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Research Article

A Novel Solution for the Glauert-Jet Problem by Variational Iteration Method-Padé Approximant

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We will consider variational iteration method (VIM) and Padé approximant, for finding analytical solutions of the Glauert-jet (self-similar wall jet over an impermeable, resting plane surface) problem. The solutions are compared with the exact solution. The results illustrate that VIM is an attractive method in solving the systems of nonlinear equations. It is predicted that VIM can have a found wide application in engineering problems.

1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. We know that most of engineering problems are nonlinear, and it is difficult to solve them analytically. Various powerful mathematical methods have been proposed for obtaining exact and approximate analytic solutions.

The VIM was first proposed by He [1, 2] systematically illustrated in 1999 [3], and used to give approximate solutions of the problem of seepage flow in porous media with fractional derivatives. The VIM is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. In this method, general Lagrange multipliers are introduced to construct correction functionals for the problems. The multipliers can be identified optimally via the variational theory. There is no need of linearization or discretization, and large computational work and round-off errors are avoided. It has been used to solve effectively, easily, and accurately a large class of nonlinear problems with approximation [4, 5]. It was shown by many authors [6–16] that this method is more powerful than existing techniques such as the Adomian method [17, 18]. He et al. [19] proposed three standard variational iteration algorithms for solving differential equations, integro-differential equations, fractional differential equations, differential-difference equations, and fractional/fractal differential-difference equations. The algorithm used in this paper belongs to the variational iteration algorithm-I according to [19]. Recently Herisanu and Marinca [20] suggested an optimal variational iteration algorithm using the variational iteration method for nonlinear oscillator. Ismail and Abde Rabboh [21] presented a restrictive Padé approximation for the generalized Fisher and Burger-Fisher equations. Rashidi and Shahmohamadi [22] provided a new technique for solution of three-dimensional Navier-Stokes equations for the flow near an infinite rotating disk by combination of variational iteration method and Padé approximant.

The motivation of this letter is to extend variational iteration method and Padé approximant to solve Glauert-jet problem [23], also known in the Russian literature as the Akatnov results [24]. The Glauert-jet named also the plane wall jet is a thin jet of fluid that flows tangentially to an impermeable, resting wall surrounded by fluid of the same type in a quiescent ambient flow. It consists of an inner region wherein the flow resembles the boundary layer and an outer region wherein the flow is like a free shear layer. The wall jets are of great engineering importance with many applications. Main applications are turbine blade cooling, paint spray, and air-foils in high-lift configurations [25].

We consider the self-similar plane wall jet formed over an impermeable resting wall governed by the equation of a steady boundary layer over a flat plate (see [26])

$$\frac{\partial\psi(x;y)}{\partial y}\frac{\partial^2\psi(x;y)}{\partial x\partial y}-\frac{\partial\psi(x;y)}{\partial x}\frac{\partial^2\psi(x;y)}{\partial y^2}=\frac{\partial^3\psi(x;y)}{\partial y^3}, \tag{1.1}$$

with the dimensionless stream function

$$\psi(x,y)=4x^{1/4}f(t), \quad t=x^{-3/4}y, \tag{1.2}$$

where $\psi(x,y)$ is the stream function. The system deals with the following impermeability, no-slip, and asymptotic conditions:

$$\psi(x,0)=0, \quad \frac{\partial\psi(x;0)}{\partial y}=0, \quad \frac{\partial\psi(x;y)}{\partial y}\longrightarrow 0 \quad \text{as } y\longrightarrow\infty. \tag{1.3}$$

The self-similar part $f(t)$ satisfies the ordinary differential equation

$$f'''+ff''+2f'^2=0, \tag{1.4}$$

along with the boundary conditions

$$f(0)=0, \qquad f'(0)=0, \qquad f'(t)\longrightarrow 0 \quad \text{as } t\longrightarrow\infty. \tag{1.5}$$

The Glauert-jet solution corresponds to the normalisation $f(\infty)=1$ of the stream function. It leads to the well-known implicit form of the analytical solution of the problem (1.4) and (1.5) found by Glauert [23]

$$t=3^{1/2}\arctan\left[\frac{(3f)^{1/2}}{2+f^{1/2}}\right]+\ln\left[\frac{\left(1+f+f^{1/2}\right)^{1/2}}{1-f^{1/2}}\right]. \tag{1.6}$$

The downstream velocity profile $f'(t)$ of the Glauert-jet solution is

$$f'(t) = \frac{2}{3}\sqrt{f}\left(1-\left(\sqrt{f}\right)^{3}\right), \tag{1.7}$$

with the skin friction

$$f''(0) = \frac{2}{9}. \tag{1.8}$$

2. Basic Concepts of VIM

To illustrate the basic concepts of VIM, we consider the following differential equation:

$$Lu + Nu = g(t), \tag{2.1}$$

where L, N, and $g(t)$ are the linear operator, the nonlinear operator, and a heterogeneous term, respectively. The variational iteration method was proposed by He where a correction functional for (2.1) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left[Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right]d\tau, \quad n \geq 0. \tag{2.2}$$

It is obvious that the successive approximations, u_j, $j \geq 0$ can be established by determining λ, a general Lagrangian multiplier, which can be identified optimally via the variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(t)$, $n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. With λ determined, then several approximations $u_j(t)$, $j \geq 0$, follow immediately. Consequently, the exact solution may be obtained by using

$$u = \lim_{n\to\infty} u_n. \tag{2.3}$$

3. Analytical Solution

In order to obtain VIM solution of (1.4), we construct a correction functional which reads

$$f_{n+1}(t) = f_n(t) + \int_0^t \lambda\left(\frac{\partial^3 f_n(\tau)}{\partial\tau^3} + \tilde{f}_n(\tau)\frac{\partial^2 \tilde{f}_n(\tau)}{\partial\tau^2} + 2\left(\frac{\partial\tilde{f}_n(\tau)}{\partial\tau}\right)^2\right)d\tau, \tag{3.1}$$

where λ is the general Lagrangian multiplier which is to be determined later and $\tilde{f}_n(\tau)$ is considered as a restricted variation, that is, $\delta\tilde{f}_n(\tau) = 0$.

Its stationary conditions can be obtained as follows:

$$1+\lambda''(\tau)\big|_{\tau=t}=0\ ,\qquad \lambda(\tau)|_{\tau=t}=0\ ,\qquad \lambda'(\tau)\big|_{\tau=t}=0,\qquad \lambda'''(\tau)=0. \tag{3.2}$$

The Lagrange multiplier can be identified as

$$\lambda=-\frac{1}{2}(\tau-t)^2. \tag{3.3}$$

As a result, the following variational iteration formula can be obtained

$$f_{n+1}(t)=f_n(t)-\frac{1}{2}\int_0^t(\tau-t)^2\left(\frac{\partial^3 f_n(\tau)}{\partial\tau^3}+\tilde{f}_n(\tau)\frac{\partial^2\tilde{f}_n(\tau)}{\partial\tau^2}+2\left(\frac{\partial\tilde{f}_n(\tau)}{\partial\tau}\right)^2\right)d\tau. \tag{3.4}$$

Now we must start with an arbitrary initial approximation. Therefore according to (1.5) and (1.8), it is straight-forward to choose an initial guess

$$f_o(t)=\frac{1}{9}t^2. \tag{3.5}$$

Using the above variational formula (3.4), we have

$$f_1(t)=f_0(t)-\frac{1}{2}\int_0^t(\tau-t)^2\left(\frac{\partial^3 f_0(\tau)}{\partial\tau^3}+\tilde{f}_0(\tau)\frac{\partial^2\tilde{f}_0(\tau)}{\partial\tau^2}+2\left(\frac{\partial\tilde{f}_0(\tau)}{\partial\tau}\right)^2\right)d\tau. \tag{3.6}$$

Substituting (3.5) into (3.6), we have

$$f_1(t)=\frac{1}{9}t^2-\frac{1}{486}t^5. \tag{3.7}$$

By the same way, we can obtain $f_2(t),\ f_3(t),\ldots.$

After obtaining the result of 7th iteration, we will apply the padé approximation using symbolic software such as Mathematica; we have the following:

$$\begin{aligned} f(t)_{[12,12]}={}&2184t^2\Big(787679232592245840+17256587913623520t^3+99910515826755t^6\\ &+135721670638t^9\Big)\Big/\Big(1548262299583318423104 0+625910732693761394880t^3\\ &+7676359653888591120t^6+293813628616 83924t^9+19389517508143t^{12}\Big). \end{aligned} \tag{3.8}$$

Therefore, we are able to give an approximate solution of the considered problem.

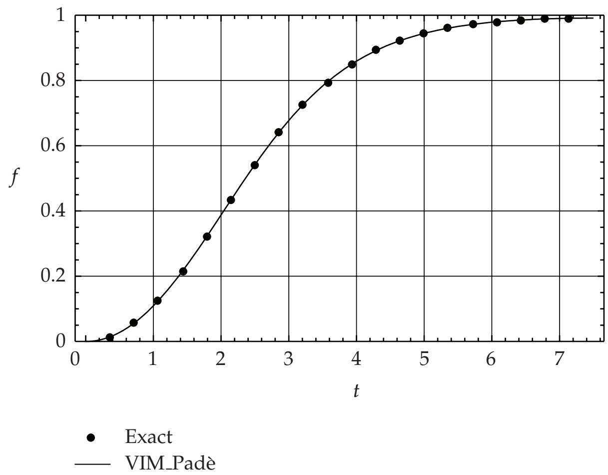

Figure 1: Comparison of the exact solution (1.6) with the VIM-Padé solution (3.8).

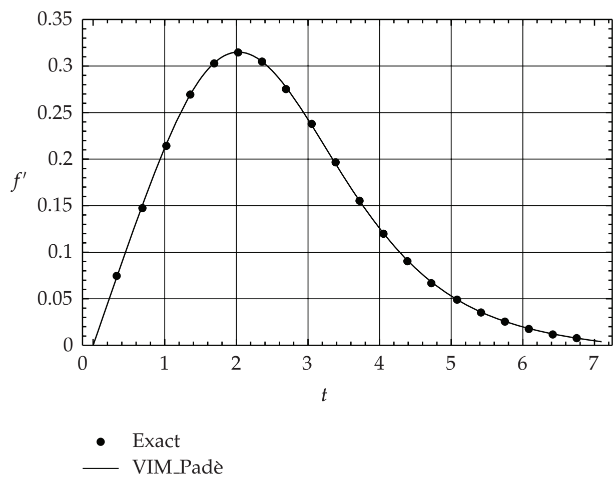

Figure 2: Comparison of the exact derivative solution (1.7) with the derivative of the VIM-Padé solution (3.8).

4. Discussion

In this paper, VIM and Padé approximants are used to find approximate solutions of the famous Glauert-jet problem. The problem of fluid jet along an impermeable, resting wall surrounded by fluid of the same type at rest has been considered. The closed-form solution of the corresponding boundary layer equations (1.6) was given by Glauert [23], known as the plane wall jet, e-jet, or the exponentially decaying wall jet [25].

The VIM-Padé solution of $f(t)$ has been compared with the exact solution in Figure 1. Table 1 demonstrates the values of absolute error between VIM-Padé solution and the exact solution for different values of t. In order to give a comprehensive approach of the problem, the derivative of the VIM-Padé solution (3.8) graph is given as well as the exact downstream velocity profile (1.7) graph, as shown in Figure 2. Table 2 demonstrates the values of absolute error between the derivative of the VIM-Padé solution (3.8) and the exact derivative solution

Table 1: Comparison of the exact solution (1.6) with the VIM-Padé solution (3.8).

t	VIM-Padé	Exact results	Absolute error
0.5	0.0277	0.0277	0
1	0.1091	0.1091	0
1.5	0.2354	0.2354	0
2	0.3879	0.3879	0
2.5	0.5421	0.5421	0
3	0.6774	0.6774	0
3.5	0.7833	0.7833	0
4	0.8595	0.8595	0
4.5	0.9111	0.9111	0
5	0.9446	0.9446	0
5.5	0.9657	0.9658	0.0001
6	0.9786	0.9791	0.0005
6.5	0.9863	0.9872	0.0009
7	0.9901	0.9922	0.0021

Table 2: Comparison of the exact derivative solution (1.7) with the derivative of the VIM-Padé solution (3.8).

t	VIM-Padé	Exact results	Absolute error
0.5	0.1105	0.1105	0
1	0.2123	0.2123	0
1.5	0.2865	0.2865	0
2	0.3149	0.3149	0
2.5	0.2949	0.2949	0
3	0.2428	0.2428	0
3.5	0.1810	0.1810	0
4	0.1256	0.1256	0
4.5	0.0829	0.0830	0.0001
5	0.0530	0.0531	0.0001
5.5	0.0329	0.0333	0.0004
6	0.0198	0.0206	0.0008
6.5	0.0111	0.0126	0.0015
7	0.0049	0.0077	0.0028

(1.7) for different values of t.The accuracy of the method is very good, and the obtained results are near to the exact solution. The result show that the VIM is a powerful methods has high accuracy, and, very efficient.We sincerely hope this method can be applied in a wider range.

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