\|P^0_{vk}(\cos\theta)\right\|^2}g_k(\xi), \\
\alpha_2(\xi,\theta) &= R\sum_{k=0}^{\infty}\frac{P^1_{vk}(\cos\theta)}{\left\|P^1_{vk}(\cos\theta)\right\|^2}y_k(\xi), \\
F_3(\theta,\eta) &= \sum_{k=0}^{\infty}\sum_{j=1}^{2}\frac{\alpha(s_{j,k})}{R}\frac{(2\mu\mu_0 - (\eta\mu_0+1)s_j)}{\Delta_{j,k}}\frac{P^0_{vk}(\cos\theta)P^0_{vk}(\cos\eta)}{\left\|P^0_{vk}(\cos\theta)\right\|^2}\sin\eta \\
&\quad + \sum_{k=0}^{\infty}\sum_{j=1}^{2}\frac{\mu\mu_0}{R}\frac{g(s_{j,k})}{\Delta_{j,k}}\frac{\left((P^1_{vk}(\cos\theta))^{\bullet} + ctg\theta P^1_{vk}(\cos\theta)\right)}{\left\|P^1_{vk}(\cos\theta)\right\|^2}P^0_{vk}(\cos\eta)\sin\eta, \\
F_4(\theta,\eta) &= \sum_{k=0}^{\infty}\sum_{j=1}^{2}\frac{\beta(s_{j,k})}{\Delta_{j,k}}(2\mu\mu_0 - (\eta\mu_0+1)s_j)\frac{P^0_{vk}(\cos\theta)P^1_{vk}(\cos\eta)}{\left\|P^0_{vk}(\cos\theta)\right\|^2}\sin\eta
\end{aligned}$$

$$+\sum_{k=0}^{\infty}\sum_{j=1}^{2}\frac{\mu\mu_0}{R}\frac{l(s_{j,k})}{\Delta_{j,k}}\frac{\left((P_{vk}^1(\cos\theta))^{\bullet}+ctg\theta P_{vk}^1(\cos\theta)\right)}{\left\|P_{vk}^1(\cos\theta)\right\|^2}P_{vk}^1(\cos\eta)\sin\eta,$$

$$\alpha_3(\xi,\theta)=\frac{2\mu\mu_0}{R}\sum_{k=0}^{\infty}\frac{P_{vk}^0(\cos\theta)}{\left\|P_{vk}^0(\cos\theta)\right\|^2}g_k(\xi)$$

$$+\frac{\mu\mu_0}{R}\sum_{k=0}^{\infty}\frac{y_k(\xi)}{\left\|P_{vk}^1(\cos\theta)\right\|^2}\left(\left(P_{vk}^1(\cos\theta)\right)^{\bullet}+ctg\theta P_{vk}^1(\cos\theta)\right),$$

$$\alpha_4(\xi,\theta)=\frac{\mu\mu_0+1}{R}\sum_{k=0}^{\infty}\frac{P_{vk}^0(\cos\theta)}{\left\|P_{vk}^0(\cos\theta)\right\|^2}g_k(\xi). \tag{D.1}$$

E. The Order of the Displacement's Singularity

Guttmann's representation of the equilibrium equation solutions [26] was used to obtain the order of the displacement's singularity:

$$u(\nu,\theta)=\Phi'(\nu,\theta)-2(1-\mu)r\Delta F(\nu,\theta),\qquad \nu(r,\theta)=\frac{\Phi^{\bullet}(\nu,\theta)}{r}, \tag{E.1}$$

where

$$u(r,\theta)=2Gu_{\nu}(r,\theta),\quad \nu(r,\theta)=2Gu_{\theta}(r,\theta),\quad \Delta^2F(r,\theta)=0, \tag{E.2}$$

$$\Phi(r,\theta)=rF'(r,\theta)+\kappa F(r,\theta), \tag{E.3}$$

The operators Δ and ∇ are defined by the equalities $\Delta F=(r^2F')'/r^2-\nabla F/r^2,\nabla-\quad\nabla F=-(\sin\theta F^{\bullet})^{\bullet}/\sin\theta$.

Correspondently to [22], the function $F(r,\theta)$ is represented in the form $F(r,\theta)=r^{\lambda}g(\theta)$, where λ is the searched order of the singularity. This representation is substituted in (E.2), and operator Δ^2 is applied to it. It leads to the solving of the differential equation:

$$\lambda(\lambda+1)g(\theta)+\frac{(\sin\theta g^{\bullet}(\theta))^{\bullet}}{\sin\theta}=C_0P_{\lambda-2}(\cos\theta)+C_1Q_{\lambda-2}(\cos\theta), \tag{E.4}$$

where C_0, C_1 are the unknown constants. The solution of this equation is

$$g(\theta)=C_0P_{\lambda}(\cos\theta)+C_1P_{\lambda-2}(\cos\theta)+C_2Q_{\lambda}(\cos\theta)+C_3Q_{\lambda-2}(\cos\theta). \tag{E.5}$$

Taking into consideration the regularity of the solution of variable θ, one must demand that $C_2 = C_3 = 0$. It allows one to write:

$$F(r,\theta) = r^{\lambda}\left(C_0 P_{\lambda}(\cos\theta) + C_1 P_{\lambda-2}(\cos\theta)\right),$$
$$\Phi(r,\theta) = r^{\lambda}(\lambda+\kappa)\left(C_0 P_{\lambda}(\cos\theta) + C_1 P_{\lambda-2}(\cos\theta)\right). \tag{E.6}$$

The final expression for the displacements will be acquired after the substitution of the solutions (E.6) into the formulas (E.1)

$$u(r,\theta) = r^{\lambda-1}\Big[\lambda(\lambda+\kappa)C_0 P_{\lambda}(\cos\theta) + \Big(\lambda^2+\lambda\kappa+2(1-\mu)\Big)C_1 P_{\lambda-2}(\cos\theta)\Big],$$
$$v(r,\theta) = r^{\lambda-1}(\lambda+\kappa)\left[C_0 P_{\lambda}^{\bullet}(\cos\theta) + C_1 P_{\lambda-2}^{\bullet}(\cos\theta)\right], \tag{E.7}$$

when it follows that $u_r(r,\theta) = (1/2G)u(r,\theta)$, $u_{\theta}(r,\theta) = (1/2G)v(r,\theta)$.

The conditions of the problem (2.1) should be satisfied on the conical surface $\theta = \omega$ in order for the stress to be found from the known relations of the displacements and stress connections:

$$\tau_{r\theta} = \frac{1}{2G}r^{\lambda-2}\Big[(\lambda-1)(\lambda+\kappa)C_0 P_{\lambda}^{\bullet}(\cos\theta) + \Big(\lambda^2+\lambda(\kappa-1)-\kappa+1-\mu\Big)C_1 P_{\lambda-2}^{\bullet}(\cos\theta)\Big],$$
$$v_{\theta}(r,\theta) = \frac{1}{2G}r^{\lambda-2}\left[(\mu\mu_0(\lambda+1)+1)g_1(\theta) + (\lambda+\kappa)(\mu\mu_0+1)g_2(\theta)\right],$$
$$g_1(\theta) = \lambda(\lambda+\kappa)C_0 P_{\lambda}(\cos\theta) + (\lambda(\lambda+\kappa)+2(1-\mu))C_1 P_{\lambda-2}(\cos\theta),$$
$$g_2(\theta) = C_0 P_{\lambda}^{\bullet\bullet}(\cos\theta) + C_1 P_{\lambda-2}^{\bullet\bullet}(\cos\theta). \tag{E.8}$$

One must substitute the equalities (E.6) in the conditions (2.13) and pass to $\theta = \omega$. With that, the homogenous system of equations with regard to the unknown constants C_0, C_1 is obtained. Its determinant should be equal to zero for its unique solution. It yields the transcendental equation for λ obtaining:

$$\begin{aligned}
&\Big(\lambda^2+\lambda(\kappa-1)-\kappa\Big)(\lambda\mu\mu_0+\mu\mu_0+1)\Big(\lambda^2+\lambda\kappa+2(1-\mu)\Big)P_{\lambda}^{\bullet}(\cos\omega)P_{\lambda-2}(\cos\omega)\\
&\quad+\Big(\lambda^2+\lambda(\kappa-1)-\kappa\Big)(\lambda(\mu\mu_0+1)+\kappa(\eta\mu_0+1))P_{\lambda}^{\bullet}(\cos\omega)P_{\lambda-2}^{\bullet\bullet}(\cos\omega)\\
&\quad-\lambda(\lambda+\kappa)\Big(\lambda^2+\lambda(\kappa-1)+1-\mu-\kappa\Big)(\lambda\mu\mu_0+\mu\mu_0+1)P_{\lambda-2}^{\bullet}(\cos\omega)P_{\lambda}(\cos\omega)\\
&\quad-(\lambda+\kappa)(\mu\mu_0+1)\Big(\lambda^2+\lambda(\kappa-1)-\kappa+1-\mu\Big)(\lambda\mu\mu_0+\kappa(\eta\mu_0+1))P_{\lambda-2}^{\bullet}(\cos\omega)P_{\lambda}^{\bullet\bullet}(\cos\omega)\\
&=0.
\end{aligned} \tag{E.9}$$

Equation (E.9) is solved numerically with MAPLE. By results of the roots' analysis that root, which brings the strongest singularity in the solution, gets out. The searched value is $\lambda = \lambda_*$, hence the searched order of the displacement's singularity is $v(r,\omega) \sim r^{\lambda_*-1}$.

F. The Coefficients of the Linear Algebraic Equation System with Regard to the Expansion Coefficients (4.5)

$$\tilde{F}_{kl}^{j} = \iint_0^1 \sqrt{\eta - \eta^2}\sqrt{\theta - \theta^2} V_k(2\eta - 1) V_l(2\theta - 1) \tilde{F}_j(\theta, \eta) d\eta\, d\theta, \quad j = \overline{1, 4},$$

$$f_l^1 = -\int_0^1 \sqrt{\theta - \theta^2} \tau_{2\theta}^{\theta}(\theta) V_l(2\theta - 1) d\theta, \qquad f_l^2 = -\int_0^1 \sqrt{\theta - \theta^2} \sigma_r^0(\theta) V_l(2\theta - 1) d\theta, \tag{F.1}$$

$$B_{nl}^{j_i} = \int_0^1 B_n^j(\theta) \sqrt{\theta - \theta^2} V_l(2\theta - 1) d\theta, \quad i = 1, 2.$$

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