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Research Article

Calculation of Some Integrals Arising in Heat Transfer in Grinding

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We calculate some integrals involved in the temperature field evaluation of a workpiece during the grinding process. For the case of dry continuous grinding, this calculation allows a faster computation of the temperature field on surface and inside the workpiece.

1. Introduction

In the resolution of the heat equation that models the heat transfer during the grinding process [1], there appear integrals [2] of the following type:

$$I_1(a,b,c) := \int_0^\infty \left[\operatorname{erf}\left(\frac{a}{\sigma}+b\sigma\right)-\operatorname{erf}\left(\frac{a-c}{\sigma}+b\sigma\right)\right]d\sigma, \tag{1.1}$$

$$I_2(x,y) := \frac{|y|}{\pi}\int_{-\infty}^{\infty} K_0(|x-\xi|)\frac{K_1\left(\sqrt{y^2+\xi^2}\right)}{\sqrt{y^2+\xi^2}}d\xi, \tag{1.2}$$

which have not been tabulated yet [3]. The first integral arises in the evaluation of the temperature on the surface of the workpiece, while the second integral is used for the evaluation of the field temperature inside the workpiece. On the one hand, despite the fact that these integrals can be computed by using the uniqueness of the solution to the Laplace equation, as done in [2], this is cumbersome, since the integrals to be calculated are subproducts of a wider framework. This framework is the resolution of the heat equation in the stationary regime in two different ways. This paper presents a straightforward proof, based on elementary integral calculus and complex variables. On the other hand, the computation of these improper integrals, which show a parametric dependence on a, b, and c for I_1, and x and y for I_2, is expensive in order to get the field temperature of the wokpiece being ground. The goal of this paper is the resolution of (1.1) and (1.2), so that the evaluation of the temperature field may be faster.

This paper is organized as follows. Sections 2 and 3 are devoted to the calculation of I_1 and I_2, respectively. Section 4 applies I_1 and I_2 to the solution of the stationary regime in dry continuous grinding within the Samara-Valencia model [1].

2. First Integral

In order to calculate the integral (1.1), let us solve the integral (2.1) in two different ways. Afterwards, comparing the results obtained, we will find the sought-for solution. Let us define:

$$I := \int_0^\infty \left\{ \int_0^c \exp\left(-\frac{(x-a-4bs)^2}{4s} \right) dx \right\} \frac{ds}{s}, \tag{2.1}$$

where a, b, and c are parameters within the integral.

2.1. First Calculation

Applying Fubini's theorem to (2.1), we can exchange the integration order, obtaining

$$I = \int_0^c \left\{ \int_0^\infty \exp\left(-\frac{(x-a-4bs)^2}{4s} \right) \frac{ds}{s} \right\} dx. \tag{2.2}$$

Let us develop the exponential within (2.2), as follows:

$$-\frac{(x-a-4bs)^2}{4s} = -\frac{(x-a)^2}{4s} - 4b^2 s + 2b(x-a). \tag{2.3}$$

Calling the inner integral in (2.2) $\widehat{I}$ and taking into account (2.3), we have

$$\widehat{I} := e^{2b(x-a)} \int_0^\infty \exp\left(-\frac{(x-a)^2}{4s} - 4b^2 s \right) \frac{ds}{s}. \tag{2.4}$$

Let us perform the change of variables $\sigma = 4b^2 s$ in (2.4), and let us define the variable z as

$$z := |2b(x-a)|, \tag{2.5}$$

so that (2.4) becomes

$$\widehat{I} = e^{2b(x-a)} \int_0^\infty \exp\left(-\frac{z^2}{4\sigma} - \sigma\right) \frac{d\sigma}{\sigma}. \tag{2.6}$$

Knowing the following integral representation of the zero order modified Bessel function [4]:

$$K_0(z) = \frac{1}{2} \int_0^\infty \exp\left(-\sigma - \frac{z^2}{4\sigma}\right) \, \sigma^{-1} d\sigma, \tag{2.7}$$

and substituting (2.7) in (2.6), we get

$$\widehat{I} = 2e^{2b(x-a)} K_0(z). \tag{2.8}$$

Remembering the definition of z given in (2.5), and substituting (2.8) in (2.2), we find the following expression for I:

$$I = 2 \int_0^c e^{2b(x-a)} K_0(|2b(x-a)|) dx. \tag{2.9}$$

Now, performing in (2.9) the change of variables $u = 2b(x - a)$, we obtain

$$I = \frac{1}{b} \int_{-2ba}^{2b(c-a)} e^u K_0(|u|) du. \tag{2.10}$$

In the appendix, it is shown that,

$$\mathrm{Jg}(x) := \int_0^x e^u K_0(|u|) du = \begin{cases} xe^x \left[K_0(|x|) + \operatorname{sign}(x) K_1(|x|)\right] - 1, & \forall x \neq 0, \\ 0, & \forall x = 0. \end{cases} \tag{2.11}$$

Straightforwardly from (2.11), (2.10) can be rewritten as

$$I = \frac{1}{b} \{\mathrm{Jg}(2b(c-a)) - \mathrm{Jg}(-2ba)\}. \tag{2.12}$$

2.2. Second Calculation

Let us consider now the inner integral in (2.1):

$$\overline{I} := \int_0^c \exp\left(-\frac{(x-a-4bs)^2}{4s}\right) dx. \tag{2.13}$$

Performing the following change of variables in (2.13) $u = (x - a - 4bs)/2\sqrt{s}$, we obtain that

$$\overline{I} = 2\sqrt{s}\int_{-(a+4bs)/2\sqrt{s}}^{(c-a-4bs)/2\sqrt{s}} e^{-u^2}du. \tag{2.14}$$

We can rewrite (2.14) in terms of the error function [4], so that

$$\overline{I} = \sqrt{\pi s}\left[\operatorname{erf}\left(\frac{c-a-4bs}{2\sqrt{s}}\right) + \operatorname{erf}\left(\frac{a+4bs}{2\sqrt{s}}\right)\right]. \tag{2.15}$$

Therefore, substituting (2.15) in (2.1), we obtain that

$$I = \sqrt{\pi}\int_0^\infty \frac{1}{\sqrt{s}}\left[\operatorname{erf}\left(\frac{a}{2\sqrt{s}} + 2b\sqrt{s}\right) - \operatorname{erf}\left(\frac{a-c}{2\sqrt{s}} + 2b\sqrt{s}\right)\right]ds. \tag{2.16}$$

Let us perform now the change of variables $\sigma = 2\sqrt{s}$ in (2.16), so that (2.16) becomes

$$I = \sqrt{\pi}\int_0^\infty \left[\operatorname{erf}\left(\frac{a}{\sigma} + b\sigma\right) - \operatorname{erf}\left(\frac{a-c}{\sigma} + b\sigma\right)\right]d\sigma. \tag{2.17}$$

2.3. *Comparison*

Comparing (2.12) and (2.17), and the remembering the definition given in (1.1), we finally get,

$$I_1(a,b,c) = \frac{1}{b\sqrt{\pi}}\{\mathrm{Jg}(2b(c-a)) - \mathrm{Jg}(-2ba)\}, \tag{2.18}$$

where the function $\mathrm{Jg}(x)$ has been defined in (2.11).

3. Second Integral

In order to calculate the second integral defined in (1.2), let us perform the translation of coordinates, $\xi' = x - \xi$, thus,

$$I_2(x,y) = \frac{|y|}{\pi}\int_{-\infty}^{\infty} K_0(|\xi'|)\frac{K_1\left(\sqrt{y^2+(\xi'+x)^2}\right)}{\sqrt{y^2+(\xi'+x)^2}}d\xi'. \tag{3.1}$$

Let us define the complex integral along the contour C, as presented in Figure 1:

$$I_C(x,y) := \frac{|y|}{\pi}\int_C K_0(|\xi'|)\frac{K_1\left(\sqrt{y^2+(\xi'+x)^2}\right)}{\sqrt{y^2+(\xi'+x)^2}}d\xi'. \tag{3.2}$$

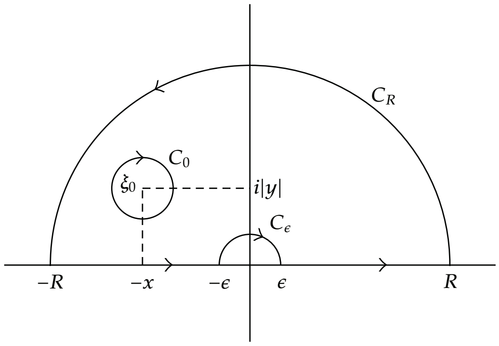

Figure 1

Observe that we can divide the integration path as follows:

$$I_C(x,y) = I_R^-(x,y) + I_R^+(x,y) + I_{C_R}(x,y) + I_{C_\epsilon}(x,y), \tag{3.3}$$

where we have defined:

$$I_R^\pm(x,y) := \mp\frac{|y|}{\pi}\int_{\pm R}^{\pm\epsilon} K_0(|\xi'|)\frac{K_1\left(\sqrt{y^2+(\xi'+x)^2}\right)}{\sqrt{y^2+(\xi'+x)^2}}d\xi', \tag{3.4}$$

$$I_{C_{R,\epsilon}}(x,y) := \frac{|y|}{\pi}\int_{C_{R,\epsilon}} K_0(|\xi'|)\frac{K_1\left(\sqrt{y^2+(\xi'+x)^2}\right)}{\sqrt{y^2+(\xi'+x)^2}}d\xi', \tag{3.5}$$

and where C_R and C_ϵ are the semicircumferences appearing in Figure 1. Taking absolute values and performing in (3.5) the changes of variables $\xi' = Re^{i\theta}$ and $\xi' = \epsilon e^{i\theta}$ along I_{C_R} and I_{C_ϵ}, respectively, we obtain that

$$\left|I_{C_R}(x,y)\right| = \left|\frac{yR}{\pi}K_0(R)\int_0^\pi \frac{K_1\left(\sqrt{y^2+\left(Re^{i\theta}+x\right)^2}\right)}{\sqrt{y^2+\left(Re^{i\theta}+x\right)^2}}d\theta\right|, \tag{3.6}$$

$$\left|I_{C_\epsilon}(x,y)\right| = \left|\frac{y\epsilon}{\pi}K_0(\epsilon)\int_0^\pi \frac{K_1\left(\sqrt{y^2+\left(\epsilon e^{i\theta}+x\right)^2}\right)}{\sqrt{y^2+\left(\epsilon e^{i\theta}+x\right)^2}}d\theta\right|. \tag{3.7}$$

Taking limits in (3.6),

$$\lim_{R\to\infty}\left|I_{C_R}(x,y)\right| = \lim_{R\to\infty}\left|\frac{yR}{\pi}K_0(R)\int_0^\pi \frac{K_1(R)}{R}d\theta\right| = \lim_{R\to\infty}\left|yK_0(R)K_1(R)\right| = 0. \tag{3.8}$$

Taking into account the asymptotic formula [4],

$$K_0(z) \underset{z\to 0^+}{\sim} \log\left(\frac{2}{z}\right), \tag{3.9}$$

we can take limits in (3.7), so that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\left|I_{C_\epsilon}(x,y)\right| &= \lim_{\epsilon\to 0^+}\left|\frac{y\epsilon}{\pi}K_0(\epsilon)\int_0^\pi \frac{K_1\left(\sqrt{y^2+x^2}\right)}{\sqrt{y^2+x^2}}d\theta\right| \\ &= \left|y\frac{K_1\left(\sqrt{y^2+x^2}\right)}{\sqrt{y^2+x^2}}\right|\left|\lim_{\epsilon\to 0^+}\epsilon\log\left(\frac{2}{\epsilon}\right)\right| = 0.\end{aligned} \tag{3.10}$$

Observe that from (3.4) and (3.10), the following limit is satisfied:

$$\lim_{R\to\infty,\ \epsilon\to 0^+} I_R^-(x,y) + I_R^+(x,y) = I_2(x,y). \tag{3.11}$$

Therefore, taking limits in (3.3) and applying the results (3.11), (3.8), we can conclude that

$$\lim_{R\to\infty,\epsilon\to 0^+} I_C(x,y) = I_2(x,y). \tag{3.12}$$

Since the integrand in (3.2) contains an unique singularity inside the contour C at the point,

$$\xi_0 = -x + i|y|, \tag{3.13}$$

applying the residue theorem [5], we can contract the contour C to the contour C_0, the latter surrounding the neighbourhood of the point ξ_0, as shown in Figure 1. Therefore, according to (3.12),

$$I_{C_0}(x,y) = \lim_{R\to\infty,\epsilon\to 0^+} I_C(x,y) = I_2(x,y), \tag{3.14}$$

where

$$I_{C_0}(x,y) := \frac{|y|}{\pi}\int_{C_0} K_0(|\xi'|)\frac{K_1\left(\sqrt{y^2+(\xi'+x)^2}\right)}{\sqrt{y^2+(\xi'+x)^2}}d\xi'. \tag{3.15}$$

Taking a circular contour C_0 around ξ_0, we can perform in (3.15) the change of variables $\xi' = \xi_0 + \eta e^{i\theta}$

$$I_{C_0}(x, y) = i\eta \frac{|y|}{\pi} \int_0^{2\pi} K_0\left(\left|\xi_0 + \eta e^{i\theta}\right|\right) \frac{K_1\left(\sqrt{y^2 + \left(\xi_0 + \eta e^{i\theta} + x\right)^2}\right)}{\sqrt{y^2 + \left(\xi_0 + \eta e^{i\theta} + x\right)^2}} e^{i\theta} d\theta. \tag{3.16}$$

Taking now into account the asymptotic formula [4],

$$K_1(z) \underset{z \to 0^+}{\sim} \frac{1}{z}, \tag{3.17}$$

we can take limits in (3.16),

$$I_{C_0}(x, y) = \lim_{\eta \to 0^+} I_{C_0}(x, y) = i \frac{|y|}{\pi} K_0(|\xi_0|) \lim_{\eta \to 0^+} \eta \int_0^{2\pi} \frac{e^{i\theta} d\theta}{y^2 + \left(\xi_0 + \eta\ e^{i\theta} + x\right)^2}. \tag{3.18}$$

Substituting the value of ξ_0 given in (3.13), and simplifying,

$$\begin{aligned} I_{C_0}(x, y) &= i \frac{|y|}{\pi} K_0(|\xi_0|) \lim_{\eta \to 0^+} \int_0^{2\pi} \frac{d\theta}{\eta e^{i\theta} + 2i|y|} \\ &= K_0(|\xi_0|) = K_0\left(\sqrt{x^2 + y^2}\right). \end{aligned} \tag{3.19}$$

Finally, from (3.14) and (3.19) we conclude that

$$I_2(x, y) = K_0\left(\sqrt{x^2 + y^2}\right). \tag{3.20}$$

4. Application to the Grinding Process

4.1. Samara-Valencia Model

The Samara-Valencia model setup is depicted in Figure 2. The workpiece moves at a constant speed v_d and is assumed to be infinite along Ox and Oz, and semiinfinte along Oy. The plane $y = 0$ is the surface being ground. The contact area between the wheel and the workpiece is an infinitely long strip of width δ located parallel to the Oz axis and on the plane $y = 0$. Both the wheel and the workpiece are assumed rigid.

The Samara-Valencia model [1] solves the convection heat equation,

$$\partial_t T(t, x, y) = k\left[\partial_{xx} T(t, x, y) + \partial_{yy} T(t, x, y)\right] - v_d \partial_x T(t, x, y), \tag{4.1}$$

subject to the initial condition

$$T(0, x, y) = 0, \tag{4.2}$$

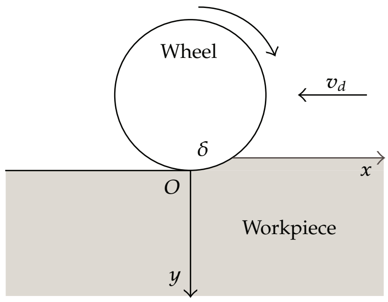

Figure 2

and the boundary condition,

$$k_0 \partial_y T(t, x, 0) = b(t, x) T(t, x, 0) + d(t, x), \tag{4.3}$$

where $-\infty < x < \infty$ and $t, y \geq 0$. The first term of (4.3) models the application of coolant over the workpiece surface considering $b(t, x)$ as the heat transfer coefficient. The second term, $d(t, x)$, represents the heat flux entering into the workpiece. This heat flux is generated on the surface by friction between the wheel and the workpiece. The solution of the Samara-Valencia model (4.1)–(4.3) may be presented as the sum of two terms,

$$T(t, x, y) := T^{(0)}(t, x, y) + T^{(1)}(t, x, y), \tag{4.4}$$

where

$$\begin{aligned} T^{(0)}(t, x, y) &:= -\frac{1}{4\pi k_0} \int_0^t \frac{ds}{s} \exp\left(\frac{-y^2}{4ks}\right) \\ &\quad \times \int_{-\infty}^{\infty} dx'\, d(t - s, x') \exp\left(-\frac{(x' - x - v_d s)^2}{4ks}\right), \\ T^{(1)}(t, x, y) &:= \frac{1}{4\pi} \int_0^t \frac{ds}{s} \exp\left(\frac{-y^2}{4ks}\right) \int_{-\infty}^{\infty} dx' \left(\frac{y}{2ks} - \frac{b(t - s, x')}{k_0}\right) \\ &\quad \times T(t - s, x', 0) \exp\left(-\frac{(x' - x - v_d s)^2}{4ks}\right). \end{aligned} \tag{4.5}$$

Notice that $T^{(0)}$ contains the friction function $d(t, x)$, and $T^{(1)}$ contains the temperature field on the surface and the heat transfer coefficient $b(t, x)$.

4.2. Stationary Regime in Dry Continuous Grinding

Let us consider a wrokpiece being ground with thermal diffusivity $k(m^2 s^{-1})$ and thermal conductivity $k_0(Wm^{-1}K^{-1})$. In the stationary regime, $t \to \infty$, let us define,

$$\begin{aligned} T^{(0)}(x, y) &:= \lim_{t\to\infty} T^{(0)}(t, x, y), \\ T^{(1)}(x, y) &:= \lim_{t\to\infty} T^{(1)}(t, x, y). \end{aligned} \tag{4.6}$$

For the case of dry grinding, $b(t, x) = 0$, and a constant flux $q(Wm^{-2})$ entering into the workpiece, we can rewrite (4.6) as, [2]

$$T^{(0)}(X, Y) = \tau \int_{X-\Delta}^{X} e^{-u} \; K_0\left(\sqrt{Y^2 + u^2}\right) du, \tag{4.7}$$

$$T^{(1)}(X, Y) = \frac{Y}{2\pi} \int_{-\infty}^{\infty} T(X-u, 0) e^{-u} \frac{K_1\left(\sqrt{Y^2 + u^2}\right)}{\sqrt{Y^2 + u^2}} du, \tag{4.8}$$

where we have used dimensionless variables,

$$\begin{aligned} Y &:= \frac{v_d}{2k} y, \\ X &:= \frac{v_d}{2k} x, \\ \Delta &:= \frac{v_d}{2k} \delta, \end{aligned} \tag{4.9}$$

and a normalized temperature,

$$\tau := \frac{qk}{\pi k_0 v_d}. \tag{4.10}$$

Notice that the numerical solution of the temperature field

$$T(X, Y) = T^{(0)}(X, Y) + T^{(1)}(X, Y), \tag{4.11}$$

is cumbersome since (4.8) involves the temperature field at the workpiece surface inside the integrand. We will see in the following section that this integral equation can be solved by using $I_2(x, y)$.

4.3. Application of I_2

For simplicity, let us define the functions, for $Y \geq 0$,

$$n(u, Y) := Ye^{-u} \frac{K_1\left(\sqrt{Y^2 + u^2}\right)}{\sqrt{Y^2 + u^2}},$$
$$m(u, Y) := e^{-u} K_0\left(\sqrt{Y^2 + u^2}\right). \tag{4.12}$$

so that we can rewrite (1.2), (4.7), and (4.8) into a simpler form,

$$m(X, Y) = \frac{1}{\pi} \int_{-\infty}^{\infty} m(X - u, 0) n(u, Y) du, \tag{4.13}$$

$$T^{(0)}(X, Y) = \tau \int_{X-\Delta}^{X} m(u, Y) du, \tag{4.14}$$

$$T^{(1)}(X, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(X - u, 0) n(u, Y) du. \tag{4.15}$$

Defining the Fourier transform as

$$\mathcal{F}[f(x)](\omega) := \widehat{f}(\omega) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx, \tag{4.16}$$

we may apply the convolution theorem of the Fourier transform to (4.13) and (4.15) [3], obtaining

$$\widehat{m}(\omega, Y) = 2\widehat{m}(\omega, 0)\widehat{n}(\omega, Y), \tag{4.17}$$

$$\widehat{T}^{(1)}(\omega, Y) = \widehat{T}(\omega, 0)\widehat{n}(\omega, Y). \tag{4.18}$$

Performing the derivative with respect to X in (4.14),

$$\frac{\partial T^{(0)}}{\partial X} = \tau[m(X - \Delta, Y) - m(X, Y)], \tag{4.19}$$

and applying the Fourier transform to (4.19), we eventually obtain

$$\widehat{T}^{(0)}(\omega, Y) = h(\omega)\widehat{m}(\omega, Y), \tag{4.20}$$

where we have defined

$$h(\omega) := \tau \frac{1 - e^{i\Delta\omega}}{i\omega}. \tag{4.21}$$

Multiplying by $h(\omega)$ (4.17) and taking into account (4.20), we get,

$$\widehat{T}^{(0)}(\omega, Y) = 2\widehat{T}^{(0)}(\omega, 0)\widehat{n}(\omega, Y). \tag{4.22}$$

Let us divide now (4.18) and (4.22) for $Y = 0$,

$$\frac{\widehat{T}^{(1)}(\omega, 0)}{\widehat{T}^{(0)}(\omega, 0)} = \frac{\widehat{T}(\omega, 0)}{2\widehat{T}^{(0)}(\omega, 0)}. \tag{4.23}$$

But straightforwardly from (4.11),

$$\widehat{T}(\omega, 0) = \widehat{T}^{(0)}(\omega, 0) + \widehat{T}^{(1)}(\omega, 0), \tag{4.24}$$

so that (4.23) can be rearranged, leading to,

$$\widehat{T}^{(1)}(\omega, 0) = \widehat{T}^{(0)}(\omega, 0), \tag{4.25}$$

thus,

$$\widehat{T}(\omega, 0) = 2\widehat{T}^{(0)}(\omega, 0). \tag{4.26}$$

Substituting (4.26) into (4.18) and comparing the result to (4.22),

$$\widehat{T}^{(1)}(\omega, Y) = 2\widehat{T}^{(0)}(\omega, 0)\widehat{n}(\omega, Y) = \widehat{T}^{(0)}(\omega, Y), \tag{4.27}$$

so that,

$$\widehat{T}(\omega, Y) = 2\widehat{T}^{(0)}(\omega, Y). \tag{4.28}$$

Performing now the Fourier antitransformation in (4.28),

$$T(X, Y) = 2T^{(0)}(X, Y), \tag{4.29}$$

and according to (4.7),

$$T(X, Y) = 2\tau \int_{X-\Delta}^{X} e^{-u} K_0\left(\sqrt{Y^2 + u^2}\right) du. \tag{4.30}$$

Equation (4.29) agrees with [6]. Notice that the numerical computation of $T(X, Y)$ in (4.30) is now quite more simple.

4.4. Application of I_1

According to the integral representation of the K_0 Bessel function (2.7), we can rewrite (4.30) as

$$T(X,Y) = \tau \int_{X-\Delta}^{X} e^{-u} du \int_0^\infty \exp\left(-t - \frac{Y^2 + u^2}{4t}\right) \frac{dt}{t}. \tag{4.31}$$

By Fubini's theorem, we may rearrange (4.31) as

$$T(X,Y) = \tau \int_0^\infty \exp\left(-\frac{Y^2}{4t}\right) \frac{dt}{t} \int_{X-\Delta}^{X} \exp\left[-\left(\frac{u}{2\sqrt{t}} + \sqrt{t}\right)^2\right] du. \tag{4.32}$$

The inner integral in (4.32) can be expressed in terms of the error function [4],

$$\begin{aligned} T(X,Y) = \sqrt{\pi}\tau \int_0^\infty &\exp\left(-\frac{Y^2}{4t}\right) \\ &\times \left[\operatorname{erf}\left(\frac{X}{2\sqrt{t}} + \sqrt{t}\right) - \operatorname{erf}\left(\frac{X-\Delta}{2\sqrt{t}} + \sqrt{t}\right)\right] \frac{dt}{\sqrt{t}}. \end{aligned} \tag{4.33}$$

Performing the change of variables $\sigma = 2\sqrt{t}$ we get

$$\begin{aligned} T(X,Y) = \sqrt{\pi}\tau \int_0^\infty &\exp\left(-\frac{Y^2}{\sigma^2}\right) \\ &\times \left[\operatorname{erf}\left(\frac{X}{\sigma} + \frac{\sigma}{2}\right) - \operatorname{erf}\left(\frac{X-\Delta}{\sigma} + \frac{\sigma}{2}\right)\right] d\sigma, \end{aligned} \tag{4.34}$$

so that the temperature field on the surface, $Y = 0$, is

$$T(X,0) = \sqrt{\pi}\tau \int_0^\infty \left[\operatorname{erf}\left(\frac{X}{\sigma} + \frac{\sigma}{2}\right) - \operatorname{erf}\left(\frac{X-\Delta}{\sigma} + \frac{\sigma}{2}\right)\right] d\sigma. \tag{4.35}$$

Applying now (2.18), taking $a = X, b = 1/2$, and $c = \Delta$, we may calculate (4.35),

$$T(X,0) = 2\tau\left[\mathrm{Jg}(\Delta - X) - \mathrm{Jg}(-X)\right]. \tag{4.36}$$

Appendix

A. The Function Jg(x)

In order to solve the integral (2.10), we can take advantage of the following result, known as *King's integral* [7]:

$$\int_0^x e^{\pm u} K_0(u)du = xe^{\pm x}[K_0(x) \pm K_1(x)] \mp 1. \tag{A.1}$$

Let us define the function Jg(x) as follows:

$$\mathrm{Jg}(x) := \int_0^x e^u K_0(|u|)du. \tag{A.2}$$

We can distinguish three different cases, $x > 0$, $x < 0$, and $x = 0$.

A.1. Jg(x) *for* $x > 0$

Straightforwardly from *King's integral*, (A.1), we obtain that

$$\mathrm{Jg}(x) = \int_0^x e^u K_0(u)du = xe^x[K_0(x) + K_1(x)] - 1. \tag{A.3}$$

A.2. Jg(x) *for* $x < 0$

We can perform in (A.2) the change of variables $u = -u'$, so that

$$\mathrm{Jg}(x) = \int_0^x e^u K_0(-u)du = -\int_0^{-x} e^{-u'} K_0(u')du'. \tag{A.4}$$

Now, we can apply in (A.4) the result given by *King's integral* (A.1),

$$\mathrm{Jg}(x) = xe^x[K_0(-x) - K_1(-x)] - 1. \tag{A.5}$$

A.3. Jg(x) *for* $x = 0$

We can rewrite (A.3) and (A.5) as a single expression, for $x \neq 0$,

$$\mathrm{Jg}(x) = xe^x\left[K_0(|x|) + \mathrm{sign}(x)K_1(|x|)\right] - 1, \tag{A.6}$$

where

$$\mathrm{sign}(x) = \frac{|x|}{x}, \quad x \neq 0 \tag{A.7}$$

In order to evaluate $\mathrm{Jg}(x)$ at $x = 0$, we can perform the following limit

$$\mathrm{Jg}(0) = \lim_{x\to 0} xe^{x}\left[K_0(|x|) + \mathrm{sgn}(x)K_1(|x|)\right] - 1. \tag{A.8}$$

Applying now the asymptotic representation of K_0 and K_1 [4],

$$K_0(z) \underset{z\to 0^+}{\sim} \log\left(\frac{2}{z}\right), \qquad K_1(z) \underset{z\to 0^+}{\sim} \frac{1}{z}, \tag{A.9}$$

and taking into account (A.7), we can evaluate (A.8),

$$\mathrm{Jg}(0) = \lim_{x\to 0} xe^{x}\left[\log\left(\frac{2}{|x|}\right) + \frac{|x|}{x}\frac{1}{|x|}\right] - 1 = \lim_{x\to 0} -x\log\left(\frac{|x|}{2}\right) = 0. \tag{A.10}$$

Finally, according to (A.6) and (A.10), we can express $\mathrm{Jg}(x)$ in the following terms:

$$\mathrm{Jg}(x) := \int_0^x e^{u}K_0(|u|)du = \begin{cases} xe^{x} \quad \left[K_0(|x|) + \mathrm{sign}(x)K_1(|x|)\right] - 1, & \forall x \neq 0, \\ 0, & \forall x = 0. \end{cases} \tag{A.11}$$

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References

[1] D. L. Skuratov, Yu. L. Ratis, I. A. Selezneva, J. Pérez, P. Fernández de Córdoba, and J. F. Urchueguía, “Mathematical modelling and analytical solution for workpiece temperature in grinding,” *Applied Mathematical Modelling*, vol. 31, no. 6, pp. 1039–1047, 2007.

[2] J. L. González-Santander, *Modelización matemática de la transmisión de calor en el proceso del rectificado industrial plano*, Ph.D. thesis, Universidad Politécnica de Valencia, Valencia, Spain, 2009, http://hdl.handle.net/10251/4769.

[3] I. S. Gradsthteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, NY, USA, corrected and enlarged edition, 1980.

[4] N. N. Lebedev, *Special Functions and Their Applications*, Dover, New York, NY, USA, 1972.

[5] J. B. Conway, *Functions of One Complex Variable*, vol. 11 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1978.

[6] J. L. González-Santander, J. Pérez, P. Fernández de Córdoba, and J. M. Isidro, “An analysis of the temperature field of the workpiece in dry continuous grinding,” *Journal of Engineering Mathematics*, vol. 67, no. 3, pp. 165–174, 2009.

[7] Y. L. Luke, “Some notes on integrals involving Bessel functions,” *Journal of Mathematical Physics*, vol. 29, pp. 27–30, 1950.