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Research Article

A Leontief-Type Input-Output Inclusion

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A Leontief-type input-output inclusion problem based on a set-valued consuming map is studied. By applying nonlinear analysis approach, in particular using the surjection and continuity technique with respect to set-valued maps, solvability and stability results with and without continuity assumption concerning this inclusion are obtained.

1. Introduction

In this paper, we study the solvability and stability of the following input-output inclusion:

$$\begin{gathered} x \in X \subset R_+^n, \\ c \in (I - A)x = x - Ax, \end{gathered} \tag{1.1}$$

where R_+^n is the set of nonnegative vectors of the n-dimensional Euclidean space R^n,

$$\begin{aligned} &X \subset R_+^n \text{ is a nonempty convex compact subset}, c \in R_+^n \text{ is a net output vector}, \\ &I \text{ is the identity map from } R^n \text{ to itself, and } A : X \longrightarrow 2^{R_+^n} \text{ is a set-valued map} \\ &\text{from } X \text{ to } R_+^n \text{ with nonempty convex compact values (i.e., for each } x \in X, \\ &Ax \text{ is a nonempty and convex compact subset of } R_+^n). \end{aligned} \tag{1.2}$$

The present study is essentially a continuation of the investigation initiated in [1–5] where the classical Leontief input-output model was briefly reviewed, several generalized Leontief input-output models were introduced, numerous key references were cited, and some arguments about the assumptions on X and A were made. For the necessary background material and preliminaries, the reader is referred to [1–5]. Here we will make use of the Rogalski-Cornet Theorem in [6] and the Rogalski-Cornet-type Theorem proved in [5] to prove several solvability and stability theorems with and without continuous conditions concerning A for model (1.1). Obviously, for some $c \in R^n_+$, inclusion (1.1) may not have solutions. Specifically, if c makes (1.1) solvable, then

$$c \in C(A) \mathrel{\hat{=}} R^n_+ \bigcap (I - A)X, \quad \text{where } (I - A)X = \bigcup_{x \in X} (x - Ax). \tag{1.3}$$

While $C(A)$ gives us an expression for all possible c for which (1.1) has solutions, it is required that all the information regarding A is available. It is our intention in this paper to discover some conditions under which (1.1) has solutions for the situation that the information of A is only available near the boundary of X. We also provide a stability analysis for the solution set in terms of closeness, upper semicontinuity and, upper hemicontinuity of certain related set-valued maps.

The paper is organized as follows. In the rest of this section, we review some necessary concepts and several useful results, which are used throughout this paper. In Section 2, we study (1.1) under the assumption that A is upper semicontinuous. In Section 3, we recall a Rogalski-Cornet-type theorem appearing in [5], and use it to obtain three solvability and stability results. We give our concluding remarks in Section 4.

In the sequel, we use several classes of maps, including upper and lower semicontinuous (in short, u.s.c. and l.s.c.), upper hemicontinuous (in short, u.h.c.), continuous, and closed set-valued maps between Hausdorff topological (or Hausdorff locally convex) spaces, whose definitions and some other related concepts are given below and can also be found in [6–9].

Definition 1.1. Let U and V be two Hausdorff topological spaces and $F : U \to 2^V$ a set-valued map from U to V. The domain of F is the set $\{x \in U : Fx \neq \emptyset\}$ denoted by $\operatorname{dom} F$, and the graph of F is the set $\{(x, y) \in U \times V : x \in U,\ y \in Fx\}$ denoted by $\operatorname{graph} F$.

(1) We say that F is strict if $\operatorname{dom} F = U$, and F is closed if $\operatorname{graph} F$ is closed in $U \times V$.

(2) We say that F is u.s.c. at $x^0 \in U$ if for any neighborhood $N(Fx^0)$ of Fx^0, there exists a neighborhood $N(x^0)$ of x^0 such that $F(N(x^0)) \subseteq N(Fx^0)$. F is said to be u.s.c. if F is u.s.c. at every point $x \in U$.

(3) We say that F is l.s.c. at $x^0 \in U$ if for any $y^0 \in Fx^0$ and any neighborhood $N(y^0)$ of y^0, there exists a neighborhood $N(x^0)$ of x^0 such that $Fx \bigcap N(y_0) \neq \emptyset$ for any $x \in N(x^0)$. F is said to be l.s.c. if F is l.s.c. at every point $x \in U$.

(4) If V is a Hausdorff locally convex vector space, V^* is its dual and $\langle \cdot, \cdot \rangle$ is the duality paring on $V^* \times V$. We say that F is u.h.c. at $x^0 \in \operatorname{dom} F$ if for any $p \in V^*$, the function $x \mapsto \sigma^{\#}(Fx, p) = \sup_{y \in Fx} \langle p, y \rangle$ is upper semicontinuous (in short, u.s.c.) at x^0. F is said to be u.h.c. if it is u.h.c. at every point of $\operatorname{dom} F$.

We now recall a number of auxiliary results that will be needed in proving our main theorems. They are stated below as lemmas.

Lemma 1.2 (see [8]). *Let F be an u.s.c. set-valued map from a Hausdorff topological space U to a Hausdorff topological space V with closed values. Then F is closed.*

Lemma 1.3 (see [8]). *Let F be a closed set-valued map from a Hausdorff topological space U to a compact Hausdorff topological space V. Then F is u.s.c..*

Lemma 1.4 (see [8]). *Suppose that U is a Hausdorff topological space, V a Hausdorff locally convex vector space equipped with the weak topology $\sigma(V,V^*)$, and $F : U \to 2^{(V,\sigma(V,V^*))}$ a set-valued map that is u.s.c. at $x_0 \in U$. Then F is u.h.c. at $x_0 \in U$.*

Remark 1.5. If $V = R^n$, then the weak topology $\sigma(R^n, R^{n*})$ on R^n coincides with the norm topology.

Lemma 1.6 (see [8]). *Suppose that U is a complete metric space, and V a compact metric space. If $F : U \to 2^V$ is a closed and strict set-valued map, then the subset of points at which F is continuous is residual, that is, the interior of the discontinuous point set of F is empty.*

In order to use the Rogalski-Cornet Theorem and Rogalski-Cornet type Theorem to discuss the solvability and stability of (1.1), we need some further concepts.

Let U be a Hausdorff locally convex vector space (U^* its dual, $\langle\cdot,\cdot\rangle$ the duality paring on $\langle U^*, U\rangle$), X a subset of U, $\operatorname{int}X$ the interior of X, $\partial X = X \setminus \operatorname{int} X$ the boundary of X, and $S : X \to 2^U$ a set-valued map from X to U. Let $p \in U^*$. The normal cone $N_X(x)$ to X at $x \in X$, the supporting set $\partial(X,p)$ of X, and the upper and lower supporting functions $x \mapsto \sigma^\#(Sx,p)$ and $x \mapsto \sigma^\flat(Sx,p)$ on X are defined by

$$\begin{gathered}
N_X(x) = \left\{p \in U^* : \langle p,x\rangle = \sigma^\#(X,p) \mathrel{\widehat{=}} \sup_{y\in X}\langle p,y\rangle\right\},\\
\partial(X,p) = \left\{x \in X : \langle p,x\rangle = \sigma^\flat(X,p) \mathrel{\widehat{=}} \inf_{y\in X}\langle p,y\rangle\right\},\\
\sigma^\#(Sx,p) = \sup_{y\in Sx}\langle p,y\rangle, \quad \sigma^\flat(Sx,p) = \inf_{y\in Sx}\langle p,y\rangle \quad \text{for } x \in X.
\end{gathered} \tag{1.4}$$

We say that

$$\begin{gathered}
S \text{ is outward if } \forall p \in U^*, \forall x \in \partial(X,p), \quad \langle p,x\rangle \ge \sigma^b(Sx,p),\\
S \text{ is inward if } \forall p \in U^*, \forall x \in \partial(X,p), \quad \langle p,x\rangle \le \sigma^\#(Sx,p),\\
S \text{ is strongly inward if } \forall p \in U^*, \forall x \in \partial(X,p), \quad \sigma^\#(Sx,p) \ge \sigma^\#(X,p).
\end{gathered} \tag{1.5}$$

With the help of these concepts, the Rogalski-Cornet theorem can be stated as follows.

Theorem 1.7 (see [6] Rogalski-Cornet). *Suppose that X is a convex compact subset of U supplied with the weak topology, and S is an u.h.c. set-valued map from X to U with nonempty closed convex values. If S is either outward or strongly inward, then for any $y \in X$, there exists a solution $x \in X$ to $y \in Sx$, that is, $SX = \bigcup_{x\in X} Sx \supseteq X$.*

We use the following notations throughout this paper:

$$|Ax|_\infty = \sup_{z\in Ax}\|z\|, \qquad \mu_\infty = \sup_{x\in\partial X}|Ax|_\infty,$$
$$d(x,F) = \inf_{z\in F}\|x-z\| \quad \text{for each } x\in R^n, F\subseteq R^n \text{ with } F\neq\emptyset. \tag{1.6}$$

2. Theorems With u.h.c. Condition

In this section, we assume that A of (1.1) is u.s.c. on X.

Associating this assumption with (1.2), we can show that $AX = \bigcup_{x\in X} Ax$ and $A(\partial X) = \bigcup_{x\in\partial X} Ax$ are compact. Therefore, A is a strict and u.s.c. set-valued map from X (a convex compact subset of R^n_+) to AX (a compact subset of R^n_+) with convex compact values, and μ_∞ defined by (1.6) is finite. Moreover, by Lemmas 1.2 and 1.4 and Remark 1.5, A is also closed and u.h.c. on X. Suppose that G^A is a set-valued map from $C(A)$ to X defined by

$$C(A) = R^n_+ \bigcap (I-A)X,$$
$$G^A c = \{x\in X : c\in x - Ax\} \quad \text{for } c\in C(A). \tag{2.1}$$

Then we have the following results.

Theorem 2.1. *If $C(A)\neq\emptyset$, then $C(A)$ is compact and G^A is closed, u.s.c., and u.h.c., and the subset of points at which G^A is continuous is residual.*

Proof. Since X and AX are compact, we know that $(I-A)X$ is bounded, so is $C(A)$. Suppose that $\{c_k : k\geq 1\}\subseteq C(A)$ and $c_k \to c_0 \in R^n_+$ as $k\to\infty$. Then there exist $\{x_k : k\geq 1\}\subset X$ such that $c_k\in x_k - Ax_k$. Since $\{x_k : k\geq 1\}$ has a convergent subsequence, we may assume that $x_k \to x_0\in X$ as $k\to\infty$. As A is closed, we then obtain $c_0\in x_0 - Ax_0$, which shows that $C(A)$ is closed, and hence, also compact.

For the continuity results of G^A, according to Lemmas 1.3–1.6 and Remark 1.5, we only need to prove that G^A is closed because $C(A)$ and X are compact. Suppose that $\{(c_k, x_k) : k\geq 1\}\subseteq \operatorname{graph} G^A$ such that $(c_k, x_k)\to(c,x)$ as $k\to\infty$. Then $c_k\in x_k - Ax_k (k\geq 1)$. Since A is closed, $c_k\to c\in C(A)$ and $x_k\to x\in X$, we get $c\in x - Ax$. Hence, G^A is closed and has all the continuity results stated in the theorem. □

Next, we use Theorem 1.7 to obtain two solvability and stability results by means of the interior and exterior approximation methods used in [2] (three approximation methods have been used to study the single-valued input-output equation in [2]). Besides the assumptions that X is convex compact and A is strict and u.s.c. with convex compact values, we further assume that $\operatorname{int} X\neq\emptyset$ in this section. We have the following lemma. The proof is straightforward and hence omitted.

Lemma 2.2. *Let $Y\subseteq R^n$ be a nonempty convex set, $y\in Y$, and $\partial Y = Y\setminus \operatorname{int} Y$ the boundary of Y. If $p\neq 0$ and $y\in\partial(Y,p)$ (or $p\in N_Y(y)$), then $y\in\partial Y$.*

2.1. Interior Approximation Method

Define a subset X_∞ of X and a set-valued map F_1 from X_∞ to X by

$$X_\infty = X \bigcap \left[\mu_\infty, +\infty\right)^n, \quad \text{where } \left[\mu_\infty, +\infty\right)^n = \left\{(x_1, x_2, \ldots, x_n) \in R^n_+ : x_i \geq \mu_\infty\right\},$$
$$F_1 c = \left\{x \in X : \exists \Delta c \in R^n \text{ s.t. } c + \Delta c \in x - Ax, \|\Delta c\| \leq \mu_\infty\right\}, \quad \text{for } c \in X_\infty. \tag{2.2}$$

Then we have the first solvability and stability result for (1.1) as follows.

Theorem 2.3. *Suppose that $X_\infty \neq \emptyset$ and $c \in X_\infty$. Then there exist $x \in X$, $\Delta c \in R^n$ such that $c + \Delta c \in x - Ax$ and $\|\Delta c\| \leq \mu_\infty$. Moreover, F_1 defined by* (2.2) *is closed, u.s.c., and u.h.c., and the subset of points at which F_1 is continuous is residual.*

Proof. Since Ax is compact for $x \in X$ and A is u.s.c., it is easy to see that for $\varepsilon > 0$, $B(Ax, \varepsilon) = \{y \in R^n : d(y, Ax) < \varepsilon\}$ is an open neighborhood of Ax, and that for each $k \geq 1$ and $x \in \partial X$, there exists a neighborhood $U(x)$ of x with $A(U(x) \cap X) \subseteq B(Ax, 1/k)$. As ∂X is compact and $\partial X \subseteq \bigcup_{x \in \partial X} U(x)$, there exist $\{z^i : 1 \leq i \leq m\} \subseteq \partial X$ such that

$$\partial X \subseteq \bigcup_{i=1}^{m} U_i, \quad A(U_i \cap X) \subseteq B\left(Az^i, \frac{1}{k}\right), \quad \text{for } i = 1, 2, \ldots, m, \tag{2.3}$$

where $U_i \mathrel{\widehat{=}} U(z^i)$. Let $Y = X \setminus (\bigcup_{i=1}^m U_i)$. Then it is a closed subset of X with $\partial X \cap Y = \emptyset$. Let $d = d(\partial X, Y) = \inf_{x \in \partial X, y \in Y} d(x, y)$. Then we have $d > 0$. Otherwise, there exist sequences $\{x^j : j \geq 1\} \subseteq \partial X$ and $\{y^j : j \geq 1\} \subseteq Y$, such that $d(x^j, y^j) \to 0$ as $j \to \infty$. Since X is compact, so are ∂X and Y, which imply that there exist convergent subsequences of $\{x^j : j \geq 1\}$ and $\{y^j : j \geq 1\}$. Without loss of generality, we may assume that $\{x^j : j \geq 1\}$ and $\{y^j : j \geq 1\}$ are convergent to the same point x as $j \to \infty$. Since ∂X and Y are closed, we obtain $x \in \partial X \cap Y$, a contradiction. Therefore, $d > 0$. Let $U_0 = \bigcup_{y \in Y} B(y, d/2)$. Then it is easy to see that U_0 is an open subset of X, $X \setminus U_0 \subseteq \bigcup_{i=1}^m U_i$ and

$$A(X \setminus U_0) \subseteq \bigcup_{i=1}^{m} B\left(Az^i, \frac{1}{k}\right). \tag{2.4}$$

For each $k \geq 1$, by the truncation technique of generalized functions in partial differential equations [10], there is a continuous function g_k from X to R_+ such that $0 \leq g_k(x) \leq 1 \ (x \in X)$, $g_k|_{U_0} = 1$, $\operatorname{supp} g_k \equiv \overline{\{x \in X : g_k(x) \neq 0\}} \subseteq \operatorname{int} X$, and hence $g_k|_{\partial X} = 0$. Here $\overline{M}$ denotes the closure of M. Define $A_k : X \to 2^{R^n}$ by

$$A_k x = g_k(x) Ax, \quad \text{for } x \in X. \tag{2.5}$$

We claim that for each $k \geq 1$, A_k is a closed, u.s.c., and u.h.c. set-valued map with nonempty convex compact values. Indeed, $A_k x$ is clearly a convex compact subset of R^n for each $x \in X$. Assume that $(x^j, y^j) \in \operatorname{graph} A_k$ (i.e., $y^j \in g_k(x^j)Ax^j$) such that $(x^j, y^j) \to (x^0, y^0)$ as $j \to \infty$. Then $x^j \to x^0, y^j \to y^0$ $(j \to \infty)$, and for each $j \geq 1$ there exists $w^j \in Ax^j$ with $y^j = g_k(x^j)\mathrm{w}^j$. Since AX is compact, we may suppose that $w^j \to w^0$ as $j \to \infty$. By letting $j \to \infty$ and using the fact that A is closed, we get $w^0 \in Ax^0$ and $y^0 = g_k(x^0)w^0 \in g_k(x^0)Ax^0$, that is, $(x^0, y^0) \in \operatorname{graph} A_k$. Hence, A_k is closed. On the other hand, by (2.5), it is easy to see that $\overline{A_k(X)}$ is compact. Hence by Lemmas 1.3 and 1.4 and Remark 1.5, A_k is also u.s.c. and u.h.c..

Since $\sigma^{\#}((I - A_k)x, p) = \langle p, x\rangle + \sigma^{\#}(A_k x, -p)$ for $p \in R^n$ and $x \in X$, it follows that for each $p \in R^n$, the function $x \to \sigma^{\#}((I - A_k)x, p)$ is u.s.c. on X. And so, by Definition 1.1, $S = I - A_k$ is u.h.c.

Assume that $p \in R^n$ and $x \in \partial(X, p)$. If $p = 0$, then $\langle p, x\rangle = 0 = \sigma^b(x - A_k x, p)$. If $p \neq 0$, then by Lemma 2.2, $x \in \partial X$, which further implies by $g_k|_{\partial X} = 0$ that $A_k x = \{0\}$ and hence $\langle p, x\rangle = \sigma^b(x - A_k x, p)$. Therefore, $S = I - A_k$ satisfies the outward condition stated in (1.5). In view of Theorem 1.7, we have $(I - A_k)X \supseteq X$. Hence for each $c \in X$, there exists $x^k \in X$ such that

$$c \in x^k - A_k x^k. \tag{2.6}$$

If $x^k \in U_0$, by (2.5) and $g_k|_{U_0} = 1$, we have $c \in x^k - Ax^k$. Set $\Delta c^k = 0$, then we get

$$c + \Delta c^k \in x^k - Ax^k, \quad \|\Delta c^k\| \leq \mu_\infty + \frac{1}{k}. \tag{2.7}$$

If $x^k \in X \setminus U_0$, by (2.5) and (2.6) there exists $w^k \in Ax^k$ such that $c = x^k - g_k(x^k)w^k$. Let $\Delta c^k = (g_k(x^k) - 1)w^k$. Then $c + \Delta c^k = x^k - w^k \in x^k - Ax^k$ and $\Delta c^k \in (g_k(x^k) - 1)Ax^k$. By virtue of (2.4), (2.5), and (1.6), we see that (2.7) is also true.

Now, we obtain two sequences $\{x^k : k \geq 1\} \subseteq X$ and $\{\Delta c^k : k \geq 1\} \subseteq R^n$ with $\|\Delta c^k\| \leq \mu_\infty + (1/k)$. Since X is compact and $\{\Delta c^k : k \geq 1\}$ is bounded, we may assume that $x^k \to x \in X$ and $\Delta c^k \to \Delta c$ as $k \to \infty$. Combining this with (2.7) and also using the fact that A is closed, we obtain that $c + \Delta c \in x - Ax$ and $\|\Delta c\| \leq \mu_\infty$.

Next, we prove the second part of the theorem. Since X_∞ and X are compact, also by Lemmas 1.3–1.6 and Remark 1.5, we only need to prove that F_1 is closed. Assume that $\{(c^k, x^k) : k \geq 1\} \subseteq \operatorname{graph} F_1$ with $(c^k, x^k) \to (c, x)(k \to \infty)$. Then for each $k \geq 1$ there exists $\Delta c^k \in R^n$ such that

$$c^k + \Delta c^k \in x^k - Ax^k, \quad \|\Delta c^k\| \leq \mu_\infty. \tag{2.8}$$

We may suppose that $\Delta c^k \to \Delta c$ as $k \to \infty$. This implies by (2.8) and the closeness of A that $c + \Delta c \in x - Ax$ and $\|\Delta c\| \leq \mu_\infty$. Hence, F_1 is closed and has all the continuity properties stated in the theorem. This completes the proof. □

Remark 2.4. In the proof of the theorem, the condition $c \in X_\infty$ is not used. We impose this requirement in order to make sure that $c + \Delta c \in R^n_+$.

2.2. Exterior Approximation Method

Define a function $\delta_\infty(x)$ on X and a set-valued map F_2 from X_∞ to X by

$$\delta_\infty(x) = \begin{cases} 0 & \text{if } x \in \operatorname{int} X, \\ |Ax|_\infty & \text{if } \ x \in \partial X, \end{cases} \tag{2.9}$$

$$F_2 c = \{x \in X : \exists \Delta c \in R^n \text{ s.t. } c + \Delta c \in x - Ax, \|\Delta c\| \le \delta_\infty(x)\} \quad \text{for } c \in X_\infty,$$

where X_∞ is defined as in (2.2). Then we have the following result.

Theorem 2.5. *Suppose that $X_\infty \neq \emptyset$ and $c \in X_\infty$. Then there exist $x \in X$, $\Delta c \in R^n$ such that $c + \Delta c \in x - Ax$, and $\|\Delta c\| \le \delta_\infty(x)$. Furthermore, F_2 defined by* (2.9) *is closed, u.s.c. and u.h.c., and the subset of points at which F_2 is continuous is residual.*

Proof. Let $\overline{B}(X, \varepsilon) = \{x \in R^n : d(x, X) \le \varepsilon\}$ for $\varepsilon > 0$. For each $k \ge 1$, let $\overline{B}_k(X) \mathrel{\hat=} \overline{B}(X, 1/k)$ and let P_k be the projection from $\overline{B}_k(X)$ to X, that is, $P_k x \in X$ such that $\|P_k x - x\| = \inf_{y \in X}\|y - x\|$ for $x \in \overline{B}_k(X)$. It is easy to see that $\overline{B}_k(X)$ is a convex compact set with nonempty interior, $P_k x = x$ if $x \in X$ and $P_k x \in \partial X$ if $x \in \overline{B}_k(X) \setminus X$, and $\|P_k x - x\| \le 1/k$ for all $x \in \overline{B}_k(X)$.

As in the proof of Theorem 2.3, we assume that g_k is a continuous function from $\overline{B}_k(X)$ to R_+ with compact support set such that

$$0 \le g_k(x) \le 1 \quad \text{for } x \in \overline{B}_k(X), \ \ g_k\big|_X = 1, \ \ g_k\big|_{\partial(\overline{B}_k(X))} = 0. \tag{2.10}$$

Let A_k be a set-valued map from $\overline{B}_k(X)$ to R^n defined by

$$A_k x = g_k(x) A(P_k x) \quad \text{for } x \in \overline{B}_k(X). \tag{2.11}$$

Utilizing the similar method as in the proof of Theorem 2.3, we can show that A_k is a closed, u.s.c., and u.h.c. set-valued map with nonempty convex compact values and satisfies

$$A_k|_X = A, \qquad A_k|_{\partial(\overline{B}_k(X))} = \{0\}. \tag{2.12}$$

In fact, if $(x^m, y^m) \in \operatorname{graph} A_k$ with $(x^m, y^m) \to (x^0, y^0)$ $(m \to \infty)$, then $x^0 \in \overline{B}_k X$, $P_k x^m \to P_k x^0 \in X$ $(m \to \infty)$, and for each $m \ge 1$, there exists $w^m \in A(P_k x^m)$ such that $y^m = g_k(x^m) w^m$. Since $A(P_k(\overline{B}_k(X))) \subseteq AX$ and AX is compact, we may suppose that $w^m \to w^0$ $(m \to \infty)$. By letting $m \to \infty$, from the closeness of A, Lemmas 1.3 and 1.4, and Remark 1.5, we conclude that $w^0 \in A(P_k x^0)$, $y^0 = g_k(x^0) w^0 \in A_k x^0$, that is, $(x^0, y^0) \in \operatorname{graph} A_k$, and thus A_k is closed, u.s.c. and u.h.c..

Also as in the proof of Theorem 2.3, we can prove that $I - A_k$ satisfies the outward condition stated in (1.5). (In fact, assume that $p \in R^n$ and $x \in \partial(\overline{B}_k(X), p)$. If $p = 0$, then $\langle p, x\rangle = 0 = \sigma^b(x - A_k x, p)$. If $p \neq 0$, then by Lemma 2.2, $x \in \partial(\overline{B}_k(X))$, which implies by (2.12) that $A_k x = \{0\}$, and hence $\langle p, x\rangle = \sigma^b(x - A_k x, p)$.) By Theorem 1.7, we obtain

$$(I - A_k)\overline{B}_k(X) \supseteq \overline{B}_k(X) \supseteq X (\supseteq X_\infty). \tag{2.13}$$

Combining this with (2.11), we know that for each $c \in X$, there exists $\overline{x}^k \in \overline{B}_k(X)$ such that

$$c \in \overline{x}^k - g_k\left(\overline{x}^k\right) A\left(P_k \overline{x}^k\right). \tag{2.14}$$

Set $x^k = P_k \overline{x}^k$. Then $x^k \in X$. If there is $k \geq 1$ such that $\overline{x}^k \in \operatorname{int} X$, then from (2.10), (2.12), and (2.14) we have $\overline{x}^k = x^k$ and $c \in x^k - Ax^k$. Set $x = x^k$ and $\Delta c = 0$. Then

$$c + \Delta c \in x - Ax, \quad \|\Delta c\| = 0 \leq \delta_\infty(x). \tag{2.15}$$

If for all $k \geq 1$, $\overline{x}^k \in \overline{B}_k(X) \setminus \operatorname{int}\ X$, then $x^k = P_k \overline{x}^k \in \partial X$ and $\|x^k - \overline{x}^k\| \leq 1/k$. By (2.14), there exists $z^k \in Ax^k$ such that $c = \overline{x}^k - g(\overline{x}^k) z^k$. Let $\Delta c^k = x^k - \overline{x}^k + [g(\overline{x}^k) - 1] z^k$. Then we obtain

$$\begin{aligned} &\text{(a) } c + \Delta c^k = x^k - z^k \in x^k - Ax^k, \\ &\text{(b) } \left\|\Delta c^k\right\| \leq \left\|x^k - \overline{x}^k\right\| + \left|g\left(\overline{x}^k\right) - 1\right| \left\|z^k\right\| \leq \frac{1}{k} + \left|Ax^k\right|_\infty. \end{aligned} \tag{2.16}$$

Since X is compact and $\{\Delta c^k : k \geq 1\}$ is bounded, we may assume that $x^k \rightarrow x$ and $\Delta c^k \rightarrow \Delta c$ as $k \rightarrow \infty$. This implies by $x^k \in \partial X (k \geq 1)$ that $x \in \partial X$. On the other hand, if $\varepsilon > 0$ and $B(Ax, \varepsilon)$ is a ε-neighborhood of Ax, then there exists a neighborhood $U(x)$ of x such that $A(U(x) \bigcap X) \subseteq B(Ax, \varepsilon)$ because A is u.s.c., and thus for any $x' \in U(x) \cap X$, $|Ax'|_\infty \leq |Ax|_\infty + \varepsilon$. Hence the function

$$x \mapsto |Ax|_\infty \quad \text{is upper semicontinuous (i.e., u.s.c.) on X.} \tag{2.17}$$

By letting $k \rightarrow \infty$ and using the closeness of A, from (2.16) and (2.17), we get $c + \Delta c \in x - Ax$ and $\|\Delta c\| \leq |Ax|_\infty = \delta_\infty(x)$. Combining this with (2.15), we conclude that the first part of the theorem is true.

As in the proof of Theorem 2.3, we also need to verify that graph F_2 is closed. Suppose that $\{(c^k, x^k) : k \geq 1\} \subseteq \operatorname{graph} F_2$ and $(c^k, x^k) \rightarrow (c, x)$ as $k \rightarrow \infty$. Then for each $k \geq 1$ there exists $\Delta c^k \in R^n$ such that

$$\begin{aligned} &\text{(a) } c^k + \Delta c^k \in x^k - Ax^k, \\ &\text{(b) } \|\Delta c^k\| \leq \delta_\infty\left(x^k\right). \end{aligned} \tag{2.18}$$

We may suppose $\Delta c^k \rightarrow \Delta c$ as $k \rightarrow \infty$. By (2.18)(a) we have $c + \Delta c \in x - Ax$. On the other hand, by (2.9) and (2.17), it is easy to see that the function $x \mapsto \delta_\infty(x)$ is also u.s.c. on X. Combining this with (2.18)(b), we obtain $\|\Delta c\| \leq \delta_\infty(x)$. Therefore, F_2 is closed and has the continuity results stated in the theorem. □

Remark 2.6. As the discussion in Remark 2.4, the assumption that $c \in X_\infty$ is used to make sure that $c + \Delta c$ is a nonnegative net output vector.

Remark 2.7. Another approach to obtain Theorem 2.5 is to use the so-called neighborhood approximation method discussed in [2]. Indeed, if we define the set-valued map T from X to R^n by $Tx = Ax + \overline{B}(x, \delta_\infty(x))$, then we can prove that T is also an upper hemicontinuous set-valued map with convex compact values and satisfies the outward condition stated in (1.5). Hence, by Theorem 1.7, we obtain that $T(X) \supseteq X \supseteq X_\infty$, which can be used to prove Theorem 2.5.

3. Theorems Without Continuity Assumption

In Theorem 1.7 the associated set-valued map is assumed to be u.h.c.. Recently, a Rogalski-Cornet-type theorem without any continuity conditions was proved in [5]. As a simple application, it is briefly applied to (1.1) in [5]. In this section, we develop more solvability and stability results for (1.1). We first review this theorem in the framework of n-dimensional Euclidean space.

Let X be a nonempty convex compact subset of R^n, $S : X \to 2^{R^n}$ a set-valued map from X to R^n such that Sx is a nonempty closed convex subset of R^n for each $x \in X$, and $SX = \bigcup_{x\in X} Sx$ its range. For each $p \in R^n$, $c \in X$, and $\varepsilon \geq 0$, we set

$$\Upsilon(S,p,c,\varepsilon) = \left\{x \in X : \sigma^{\#}(Sx - c, p) + \varepsilon \geq 0\right\}, \quad \Upsilon(S,p,c) = \Upsilon(S,p,c,0). \tag{3.1}$$

Let $\Upsilon(S)$ be the set of all $c \in X$ such that

$$\begin{aligned} &\text{(a) } \Upsilon(S,p,c) \text{ is closed}, \quad \forall p \in R^n, \\ &\text{(b) } x \in \Upsilon(S,p,c) \quad \text{if } x \in X,\ p \in N_X(x). \end{aligned} \tag{3.2}$$

Let $\widetilde{\Upsilon}(S)$ be a subset of $\Upsilon(S)$ and F^S a set-valued map from $\widetilde{\Upsilon}(S)$ to X defined by

$$\begin{aligned} &\text{(a) } \widetilde{\Upsilon}(S) = \left\{c \in \Upsilon(S) : \forall p \in R^n,\ \forall \varepsilon \geq 0, \Upsilon(S,p,c,\varepsilon) \text{ is closed}\right\}, \\ &\text{(b) } F^S c = \{x \in X : c \in Sx\} = X \cap S^{-1}c, \quad \text{for } c \in \widetilde{\Upsilon}(S). \end{aligned} \tag{3.3}$$

Then the following Rogalski-Cornet-type theorem was proved in [5].

In the sequel, we assume that X in Lemma 3.1 is precisely the same subset X of R^n_+ as stated in (1.2) and use Lemma 3.1 to obtain three results for (1.1).

Lemma 3.1. *(i) (see [5, Theorem 2.8]) If $\Upsilon(S) \neq \emptyset$, then $\Upsilon(S) \subseteq SX$.*

(ii) (see [5, Theorem 2.12]) If $\widetilde{\Upsilon}(S) \neq \emptyset$, then $\widetilde{\Upsilon}(S)$ is compact. Moreover, F^S defined by (3.3) is closed, u.s.c. and u.h.c., and the subset of points at which F^S is continuous is residual.

Remark 3.2. (i) Two counter-examples were presented in *[5]* to show that the set-valued map S in Lemma 3.1 does not need the u.s.c., l.s.c., and u.h.c. conditions.

(ii) In case S is u.h.c., then for each $p \in R^n$, $c \in R^n$, and $\varepsilon \geq 0$, the function $f(x) = \sigma^{\#}(Sx - c, p) + \varepsilon = \sigma^{\#}(Sx, p) - \langle p, c\rangle + \varepsilon$ is u.s.c. on X, and thus the upper section $\Upsilon(S,p,c,\varepsilon) = \{x \in X \mid f(x) \geq 0\}$ is closed. This proves that $\Upsilon(S,p,c,\varepsilon)$ is closed for all $p \in R^n$, $c \in R^n$, and $\varepsilon \geq 0$ provided S is u.h.c..

3.1. The First Result

The first result is similar to Theorem 2.5. However, the u.s.c. assumption concerning A introduced in Section 2 has been removed. Let $\mu_\infty, X_\infty, \delta_\infty(x)$, and F_2 be defined as in (1.6), (2.2), and (2.9), respectively. Define

$$A_\infty : X \longrightarrow 2^{R^n} \quad \text{by } A_\infty x = \overline{B}(0, \delta_\infty(x)) \quad \text{for } x \in X. \tag{3.4}$$

Applying Lemma 3.1 to $S = I - (A + A_\infty)$, we have the following results.

Theorem 3.3. *Assume that μ_∞ is finite, $X_\infty \neq \emptyset$ and $S = I - (A + A_\infty)$.*

(i) *Let $c^0 \in X_\infty$. If* (3.2)*(a) holds for c^0, then $F_2 c^0 \neq \emptyset$, that is,*

$$\exists x^0 \in X, \quad \exists \Delta c^0 \in R^n \quad s.t.\ c^0 + \Delta c^0 \in x^0 - Ax^0, \quad \|\Delta c\| \leq \sigma_\infty\left(x^0\right). \tag{3.5}$$

(ii) *If $Y(S, p, c, \varepsilon)$ defined by* (3.1) *is closed for all $p \in R^n$, $c \in X_\infty$, and $\varepsilon \geq 0$, then F_2 defined by* (2.9) *is closed, u.s.c., and u.h.c., and the subset of points at which F_2 is continuous is residual.*

(iii) *If A is u.s.c., then F_2 has the same properties as stated in (ii).*

Proof. (i) Since A has convex compact values, X is compact and μ_∞ is finite; by (3.4) it is easy to see that $S = I - (A + A_\infty)$ is a strict set-valued map from X to R^n with convex compact values. We shall prove $c^0 \in Y(S)$ by showing c^0 satisfying (3.2). By the assumption of (i), it is sufficient to prove that (3.2)(b) holds for c^0. Indeed, we can prove that (3.2)(b) holds for any $c \in X$. Suppose that $x \in X, p \in N_X(x)$, and $c \in X$. If $p = 0$, then $\sigma^\#(Sx - c, 0) = 0$. If $p \neq 0$, then $x \in \partial X$ by Lemma 2.2. Hence by (2.9), $\delta_\infty(x) = |Ax|_\infty$, $0 \in Ax + \overline{B}(0, |Ax|_\infty)$, $\sigma^b(Ax + A_\infty x, p) \leq 0$, and thus $\langle p, x\rangle = \sup_{y \in X}\langle p, y\rangle \geq \langle p, c\rangle \geq \sigma^b(Ax + A_\infty x, p) + \langle p, c\rangle$, that is, $\sigma^\#(Sx - c, p) = \sigma^\#(x - Ax - A_\infty x - c, p) \geq 0$. So (3.2)(b) holds for all $c \in X$. By Lemma 3.1(i) and $X_\infty \subseteq X$, we have $c^0 \in Y(S) \subseteq SX = \bigcup_{x \in X}(x - Ax - A_\infty x)$. This implies that (3.5) is true, and thus (i) follows.

(ii) By the proof of (i), we have known that (3.2)(b) holds for all $c \in X$. Combining this with the assumption of (ii) and using (3.3)(a), we obtain that $X_\infty \subseteq \widetilde{Y}(S)$. So $\widetilde{Y}(S)$ is nonempty and also compact by Lemma 3.1(ii). In view of Lemmas 1.3–1.6 and Remark 1.5, to complete the proof of (ii), it is enough to show that F_2 is closed since X_∞ and X are compact. Suppose that $\{(c^m, x^m) : m \geq 1\} \subseteq \operatorname{graph} F_2$ with $(c^m, x^m) \to (c^0, x^0)$ $(m \to \infty)$. Then $c^0 \in X_\infty \subset \widetilde{Y}(S)$, and for each $m \geq 1$ there exists $\Delta c^m \in \overline{B}(0, \delta_\infty(x^m))$ such that $c^m + \Delta c^m \in x^m - Ax^m$. By $\operatorname{graph} F_2 \subseteq X_\infty \times X \subseteq \widetilde{Y}(S) \times X$ and (3.4), we can see that $c^m \in Sx^m$, that is, $(c^m, x^m) \in \operatorname{graph} F^S$, where F^S is defined by (3.3) for $S = I - (A + A_\infty)$. By Lemma 3.1(ii), F^S is closed, and thus $(c^0, x^0) \in \operatorname{graph} F^S$, which implies that $x^0 \in X$ and $c^0 \in Sx^0 = (I - A - A_\infty)x^0$. So we can select $\Delta c^0 \in \overline{B}(0, \delta_\infty(x^0))$ such that $c^0 + \Delta c^0 \in x^0 - Ax^0$, that is, $(c^0, x^0) \in \operatorname{graph} F_2$. Hence F_2 is closed.

(iii) Since A is u.s.c, by the proof of Theorem 2.5, we have known that the function $x \mapsto \delta_\infty(x)$ is u.s.c. on X. Combining this with (3.4), it is easy to see that the set-valued map A_∞ is u.s.c., and, by Lemma 1.4, also u.h.c.. Thus for each $p \in R^n$, the function $x \mapsto \sigma^\#(Sx, p) = \langle p, x\rangle + \sigma^\#(Ax, -p) + \sigma^\#(A_\infty x, -p)$ is u.s.c. on X. This implies that $S = I - (A + A_\infty)$ is u.h.c.

on X. Hence by Remark 3.2 and $X_\infty \subseteq X$, the condition of (ii) is true. This completes the proof. □

3.2. The Second Result

Next we use a new number μ_0, a new set X_0, and a new set-valued map F_0 from X_0 to X to obtain further solvability and stability result for (1.1). Set

$$|Ax|_0 = \inf_{z\in Ax}\|z\|, \quad (x \in X), \quad \mu_0 = \sup_{x\in\partial X}|Ax|_0,$$
$$\delta_0(x) = \begin{cases} 0 & \text{if } x \in \operatorname{int} X, \\ |Ax|_0 & \text{if } x \in \partial X, \end{cases} \qquad X_0 = X \bigcap [\mu_0, +\infty)^n, \tag{3.6}$$

and define $A_0 : X \to 2^{R^n}$ and $F_0 : X_0 \to 2^X$ by

$$A_0 x = \overline{B}(0, \delta_0(x)) \quad \text{for } x \in X,$$
$$F_0 c = \{x \in X : \exists \Delta c \in R^n \text{ s.t. } c + \Delta c \in x - Ax, \|\Delta c\| \le \delta_0(x)\} \quad \text{for } c \in X_0. \tag{3.7}$$

Applying Lemma 3.1 to $S = I - (A + A_0)$, we have the following results.

Theorem 3.4. *Assume that μ_0 is finite, $X_0 \neq \emptyset$, and $S = I - (A + A_0)$.*

(i) *Let $c^0 \in X_0$. If* (3.2)*(a) holds for c^0, then $F_0 c^0 \neq \emptyset$, that is,*

$$\exists x^0 \in X, \quad \exists \Delta c^0 \in R^n \quad \text{s.t. } c^0 + \Delta c^0 \in x^0 - Ax^0, \ \|\Delta c^0\| \le \delta_0(x). \tag{3.8}$$

(ii) *If $Y(S, p, c, \varepsilon)$ defined by* (3.1) *is closed for all $p \in R^n$, $c \in X_0$ and $\varepsilon \ge 0$, then F_0 defined by* (3.7) *is closed, u.s.c., and u.h.c., and the subset of points at which F_0 is continuous is residual.*

(iii) *If A is u.s.c. on X and also l.s.c. on ∂X, then the result of (ii) for F_0 is also true.*

Proof. (i) It is easy to see that S is a strict set-valued map with convex compact values. As in the proof of Theorem 3.3, we can prove that (3.2)(b) holds for all $c \in X$. Thus we can follow the proof of Theorem 3.3(i) to obtain the conclusion of (i). In fact, let $x \in X$, $p \in N_X(x)$, and $c \in X$. If $p = 0$, then $\sigma^\#(Sx-c, p) = 0$. If $p \neq 0$, then $x \in \partial X$. So $\delta_0(x) = |Ax|_0$, $0 \in Ax+\overline{B}(0, \delta_0(x))$ and $\sigma^b(Ax+A_0x, p) \le 0$. This implies $\langle p, x\rangle = \sup_{y\in X}\langle p, y\rangle \ge \langle p, c\rangle \ge \sigma^b(Ax+A_0x, p)+\langle p, c\rangle$, that is, $\sigma^\#(Sx - c, p) = \sigma^\#(x - Ax - A_0x - c, p) \ge 0$. Hence, (3.2)(b) holds for all $c \in X$, and by Lemma 3.1(i), $c^0 \in Y(S) \subseteq SX = \bigcup_{x\in X}(x - Ax - A_0x)$. Thus (3.8) is true.

(ii) The proof of (ii) is similar to that of Theorem 3.3(ii). Indeed, by the proof of (i) and the assumption of (ii), we can see that $X_0 \subseteq \widetilde{Y}(S)$, and thus $\widetilde{Y}(S)$ is compact by Lemma 3.1(ii). If $\{(c^m, x^m) : m \ge 1\} \subseteq \operatorname{graph} F_0$ such that $(c^m, x^m) \to (c^0, x^0)$ as $m \to \infty$, then $c^0 \in X_0 \subset \widetilde{Y}(S)$, $x^0 \in X$, and there exists $\Delta c^m \in \overline{B}(0, \delta_0(x^m))$ with $c^m + \Delta c^m \in x^m - Ax^m$. Hence by (3.3) and (3.7), $(c^m, x^m) \in \operatorname{graph} F^S$, where F^S is defined by (3.3) for $S = I - (A + A_0)$. Also by Lemma 3.1(ii), we have $(c^0, x^0) \in \operatorname{graph} F^S$, that is, $c^0 \in (I - A - A_0)x^0$. So we can select

$\Delta c^0 \in \overline{B}(0, \delta_0(x^0))$ such that $c^0 + \Delta c^0 \in x^0 - Ax^0$. Therefore, $(c^0, x^0) \in \operatorname{graph} F_0$ and F_0 is closed.

(iii) Similar to the proof of Theorem 3.3(iii), it suffices to show that $S = I - (A + A_0)$ is u.h.c.. For $x \in \operatorname{int} X$ and $p \in R^n$, we have $A_0 x = \{0\}$. In this case, since $\sigma^\#(Sx, p) = \langle x, p\rangle + \sigma^\#(Ax, -p)$ holds for any $x \in \operatorname{int} X$ and A is u.h.c., we obtain that S is u.h.c. in $\operatorname{int} X$. Now, we prove that S is also u.h.c. at any point $x^0 \in \partial X$. Because $\sigma^\#(Sx, p) = \langle x, p\rangle + \sigma^\#(Ax, -p) + \sigma^\#(A_0 x, -p)$ holds for any $x \in X$ and $p \in R^n$, it is sufficient to show that A_0 is u.h.c. at $x^0 \in \partial X$. Since Ax^0 is compact, there is $z^0 \in Ax^0$ such that $\|z^0\| = |Ax^0|_0 = \delta_0(x^0)$. As A is l.s.c. at x^0 by the assumption, for $\varepsilon > 0$, there exists an open neighborhood $U(x^0)$ of x^0 such that $Ax \cap B(z^0, \varepsilon) \neq \emptyset$ for all $x \in X \cap U(x^0)$. This implies that for each $x \in X \cap U(x^0)$, $\delta_0(x) \leq |Ax|_0 \leq \inf_{y \in Ax \cap B(z^0, \varepsilon)} \|y\| \leq \sup_{y \in B(z^0, \varepsilon)} \|y\| \leq \delta_0(x^0) + \varepsilon$, and hence $\overline{B}(0, \delta_0(x)) \subseteq \overline{B}(\overline{B}(0, \delta_0(x^0)), \varepsilon)$ for all $x \in X \cap U(x^0)$. So we conclude that A_0 is u.s.c. at x^0 and, by Lemma 1.4, also u.h.c.. Therefore, $S = I - (A + A_0)$ is u.h.c. on X, and the conclusion of (iii) follows. This completes the proof. □

Remark 3.5. As in the previous theorems, the assumptions $c^0 \in X_\infty$ in Theorem 3.3 and $c^0 \in X_0$ in Theorem 3.4 are used to make sure that the net output vectors are nonnegative.

3.3. *The Third Result*

Finally, we use the values of A at the boundary ∂X of X to consider the solvability and stability of (1.1). Define $\widehat{F} : \widehat{X} \to 2^X$ by

$$\begin{gathered} \widehat{X} = \{c \in X : \forall x \in \partial X, (Ax + c) \cap X \neq \emptyset\}, \\ \widehat{F}c = \{x \in X : c \in x - Ax\} \quad \text{for } c \in \widehat{X}. \end{gathered} \tag{3.9}$$

Applying Lemma 3.1 to $S = I - A$, we have the following results.

Theorem 3.6. *Assume that $\widehat{X} \neq \emptyset$ and $S = I - A$.*

(i) *If* (3.2)*(a) holds for all $c \in \widehat{X}$, then $\widehat{X} \subseteq Y(S) \subseteq (I - A)X$, and $\widehat{X}$ is compact. In particular, if $\widehat{X} \cap R^n_+ \neq \emptyset$, then for any $c \in \widehat{X} \cap R^n_+$,* (1.1) *is solvable.*

(ii) *If $Y(S, p, z, \varepsilon)$ defined by* (3.1) *is closed for all $p \in R^n$, $c \in \widehat{X}$, and $\varepsilon \geq 0$, then $\widehat{F}$ defined by* (3.9) *is closed, u.s.c., and u.h.c., and the subset of points at which $\widehat{F}$ is continuous is residual.*

(iii) *If A is u.s.c., then both results of (i) and (ii) are also true.*

Proof. (i) Let $c \in \widehat{X}$, $x \in X$, and $p \in N_X(x)$. If $p = 0$, then $\sigma^\#(x - Ax - c, 0) = \langle 0, x\rangle$. If $p \neq 0$, then $x \in \partial X$. From (3.9), we know that $(Ax + c) \cap X \neq \emptyset$. This implies that $\sigma^b(Ax + c, p) \leq \sigma^b((Ax + c) \cap X, p) \leq \sigma^\#(X, p) = \langle p, x\rangle$. Hence (3.2)(b) holds for all $c \in \widehat{X}$, and then $c \in Y(S)$. By Lemma 3.1(i), it follows that $\widehat{X} \subseteq Y(S) \subseteq SX = (I - A)X$.

To prove that $\widehat{X}$ is compact, we only need to prove that $\widehat{X}$ is closed. Suppose that $\{c^m : m \geq 1\} \subseteq \widehat{X}$ such that $c^m \to c^0 \in X$ as $m \to \infty$. Let $x \in \partial X$. Then by (3.9), $(Ax + c^m) \cap X \neq \emptyset$ for $m \geq 1$. Hence there exists a sequence $\{y^m : m \geq 1\} \subseteq X$ such that $y^m \in Ax + c^m$. Therefore, $y^m - c^m \in Ax$. Because Ax is compact, there exists a convergent subsequence of $\{y^m - c^m : m \geq 1\}$. Since $c^m \to c^0$ as $m \to \infty$, we may assume that $y^m \to y^0 \in X$, which implies $y^0 \in (Ax + c^0) \cap X$. Hence $c^0 \in \widehat{X}$ and $\widehat{X}$ is closed. The rest of (i) is clear because $\widehat{X} \cap R^n_+ \subset (I - A)X$.

(ii) By assumption of (ii), the proof of (i), and Lemma 3.1, we can see that $\widehat{X} \subset \widetilde{Y}(S)$, $\widetilde{Y}(S)$ is compact, and F^S defined by (3.3) for $S = I - A$ is closed. With the same method as in proving Theorem 3.3(ii), we can show that $\widehat{F}$ is closed and has all the continuous properties stated in this theorem (ii).

(iii) By assumption of (iii) and $\sigma^{\#}((I - A)x, p) = \langle p, x\rangle + \sigma^{\#}(Ax, -p)$, we can see that $S = I - A$ is u.h.c., hence $Y(S, p, c, \varepsilon)$ is closed for all $p \in R^n$, $c \in \widehat{X}$ and $\varepsilon \geq 0$. Hence both conditions of (i) and (ii) are satisfied, and therefore the result of (iii) follows. This completes the proof. □

4. Conclusions

In this paper, the Leontief-type input-output inclusion has been studied. First applying Rogalski-Cornet theorem [6], we have proved two solvability and stability theorems (Theorems 2.3 and 2.5) under the assumption that the set-valued map in this inclusion is upper semicontinuous. Then utilizing a Rogalski-Cornet-type theorem proved in [5], we have also proved three solvability and stability theorems (Theorems 3.3–3.6) in which the continuity assumption regarding the set-valued map in this inclusion is no longer needed.

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