Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 537571, 17 pages
doi:10.1155/2010/537571

Research Article

Modeling and Pricing of Variance and Volatility Swaps for Local Semi-Markov Volatilities in Financial Engineering

Anatoliy Swishchuk[1] and Raimondo Manca[2]

[1] *Department of Mathematics and Statistics, University of Calgary, 2500 University Drive NW, Calgary, AB, Canada T2N 1N4*
[2] *Department of Mathematics, University of Rome "La Sapienza", Via del Castro, Laurenziano 9, 00161 Rome, Italy*

Correspondence should be addressed to Anatoliy Swishchuk, aswish@ucalgary.ca

Received 28 July 2010; Accepted 14 October 2010

Academic Editor: G. Rega

We consider a semi-Markov modulated security market consisting of a riskless asset or bond with constant interest rate and risky asset or stock, whose dynamics follow gemoetric Brownian motion with volatility that depends on semi-Markov process. Two cases for semi-Markov volatilities are studied: local current and local semi-Markov volatilities. Using the martingale characterization of semi-Markov processes, we find the minimal martingale measure for this incomplete market. Then we model and price variance and volatility swaps for local semi-Markov stochastic volatilities.

1. Introduction

Financial engineering is a multidisciplinary field involving financial theory, the methods of financing, using tools of mathematics, computation, and the practice of programming to achieve the desired end results. The financial engineering methodologies usually apply engineering methodologies and quantitative methods to finance. It is normally used in the securities, banking, and financial management and consulting industries, or by quantitative analysts in corporate treasury and finance departments of general manufacturing and service firms.

One of the recent and new financial products are variance and voaltility swaps, which are useful for volatility hedging and speculation. The market for variance and volatility swaps has been growing, and many investment banks and other financial institutions are now actively quoting volatility swaps on various assets: stock indexes, currencies, as well as commodities. A stock's volatility is the simplest measure of its riskiness or uncertainty. Formally, the volatility σ_R is the annualized standard deviation of the stock's returns during the period of interest, where the subscript R denotes the observed or "realized" volatility. Why trade volatility or variance? As mentioned in [1], "just as stock investors think they know something about the direction of the stock market so we may think we have insight into the level of future volatility. If we think current volatility is low, for the right price we might want to take a position that profits if volatility increases".

The Black-Scholes model for option pricing assumes that the volatility term is a constant. This assumption is not always satisfied by real-life options as the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see Hull [2]), and the assumption of constant volatility σ in financial model (such as the original Black-Scholes model) is incompatible with derivatives prices observed in the market. Stochastic volatility models are used in the field of quantitative finance and financial engineering to evaluate derivative securities, such as options, swaps. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to more accurately model derivatives.

In this paper, we show how to model and price variance and volatility swaps for two different types of stochastic volatility, local current and local semi-Markov volatilities.

The paper is organized as follows. Section 2 is devoted to the description of different types of volatilities and swaps. Martingale characterization of semi-Markov processes is considered in Section 3. Section 4 summaries relevant results from [3]. Minimal martingale measure for stock price with local semi-Markov volatility is constructed in Section 5. Section 6 contains pricing of variance swap for local semi-Markov stochastic volatility (Theorem 6.1). Section 7 is devoted to the pricing of volatility swap for local semi-Markov stochastic volatility (Theorem 7.1).

2. Volatilities and Swaps

Volatility σ is the standard deviation of the change in value of a financial instrument with a specific horizon. It is often used to quantify the risk of the instrument over that time period. The higher volatility, the riskier the security.

Historical volatility is the volatility of a financial instrument based on historical returns. It is a standard deviation (uses historical (daily, weekly, monthly, quarterly, yearly)) price data to empirically measure the volatility of a market or instrument in the past.

The annualized volatility σ is the standard deviation of the instrument's logarithmic returns in a year,

$$\sigma := \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(R_i - \overline{R}\right)^2}, \tag{2.1}$$

where $R_i := \ln(S_{t_i}/S_{t_{i-1}})$, $\overline{R} := (1/n)\sum_{i=1}^{n}\ln(S_{t_i}/S_{t_{i-1}})$, and S_{t_i} is a asset price at time t_i, $i = 1, 2, \ldots, n$.

Implied volatility is related to historical volatility; however, the two are distinct. Historical volatility is a direct measure of the movement of the underlier's price (realized volatility) over recent history. Implied volatility, in contrast, is set by the market price of the derivative contract itself, and not the underlier. Therefore, different derivative contracts on the same underlier have different implied volatilities. Most derivative markets exhibit persistent patterns of volatilities varying by strike. The pattern displays different characteristics for different markets. In some markets, those patterns form a smile curve. In others, such as equity index options markets, they form more of a skewed curve. This has motivated the name "volatility skew". For markets where the graph is downward sloping, such as for equity options, the term *"volatility skew"* is often used. For other markets, such as FX options or equity index options, where the typical graph turns up at either end, the more familiar term *"volatility smile"* is used. In practice, either the term "volatility smile" or "volatility skew" may be used to refer to the general phenomenon of volatilities varying by strike.

The models by Black and Scholes [4] (continuous-time (B, S)-security market) and Cox et al. [5] (discrete-time (B, S)-security market (binomial tree)) are unable to explain the negative skewness and leptokurticity (fat tail) commonly observed in the stock markets The famous implied-volatility smile would not exist under their assumptions.

Given the prices of call or put options across all strikes and maturities, we may deduce the volatility which produces those prices via the full Black-Scholes equation (Black and Scholes [4], Dupire [6] and Derman and Kani [7]). This function has come to be known as *local volatility*. Local volatility-function of the spot price S_t and time t: $\sigma \equiv \sigma(S_t, t)$ (see Dupire [6] formulae for local volatility).

Level-dependent volatility (e.g., CEV or Firm Model)-function of the spot price alone. To have a smile across strike price, we need σ to depend on S: $\sigma \equiv \sigma(S_t)$. In this case, the volatility and stock price changes are now perfectly correlated.

Black and Scholes (1973) made a major breakthrough by deriving pricing formulas for vanilla options written on the stock. The Black-Scholes model assumes that the volatility term is a constant. This assumption is not always satisfied by real-life options as the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see Hull [2]), and the assumption of constant volatility σ in financial model (such as the original Black-Scholes model) is incompatible with derivatives prices observed in the market. Stochastic volatility models are used in the field of quantitative finance to evaluate derivative securities, such as options, swaps. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to more accurately model derivatives.

The above issues have been addressed and studied in several ways, such as

(i) volatility is assumed to be a deterministic function of the time: $\sigma \equiv \sigma(t)$ (see Wilmott et al. [8], Merton [9]); extended the term structure of volatility to $\sigma := \sigma_t$ (deterministic function of time), with the implied volatility for an option of maturity T given by $\widehat{\sigma}_T^2 = (1/T) \int_0^T \sigma_u^2 du$,

(ii) volatility is assumed to be a function of the time and the current level of the stock price $S(t)$: $\sigma \equiv \sigma(t, S(t))$ (see Dupire [6], Hull [2]); the dynamics of the stock price satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma(t, S(t))S(t)dW_1(t), \tag{2.2}$$

where $W_1(t)$ is a standard Wiener process,

(iii) the time variation of the volatility involves an additional source of randomness, besides $W_1(t)$, represented by $W_2(t)$, and is given by

$$d\sigma(t) = a(t, \sigma(t))dt + b(t, \sigma(t))dW_2(t), \tag{2.3}$$

where $W_2(t)$ and $W_1(t)$ (the initial Wiener process that governs the price process) may be correlated (see Hull and White [10], Heston [11]),

(iv) the volatility depends on a random parameter x such as $\sigma(t) \equiv \sigma(x(t))$, where $x(t)$ is some random process (see Elliott and Swishchuk [12], Swishchuk [3, 13]),

(v) another approach is connected with stochastic volatility, namely, uncertain volatility scenario. This approach is based on the uncertain volatility model developed in Avellaneda et al. [14], where a concrete volatility surface is selected among a candidate set of volatility surfaces. This approach addresses the sensitivity question by computing an upper bound for the value of the portfolio under arbitrary candidate volatility, and this is achieved by choosing the local volatility $\sigma(t, S(t))$ among two extreme values $\sigma_{\min}$ and $\sigma_{\max}$ such that the value of the portfolio is maximized locally,

(vi) the volatility $\sigma(t, S_t)$ depends on $S_t := S(t + \theta)$ for $\theta \in [-\tau, 0]$, namely, stochastic volatility with delay (see Kazmerchuk et al. [15]).

In the approach (i), the volatility coefficient is independent of the current level of the underlying stochastic process $S(t)$. This is a deterministic volatility model, and the special case where σ is a constant reduces to the well-known Black-Scholes model that suggests changes in stock prices are lognormal distributed. But the empirical test by Bollerslev [16] seems to indicate otherwise. One explanation for this problem of a lognormal model is the possibility that the variance of $\log(S(t)/S(t-1))$ changes randomly.

In the approach (ii), several ways have been developed to derive the corresponding Black-Scholes formula: one can obtain the formula by using stochastic calculus and, in particular, the Ito's formula (see, e.g., Shiryaev [17]).

A generalized volatility coefficient of the form $\sigma(t, S(t))$ is said to be *level-dependent*. Because volatility and asset price are perfectly correlated, we have only one source of randomness given by $W_1(t)$. A time and level-dependent volatility coefficient makes the arithmetic more challenging and usually precludes the existence of a closed-form solution. However, the *arbitrage argument* based on portfolio replication and a completeness of the market remain unchanged.

The situation becomes different if the volatility is influenced by a second "nontradable" source of randomness. This is addressed in the approachs (iii), (iv), and (v) we usually obtains a *stochastic volatility model*, introduced by Hull and White [10], which is general enough to include the deterministic model as a special case.

Hobson and Rogers [18] suggested a new class of nonconstant volatility models, which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model. The volatility is nonconstant and can be regarded as an endogenous factor in the sense that it is defined in terms of the *past behaviour* of the stock price. This is done in such a way that the price and volatility form a multidimensional Markov process.

The Generalized Auto-Regression Conditional Heteroskedacity (GARCH) model (see Bollerslev [16]) is another popular model for estimating stochastic volatility. It assumes that the randomness of the variance process varies with the variance, as opposed to the square root of the variance as in the Heston model. The standard GARCH(1,1) model has the following form for the variance differential:

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \gamma\sigma_t dB_t. \tag{2.4}$$

The GARCH model has been extended via numerous variants, including the NGARCH, LGARCH, EGARCH, GJR-GARCH, and so forth.

Volatility swaps are forward contracts on future realized stock volatility, variance swaps are similar contract on variance, the square of the future volatility, both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

A stock's volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility σ_R is the annualized standard deviation of the stock's returns during the period of interest, where the subscript R denotes the observed or "realized" volatility.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility (and only to volatility).

Demeterfi et al. [1] explained the properties and the theory of both variance and volatility swaps. They derived an analytical formula for theoretical fair value in the presence of realistic volatility skews, and pointed out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap.

Javaheri et al. [19] discussed the valuation and hedging of a GARCH(1,1) stochastic volatility model. They used a general and exible PDE approach to determine the first two moments of the realized variance in a continuous or discrete context. Then they approximate the expected realized volatility via a convexity adjustment.

Brockhaus and Long [20] provided an analytical approximation for the valuation of volatility swaps and analyzed other options with volatility exposure.

Elliott and Swishchuk [12] studied option pricing formulae and pricing swaps for Markov-modulated Brownian with jumps.

In [3] we considered a semi-Markov modulated market consisting of a riskless asset or bond, B, and a risky asset or stock, S, whose dynamics depend on a semi-Markov process x. Using the martingale characterization of semi-Markov processes, we noted the incompleteness of semi-Markov modulated markets and found the minimal martingale measure. We priced variance and volatility swaps for stochastic volatilities driven by the semi-Markov processes.

In this paper, we study some extensions of the results obtained in [3] for models such as local current and local semi-Markov volatilities. We obtain the prices of variance (Theorem 6.1) and volatility swaps (Theorem 7.1) for these models.

3. Martingale Characterization of Semi-Markov Processes: Definitions and Preliminary Results

3.1. Markov Renewal and Semi-Markov Processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(X, \mathcal{X})$ be a measurable space and $Q(x, B, t) = P(x, B)G_x(t)$, $x \in X$, $B \in \mathcal{X}$, $t \in R_+$, be a semi-Markov kernel. Let us consider a $(X \times R_+, \mathcal{X} \otimes \mathcal{B}_+)$-valued stochastic process $(x_n, \tau_n; n \geq 0)$, with $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \tau_{n+1} \leq \cdots$, (see [21, 22]).

Definition 3.1. A Markov renewal process is a two-component Markov chain, $(x_n\tau_n;\ n \geq 0)$, homogeneous with respect to the second component with transition probabilities

$$P(x_{n+1} \in B, \tau_{n+1} - \tau_n \leq t \mid \mathcal{F}_n) \ = Q(x_n, B, t) = P(x, B)G_x(t) \ \text{a.s.}, \tag{3.1}$$

where $\mathcal{F}_n$ is a σ-algebra generated by (x_n, τ_n).

Define the counting process of jumps $\nu(t)$ by $\nu(t) = \sup\{n \geq 0\colon \tau_n \leq t\}$, that gives the number of jumps of the Markov renewal process in the time interval $(0, t]$.

Definition 3.2. A stochastic process $x(t)$, $t \geq 0$, defined by $x(t) = x_{\nu(t)}$ is called a semi-Markov process, associated to the Markov renewal process $(x_n\tau_n;\ n \geq 0)$.

Remark 3.3. Markov jump processes are special cases of semi-Markov processes with semi-Markov kernel $Q(x, B, t) = P(x, B)[1 - e^{-\lambda(x)t}]$.

Definition 3.4. The auxiliary process $\gamma(t) = t - \tau_{\nu(t)}$ is called the backward recurrence time-the time period since the last renewal epoch before t (or the current life in the terminology of reliability theory, or age random variable); see [22].

Remark 3.5. The current life is a Markov process with generator $Qf(t) = f'(t) + \lambda(t)[f(0) - f(t)]$, where $\lambda(t) = -\overline{F}'(t)/\overline{F}(t)$, $\overline{F}(t) = 1 - F(t)$, $\text{Domain}(Q) = C^1(R)$.

Remark 3.6. If we expand the state space of the semi-Markov process to include a component that records the amount of time already spent in the current state, then this additional information in the state description makes the semi-Markov process Markovian. For example, the following process $(x(t), \gamma(t))$ is a Markov process; see [21, 22].

Definition 3.7. The compensating operator Q of the Markov renewal process is defined by the following relation:

$$Qf(x_0, \tau_0) = q(x_0)E\left[f(x_1, \tau_1) - f(x_0, \tau_0)|\mathcal{F}|_0\right], \tag{3.2}$$

where $q(x) = 1/m(x)$, $m(x) = \int_0^{+\infty} \overline{G}_x(t)dt$, $\mathcal{F}_t = \sigma\{x(s), \tau_{\nu(s)};\ 0 \leq s \leq t\}$.

Lemma 3.8. *The compensating operator of the Markov renewal process can be defined by the relation*

$$Qf(x, t) = q(x)\left[\int_0^{+\infty} G_x(ds)\int_X P(x, dy)f(y, t + s) - f(x, t)\right]. \tag{3.3}$$

This statement follows directly from Definition 3.7.

Lemma 3.9. *Let (x_n, τ_n) be the Markov renewal process, Q be the compensating operator,*

$$m_n := f(x_n, \tau_n) - \sum_{i=1}^{n}(\tau_i - \tau_{i-1})Qf(x_{i-1}, \tau_{i-1}), \tag{3.4}$$

and $\mathcal{F}_n = \sigma\{x_k, \tau_k;\ k \leq n\}$. Then the process m_n is a $\mathcal{F}_n$-martingale for any function f such that $E_x|f(x_1, \tau_1)| < +\infty$.

Let $x(t)$ be a Markov process with infinitesimal generator Q.

Theorem 3.10. *The process*

$$m(t) := f(x(t)) - f(x) - \int_0^t Qf(x(s))ds \tag{3.5}$$

is an $\mathcal{F}_t^x = \sigma\{x(s);\ s \leq t\}$*-martingale (see [12, 22]).*

This statement follows from the Dynkin formula (see [23]).

Theorem 3.11. *The quadratic variation of the martingale* $m(t)$ *is the process*

$$\langle m(t)\rangle = \int_0^t \Big[Qf^2(x(s)) - 2f(x(s))Qf(x(s))\Big]ds. \tag{3.6}$$

(See [12]).

3.2. Jump Measure for a Semi-Markov Process

The *jump measure* for $x(t)$ is defined in the following way (see [22]):

$$\mu([0,t] \times A) = \sum_{n\geq 0} \mathbf{1}(x_n \in A,\ \tau_n \leq t), \quad A \in \mathcal{X},\ t \geq 0. \tag{3.7}$$

It is known (see [12]) that the predictable projection (compensator) for μ has the following form:

$$\nu(dt, dy) = \sum_{n\geq 0} \mathbf{1}(\tau_n < t \leq \tau_{n+1}) \frac{P(x_n, dy) g_{x_n}(t)}{\overline{G}_{x_n}(t)} dt, \tag{3.8}$$

where $\overline{G}_x(t) = 1 - G_x(t)$, $g_x(t) = dG_x(t)/dt$, for all $x \in X$.

3.3. Martingale Characterization of Semi-Markov Processes

Let x_t be a semi-Markov process with semi-Markov kernel $Q(x, B, t) = P(x, B)G_x(t)$ and $\gamma(t)$ be the current life.

Lemma 3.12. *The process*

$$m_t^f := f(x_t, \gamma(t)) - \int_0^t Qf(x_s, \gamma(s))ds \tag{3.9}$$

is a martingale with respect to the filtration $\mathcal{F}_t^* := \sigma\{x_s, \tau_{\nu(s)};\ 0 \le s \le t\}$, *where* Q *is the infinitesimal operator of Markov process* $(x_t, \gamma(t))$:

$$Qf(x,t) = \frac{df(x,t)}{dt} + \frac{g_x(t)}{\overline{G}_x(t)} \int_X P(x,dy)\left[f(y,t) - f(x,t)\right]. \tag{3.10}$$

This statement follows from Theorem 3.10 and the fact that $(x(t), \gamma(t))$ is a Markov process (see Remark 3.6).

Let us calculate the quadratic variation of the martingale m_t^f.

Lemma 3.13. *Let* Q *in* (3.10) *be such that if* $f \in Domain(Q)$, *then* $f^2 \in Domain(Q)$. *The quadratic variation* $\langle m_t^f \rangle$ *of the martingale* m_t^f *in* (3.9) *is equal to*

$$\left\langle m_t^f \right\rangle = \int_0^t \left[Qf^2(x_s, \gamma(s)) - 2f(x_s, \gamma(s))Qf(x_s, \gamma(s))\right] ds. \tag{3.11}$$

This statement follows from Theorem 3.11 and the fact that $(x(t), \gamma(t))$ is a Markov process (see Remark 3.6).

Lemma 3.14. *Let the following condition (Novikov's condition) is satisfied*

$$E^P \exp\left\{\frac{1}{2}\int_0^t \left[Qf^2(x_s, \gamma(s)) - 2f(x_s, \gamma(s))Qf(x_s, \gamma(s))\right] ds\right\} < +\infty, \quad \forall f^2 \in Domain(Q). \tag{3.12}$$

Then $E^P e_t^f = 1$, *where*

$$e_t^f := e^{m_t^f - (1/2)\langle m_t^f \rangle}, \tag{3.13}$$

and e_t^f *in* (3.13) *is a* P*-martingale (Doléans-Dade martingale). Here,* $\langle m_t^f \rangle$ *is defined in* (3.11) *and* Q *in* (3.10).

4. Pricing of Variance and Volatility Swaps for Semi-Markov Volatilities

In [3] we stated two main results on pricing of variance and volatility swaps for semi-Markov volatility. We considered the following (B, S)-security market (B stands for risk-free security "Bond" and S stands for risky security "Stock"). Assume the stock price S_t satisfies the stochastic differential eqaution

$$dS_t = S_t(\mu dt + \sigma(x_t, \gamma(t))dw_t), \tag{4.1}$$

where $\mu \in R$ is the appreciation rate and $\sigma(x_t, \gamma(t))$ is the semi-Markov volatility, and the bond price $B(t)$ is

$$B(t) = B_0 e^{rt}, \tag{4.2}$$

where $r > 0$ is the risk-free rate of return (interest rate). Then the folllowing two theorems are valid.

Theorem 4.1 (Price of Variance Swap [3]). *The value of a variance swap for semi-Markov stochastic volatility $\sigma(x_t, \gamma(t))$ equals to*

$$P_{\text{var}}(x) = e^{-rT}\left(\frac{1}{T}\int_0^T e^{tQ}\sigma^2(x,0)dt - K_{\text{var}}\right). \tag{4.3}$$

Theorem 4.2 (Price of Volatility Swap [3]). *The value of a volatility swap for semi-Markov stochastic volatility $\sigma(x_t, \gamma(t))$ is*

$$\begin{aligned} &P_{\text{vol}}(x) \\ &\quad\approx e^{-rT}\Bigg\{\sqrt{\frac{1}{T}\int_0^T e^{tQ}\sigma^2(x,0)dt} \\ &\qquad - \frac{\left[(1/T^2)\int_0^T\int_0^T E\sigma^2(x_t,\gamma(t))\sigma^2(x_s,\gamma(s))dt\,ds - \left((1/T)\int_0^T e^{tQ}\sigma(x,0)dt\right)^2\right]}{\left(8\left(\left(\frac{1}{T}\right)\int_0^T e^{tQ}\sigma^2(x,0)dt\right)^{3/2}\right) - K_{\text{vol}}}\Bigg\}. \end{aligned} \tag{4.4}$$

Here Q is defined as

$$Qf(x,t) = \frac{df(x,t)}{dt} + \frac{g_x(t)}{\overline{G}_x(t)}\int_X P(x,dy)\left[f(y,t) - f(x,t)\right]. \tag{4.5}$$

In this paper, we generalize these results for local semi-Markov volatilities.

5. Minimal Martingale Measure for Stock Price with Local Semi-Markov Volatility

We suppose that the stock price S_t satisfies the following stochastic differential equation

$$dS_t = S_t\left(r\,dt + \sigma_{\text{loc}}(S_t, x_t, \gamma(t))dw_t\right) \tag{5.1}$$

with the volatility $\sigma := \sigma_{\mathrm{loc}}(S_t, x_t, \gamma(t))$ depending on the process x_t, which is independent on standard Wiener process w_t, stock price S_t, and the current life $\gamma(t) = t - \tau_{\nu(t)}$. We call the volatility $\sigma(S_t, x_t, \gamma(t))$ the *local current semi-Markov volatility*.

Remark 5.1. We note that process (S_t, x_t, γ_t) is a Markov process on (R_+, X, R_+) with infinitesimal operator

$$
\begin{aligned}
Qf(s,x,t) = {} & \frac{\partial f(s,x,t)}{\partial t} + \frac{g_x(t)}{\overline{G}_x(t)} \int_X P(x,dy)\left[f(s,y,t) - f(s,x,t)\right] \\
& + rS\frac{\partial f(s,x,t)}{\partial s} + \frac{1}{2}\sigma^2(s,x,0)S^2\frac{\partial^2 f(s,x,t)}{\partial s^2}.
\end{aligned}
\tag{5.2}
$$

Remark 5.2 (Local Semi-Markov Volatility). Let's suppose that the stock price S_t satisfies the following stochastic differential equation

$$
dS_t = S_t(rdt + \sigma(S_t, x_t, t)dw_t) \tag{5.3}
$$

with the volatility $\sigma := \sigma(S_t, x_t, t)$ depending on the process x_t, which is independent on standard Wiener process w_t, stock price S_t and current time t.

Suppose also that $\sigma(S, x, t)$ is differentiable function by t with bounded derivative.

Then we can reduce the problem of calculating of swaps with local semi-Markov volatility to the previous one with local current semi-Markov volatility by the following expansion:

$$
\sigma(S_t, x_t, t) = \sigma(S_t, x_t, \gamma(t)) + \tau_{\nu(t)}\frac{d\sigma(S_t, x_t, t)}{dt} + o(\tau_{\nu(t)}). \tag{5.4}
$$

The error of estimation will be

$$
E\left[\sigma(S_t, x_t, t) - \sigma(S_t, x_t, \gamma(t))\right]^2 \leq E\left[\tau_{\nu(t)}\right]^2 \times C, \tag{5.5}
$$

where $C = \max_{0 \leq t \leq T} E[d\sigma(S_t, x_t, t)/dt]^2$.

In this way, it is enough to consider the case with the local current stochastic volatility in (5.1).

5.1. Minimal Martingale Measure

We consider the following (B, S) security market. Let the stock price S_t satisfies the following equation:

$$
dS_t = S_t(\mu dt + \sigma(S_t, x_t, \gamma(t))dw_t), \tag{5.6}
$$

where $\mu \in R$ is the appreciation rate and $\sigma(S_t, x_t, \gamma(t))$ is the local current semi-Markov volatility, and the bond price $B(t)$ is

$$B(t) = B_0 e^{rt}, \tag{5.7}$$

where $r > 0$ is the risk-free rate of return (interest rate).

As long as we have two sources of randomness, Brownian motion $w(t)$ and semi-Markov process x_t, the above (B, S)-security market (4.1)-(4.2) is incomplete (see [12, Theorem 1]) so there are many risk-neutral (or martingale) measures. We are going to construct the minimal martingale measure (see [12]). With respect to this construction (see [12, Lemma 4]), the minimal martingale measure P^* is as follows.

Using Girsanov's Theorem (see [17]), we obtain the following result concerning the minimal martingale measure in the above market (5.6)-(5.7).

Lemma 5.3. *Under the assumption $\int_0^T((r-\mu)/\sigma(S_t, x_t, \gamma(t)))^2 dt < +\infty$, a.s., the following holds:*

(1) *There is a probability measure P^* equivalent to P such that*

$$\frac{dP^*}{dP} = \exp\left\{\int_0^T \frac{r-\mu}{\sigma(S_t, x_t, \gamma(t))} dw(t) - \frac{1}{2}\int_0^T \left(\frac{r-\mu}{\sigma(S_t, x_t, \gamma(t))}\right)^2 dt\right\} \tag{5.8}$$

is its Radon-Nikodym density.

(2) *The discounted stock price $Z(t) = S_t/B(t)$ is a positive local martingale with respect to P^* and is given by*

$$Z(t) = \exp\left\{-\frac{1}{2}\int_0^t \sigma^2(S_t, x_s, \gamma(s)) ds + \int_0^t \sigma(S_t, x_s, \gamma(s)) dw^*(s)\right\}, \tag{5.9}$$

where $w^*(t) = \int_0^t((r-\mu)/\sigma(S_t, x_s, \gamma(s)))ds + w(t)$ is a standard Brownian motion with respect to P^*.

Remark 5.4. Measure P^* is called the minimal martingale measure.

Remark 5.5. We note that under the P^* measure the stock price S_t satisfies the following equation:

$$dS_t = S_t(r dt + \sigma(S_t, x_t, \gamma(t)) dw^*(t)), \tag{5.10}$$

and discounted process $Z(t)$ has the presentation

$$dZ(t) = Z(t)\sigma(S_t, x_s, \gamma(s)) dw^*(t). \tag{5.11}$$

Remark 5.6. A sufficient condition (Novikov's condition) for the right-hand side of (5.8) to be a martingale is

$$E\exp\left\{\frac{1}{2}\int_0^T\left(\frac{r-\mu}{\sigma(S_t,x_t,\gamma(t))}\right)^2dt\right\}<+\infty. \tag{5.12}$$

6. Pricing of Variance Swaps for Local Semi-Markov Stochastic Volatility

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$N\left(\sigma_R^2(x)-K_{\text{var}}\right), \tag{6.1}$$

where $\sigma_R^2(x)$ is the realized stock variance (quoted in annual terms) over the life of the contract,

$$\sigma_R^2(x):=\frac{1}{T}\int_0^T\sigma^2(S_t,x_s,\gamma(s))ds, \tag{6.2}$$

K_{var} is the delivery price for variance, and N is the notional amount of the swap in dollars per annualized volatility point squared. The holder of a variance swap receives, at expiry, N dollars for every point by which the stock's realized variance $\sigma_R^2(x)$ has exceeded the variance delivery price K_{var}. See [1, 24].

Pricing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract F on future realized variance with strike price K_{var} is the expected present value of the future payoff in the risk-neutral world:

$$P_{\text{var}}(x)=E\left\{e^{-rT}\left(\sigma_R^2(x)-K_{\text{var}}\right)\right\}, \tag{6.3}$$

where r is the risk-free discount rate corresponding to the expiration date T, E denotes the expectation with respect to the minimal martingale measure P (from now on we use simplier notation P instead of P^*), and $\sigma_R^2(x)$ is defined in (6.2).

Let us show how we can calculate $EV(x)$, where $V(x):=\sigma_R^2(x)$. For that we need to calculate $E\sigma^2(S_t,x_t,\gamma(t))$.

We note (see Section 3.3, Lemma 3.12) that for $\sigma(x)\in\text{Domain}(Q)$ the following process

$$m_t^\sigma:=\sigma(S_t,x_t,\gamma(t))-\sigma(x,0)-\int_0^t Q\sigma(S_s,x_s,\gamma(s))ds \tag{6.4}$$

is a zero-mean martingale with respect to $F_t:=\sigma\{w_s,x_s,\tau_{\nu(s)};\ 0\le s\le t\}$ and Q is the infinitesimal operator defined in (5.2).

The quadratic variation of the martingale m_t^f by Lemma 3.13 is equal to

$$\langle m_t^\sigma \rangle = \int_0^t \left[Q\sigma^2(S_s, x_s, \gamma(t)) - 2\sigma(S_s, x_s, \gamma(t)) Q\sigma(S_s, x_s, \gamma(t)) \right] ds, \\ \sigma^2(s, x, t) \in \text{Domain}(Q). \tag{6.5}$$

Since $\sigma(S_s, x_s, \gamma(s))$ satisfies the following stochastic differential equation:

$$d\sigma(S_t, x_t, \gamma(t)) = Q\sigma(S_t, x_t, \gamma(t))dt + dm_t^\sigma \tag{6.6}$$

then, we obtain from Itô formula (see [22]) that $\sigma^2(S_t, x_t, \gamma(t))$ satisfies the following stochastic differential equation:

$$d\sigma^2(S_t, x_t, \gamma(t)) = 2\sigma(S_t, x_t, \gamma(t))dm_t^\sigma + 2\sigma(S_t, x_t, \gamma(t))Q\sigma(S_t, x_t, \gamma(t))dt + d\langle m_t^\sigma \rangle, \tag{6.7}$$

where $\langle m_t^\sigma \rangle$ is defined in (6.5).

Substituting (6.3) into (6.5) and taking the expectation of both parts of (6.7), we obtain that

$$E\sigma^2(S_t, x_t, \gamma(t)) = \sigma^2(s, x, 0) + \int_0^t QE\sigma^2(S_s, x_s, \gamma(s))ds, \quad s = S_0, \tag{6.8}$$

and solving this equation we have

$$E\sigma^2(S_t, x_t, \gamma(t)) = e^{tQ}\sigma^2(s, x, 0). \tag{6.9}$$

Finally, we obtain that

$$EV(x) = \frac{1}{T}\int_0^T e^{tQ}\sigma^2(s, x, 0)dt. \tag{6.10}$$

In this way, we have obtained the following result (see (6.3)–(6.10)).

Theorem 6.1. *The value of a variance swap for local current semi-Markov stochastic volatility $\sigma(S_t, x_t, \gamma(t))$ is*

$$P_{\text{var}}(s, x) = e^{-rT}\left(\frac{1}{T}\int_0^T e^{tQ}\sigma^2(s, x, 0)dt - K_{\text{var}}\right), \tag{6.11}$$

where Q is defined in (5.2).

7. Pricing of Volatility Swaps for Local Semi-Markov Volatility

Volatility swaps are forward contracts on future realized stock volatility.

A stock's volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility $\sigma_R(S)$ is the annualized standard deviation of the stock's returns during the period of interest, where the subscript R denotes the observed or "realized" volatility for the stock S.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility (and only to volatility) (see [1, 19, 20, 24, 25]).

A stock *volatility swap* is a forward contract on the annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R(S) - K_{\text{vol}}), \tag{7.1}$$

where $\sigma_R(S)$ is the realized stock volatility (quoted in annual terms) over the life of contract,

$$\sigma_R(S) := \sqrt{\frac{1}{T}\int_0^T \sigma_s^2 ds}, \tag{7.2}$$

σ_t is a stochastic stock volatility, K_{vol} is the annualized volatility delivery price, and N is the notional amount of the swap in dollar per annualized volatility point. The holder of a volatility swap receives, at expiry, N dollars for every point by which the stock's realized volatility σ_R has exceeded the volatility delivery price K_{vol}. The holder is swapping a fixed volatility K_{vol} for the actual (floating) future volatility σ_R. We note that usually $N = \alpha I$, where α is a converting parameter such as 1 per volatility-square, and I is a long-short index (+1 for long and −1 for short).

Pricing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract F on future realized variance with strike price K_{var} is the expected present value of the future payoff in the risk-neutral world:

$$P_{\text{vol}}(x) = E\left\{e^{-rT}(\sigma_R(x) - K_{\text{vol}})\right\}, \tag{7.3}$$

where r is the risk-free discount rate corresponding to the expiration date T, E denotes the expectation with respect to the minimal martingale measure P (from now on we use simplier notation P instead of P^*) and $\sigma_R(x)$ is defined in (7.3).

Thus, for calculating variance swaps we need to know only $E\{\sigma_R^2(S)\}$, namely, mean value of the underlying variance.

To calculate volatility swaps we need more. From [20] we have the approximation (which uses the second-order Taylor expansion for function $\sqrt{x}$) (see also [19, page 16]):

$$E\left\{\sqrt{\sigma_R^2(S)}\right\} \approx \sqrt{E\{V\}} - \frac{\operatorname{Var}\{V\}}{8E\{V\}^{3/2}}, \tag{7.4}$$

where $V := \sigma_R^2(S)$ and $\operatorname{Var}\{V\}/8E\{V\}^{3/2}$ is the convexity adjustment.

Thus, to calculate volatility swaps we need both $E\{V\}$ and $\operatorname{Var}\{V\}$.

7.1. Pricing of Volatility Swap

From Brockhaus and Long approximation [20] we have (see also [19, page 16]):

$$E\left\{\sqrt{\sigma_R^2(S)}\right\} \approx \sqrt{E\{V\}} - \frac{\operatorname{Var}\{V\}}{8E\{V\}^{3/2}}, \tag{7.5}$$

where $V := \sigma_R^2(S)$ and $\operatorname{Var}\{V\}/8E\{V\}^{3/2}$ is the convexity adjustment.

As we can see from (7.5), to calculate volatility swaps we need both $E\{V\}$ and $\operatorname{Var}\{V\}$.

We have already calculated $E\{\sigma_R^2(S)\} = E[V]$ (see (6.10)). Let us calculate $\operatorname{Var}(V) := E[V]^2 - (E[V])^2$. In this way, we need $E[V]^2 = E\sigma_R^4(S)$. Taking into account the expression for $V = \sigma_R^2(S)$ we have:

$$E[V]^2 = \frac{1}{T^2}\iint_0^T E\left[\sigma^2(S_t, x_t, \gamma(t))\sigma^2(S_s, x_s, \gamma(s))\,dt\,ds\right]. \tag{7.6}$$

In this way, the variance of V, $\operatorname{Var}[V]$, is

$$\begin{aligned}\operatorname{Var}[V] &= E\left[V^2\right] - (E[V])^2 \\ &= \frac{1}{T^2}\iint_0^T E\sigma^2(S_t, x_t, \gamma(t))\sigma^2(S_s, x_s, \gamma(s))\,dt\,ds - \left(\frac{1}{T}\int_0^T e^{tQ}\sigma^2(s, x, 0)\,dt\right)^2.\end{aligned} \tag{7.7}$$

Taking into account (7.3)–(7.7), we obtain:

$$\begin{aligned}&P_{\text{vol}}(s, x) \\ &= e^{-rT}\left[E\sigma_R(S) - K_{\text{vol}}\right] \\ &= e^{-rT}\left[E\sqrt{\frac{1}{T}\int_0^T \sigma^2(S_s, x_s, \gamma(s))\,dt} - K_{\text{vol}}\right] \\ &\approx e^{-rT}\left[\sqrt{EV} - \frac{\operatorname{Var}(V)}{8(EV)^{3/2}} - K_{\text{vol}}\right] \\ &= e^{-rT}\left\{\sqrt{\frac{1}{T}\int_0^T e^{tQ}\sigma^2(s, x, 0)\,dt}\right. \\ &\qquad \left. - \frac{\left[(1/T^2)\int_0^T\int_0^T E\sigma^2(S_t, x_t, \gamma(t))\sigma^2(S_s, x_s, \gamma(s))\,dt\,ds - \left((1/T)\int_0^T e^{tQ}\sigma^2(s, x, 0)\,dt\right)^2\right]}{\left(8\left((1/T)\int_0^T e^{tQ}\sigma^2(s, x, 0)\,dt\right)^{3/2}\right) - K_{\text{vol}}}\right\}.\end{aligned} \tag{7.8}$$

Summarizing (7.3)–(7.8), we have the following.

Theorem 7.1. *The value of volatility swap for local current semi-Markov stochastic volatility* $\sigma(S_t, x_t, \gamma(t))$ *is*

$$
\begin{aligned}
&P_{\mathrm{vol}}(s,x) \\
&\quad\approx e^{-rT}\left\{\sqrt{\frac{1}{T}\int_0^T e^{tQ}\sigma^2(s,x,0)dt}\right. \\
&\qquad\left.-\frac{\left[(1/T^2)\int_0^T\int_0^T E\sigma^2(S_t,x_t,\gamma(t))\sigma^2(S_s,x_s,\gamma(s))dtds-\left((1/T)\int_0^T e^{tQ}\sigma(x,0)dt\right)^2\right]}{\left(8\left((1/T)\int_0^T e^{tQ}\sigma^2(s,x,0)dt\right)^{3/2}\right)-K_{\mathrm{vol}}}\right\},
\end{aligned}
\tag{7.9}
$$

where Q *is defined in* (5.2).

Acknowledgments

The first author would like to thank the Department of Mathematics, Faculty of Economics, University of Rome 1 “La Sapienza”, Rome, Italy, very much for their hospitality and friendly environment during his June 2009 visit. The research of The first author is supported by NSERC.

References

[1] K. Demeterfi, E. Derman, M. Kamal, and J. Zou, “A guide to volatility and variance swaps,” *The Journal of Derivatives*, vol. 6, no. 4, pp. 9–32, 1999.

[2] J. Hull, *Options, Futures and Other Derivatives*, Prentice Hall, Upper Saddle River, NJ, USA, 4th edition, 2000.

[3] A. Swishchuk, “Pricing of variance and volatility swaps with semi-Markov volatilities,” *Canadian Applied Mathematics Quarterly*. Accepted.

[4] F. Black and M. Scholes, “The pricing of options and corporate liabilities,” *Journal of Political Economy*, vol. 81, pp. 637–654, 1973.

[5] J. C. Cox, S. A. Ross, and M. Rubinstein, “Option pricing: a simplified approach,” *Journal of Financial Economics*, vol. 7, no. 3, pp. 229–263, 1979.

[6] B. Dupire, “Pricing with a smile,” *Risk*, vol. 7, no. 1, pp. 18–20, 1994.

[7] E. Derman and I. Kani, “Riding on a smile,” *Risk*, vol. 7, no. 2, pp. 32–39, 1994.

[8] P. Wilmott, S. Howison, and J. Dewynne, *Option Pricing: Mathematical Models and Computations*, Oxford Financial Press, Oxford, UK, 1995.

[9] R. C. Merton, “Theory of rational option pricing,” *The Rand Journal of Economics*, vol. 4, pp. 141–183, 1973.

[10] J. Hull and A. White, “The pricing of options on assets with stochastic volatilities,” *The Journal of Finance*, vol. 42, pp. 281–300, 1987.

[11] S. Heston, “A closed-form solution for options with stochastic volatility with applications to bond and currency options,” *Review of Financial Studies*, vol. 6, pp. 327–343, 1993.

[12] R. J. Elliott and A. V. Swishchuk, “Pricing options and variance swaps in Markov-modulated Brownian markets,” in *Hidden Markov Models in Finance*, R. Mamon and R. Elliott, Eds., vol. 104 of *Internat. Ser. Oper. Res. Management Sci.*, pp. 45–68, Springer, New York, NY, USA, 2007.

[13] A. Swishchuk, *Random Evolutions and Their Applications. New Trends*, vol. 504 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[14] M. Avellaneda, A. Levy, and A. Paras, "Pricing and hedging derivative securities in markets with uncertain volatility," *Applied Mathematical Finance*, vol. 2, pp. 73–88, 1995.

[15] Y. Kazmerchuk, A. Swishchuk, and J. Wu, "A continuous-time Garch model for stochastic volatility with delay," *Canadian Applied Mathematics Quarterly*, vol. 13, no. 2, pp. 123–149, 2005.

[16] T. Bollerslev, "Generalized autoregressive conditional heteroskedasticity," *Journal of Econometrics*, vol. 31, no. 3, pp. 307–327, 1986.

[17] A. N. Shiryaev, *Essentials of Stochastic Finance: Facts, Models, Theory*, vol. 3 of *Advanced Series on Statistical Science & Applied Probability*, World Scientific, River Edge, NJ, USA, 1999.

[18] David G. Hobson and L. C. G. Rogers, "Complete models with stochastic volatility," *Mathematical Finance*, vol. 8, no. 1, pp. 27–48, 1998.

[19] A. Javaheri, P. Wilmott, and E. Haug, "GARCH and volatility swaps," *Quantitative Finance*, vol. 4, no. 5, pp. 589–595, 2004.

[20] O. Brockhaus and D. Long, "Volatility swaps made simple," *Risk*, vol. 2, no. 1, pp. 92–96, 2000.

[21] J. Janssen, R. Manca, and E. Volpe, *Mathematical Finance: Deterministic and Stochastic Models*, Wiley, New York, NY, USA, 2008.

[22] V. Korolyuk and A. Swishchuk, *Semi-Markov Random Evolutions*, vol. 308 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1995.

[23] E. B. Dynkin, *Markov Processes. Vols. I, II*, Die Grundlehren der Mathematischen Wissenschaften, Bände 121-122, Academic Press Publishers, New York, NY, USA, 1965.

[24] P. Carr and D. Madan, "Towards a Theory of Volatility Trading," in *Volatility: New Estimation Techniques for Pricing Derivatives*, pp. 417–427, Risk Book Publications, London, UK, 1998.

[25] G. D'Amico, J. Janssen, and R. Manca, "Valuing credit default swap in a non-homogeneous semi-Markovian rating based model," *Computational Economics*, vol. 29, no. 2, pp. 119–138, 2007.