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Research Article

The Robust Pole Assignment Problem for Second-Order Systems

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Pole assignment problems are special algebraic inverse eigenvalue problems. In this paper, we research numerical methods of the robust pole assignment problem for second-order systems. The problem is formulated as an optimization problem. Depending upon whether the prescribed eigenvalues are real or complex, we separate the discussion into two cases and propose two algorithms for solving this problem. Numerical examples show that the problem of the robust eigenvalue assignment for the quadratic pencil can be solved effectively.

1. Introduction

Pole assignment problems are special algebraic inverse eigenvalue problems [1, 2]. The nature model for the vibrating systems, arising in a wide range of applications, especially in the design and analysis of vibration structures, such as bridges, buildings, and airplanes, can be described by the second-order differential equation. The properties of systems of second-order differential equation are governed by its associated quadratic eigenvalue problem (QEP). To avoid unwanted oscillations of the vibratory system, pole assignment can change the poles of the system by choosing a control force and improve its stability.

Consider the following second-order matrix differential equation:

$$M\ddot{z}(t) + C\dot{z}(t) + Kz(t) = 0, \tag{1.1}$$

where the dots denote differentiation with respect to time, and M, C, and K are $n \times n$ real symmetric matrices; M is positive definite (denoted by $M > 0$). Separation of variables

$$Z(t) = xe^{\lambda t}, \tag{1.2}$$

where $\lambda \in \mathbf{C}$, $x \in \mathbf{C}^{\mathbf{n}}$ is a constant vector, in (1.1), leads to the following quadratic eigenvalue problem:

$$Q(\lambda)x = \left(\lambda^2 M + \lambda C + K\right)x = 0, \tag{1.3}$$

where $Q(\lambda) = \lambda^2 M + \lambda C + K$ is quadratic pencil. In general, the $2n$ eigenvalues of $Q(\lambda)$ are named poles of system (1.1). In engineering, the dynamics of equation (1.1) can be modified by applying a control force $Bu(t)$, where $B \in R^{n\times m}$, $u(t) \in R^m$. The relation (1.1) now becomes

$$M\ddot{z}(t) + C\dot{z}(t) + Kz(t) = Bu(t). \tag{1.4}$$

The special choice

$$u(t) = F^T\dot{z}(t) + G^T z(t), \tag{1.5}$$

where F and G are $n \times m$ real matrices, is called state feedback control, and (1.1) becomes

$$M\ddot{z}(t) + \left(C - BF^T\right)\dot{z}(t) + \left(K - BG^T\right)z(t) = 0. \tag{1.6}$$

The associated quadratic eigenvalue problem becomes

$$Q_c(\lambda)x = \left(\lambda^2 M + \lambda\left(C - BF^T\right) + \left(K - BG^T\right)\right)x = 0, \tag{1.7}$$

where $Q_c(\lambda) = \lambda^2 M + \lambda(C - BF^T) + K - BG^T$ is a closed-loop pencil.

The quadratic eigenvalue assignment problem is stated explicitly as follows.

Problem QEA

Given real matrices $M, C, K \in R^{n\times n}$ and $B \in R^{n\times m}$, and a set of $2n$ complex numbers $\mathcal{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\}$, closed under complex conjugation, find real matrices $F, G \in R^{n\times m}$, such that the eigenvalues of $Q_c(\lambda)$ are equal to $\lambda_j (j = 1, 2, \ldots, 2n)$.

Conditions for the existence of solutions to problem QEA are known as in the following theorem.

Theorem 1.1 (see [3]). *Solutions F, G to problem QEA exist for every set $\mathcal{L}$ of self-conjugate complex numbers if and only if the system* (1.1) *is completely controllable, that is*

$$\operatorname{rank}\left[\lambda^2 M + \lambda C + K, B\right] = n, \quad \forall \lambda \in \mathbf{C}. \tag{1.8}$$

In a realistic situation, it is desirable to choose the feedback to ensure that the eigenstructure of closed-loop system is as robust, or insensitive to perturbation in the system matrices $M, C - BF^T$, and $K - BG^T$, as possible to the following inverse eigenvalue problem, known as the robust quadratic eigenvalue assignment problem.

Problem RQEA

Given real matrices M, C, K, B, and a set of $\mathcal{L}$ as in problem QEA, find real matrices $F, G \in R^{n\times m}$ and matrix $X \in C^{n\times 2n}$, such that $\widetilde{X} = [X^T, (\Lambda X)^T]^T$ is nonsingular, satisfying

$$MX\Lambda^2 + \left(C - BF^T\right)X\Lambda + \left(K - BG^T\right)X = 0, \quad \Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\}, \tag{1.9}$$

and the eigenstructure of the closed-loop system (1.6) is as robust as possible.

We remark that the requirement that the matrix Λ is diagonal, together with the invertibility of $\widetilde{X}$. These require the prescribed eigenvalues to be distinct. In the next section, we derive conditions for the solution of problem RQEA.

In the majority of methods that have been proposed for solving problem RQEA, the second-order control system (1.1) is rewritten as a first-order system. There are two difficulties in using this approach. The first is the linear system which has a double dimension of the original quadratic system and, hence, the computational work used to solve the problem is greater than necessary. The second difficulty arises because all the exploitable properties such as definiteness, and sparsity, of the coefficient matrices M, C, and K, usually offered by a practice problem, will be completely destroyed. So it is natural to wonder if solutions of the robust pole assignment problem can be obtained without resorting to a first-order reformulation.

Over the past years, many techniques for pole assignment without linearization have been proposed. Chu and Datta [4] gave a method to solve Problem RQEA. Nichols and Kautsky [5] recently have proposed a numerical method to solve this problem without linearization; the measures of robustness are subject to structured perturbations.

Datta and Sarkissian [6] proposed a direct partial modal approach to solve the partial eigenvalue assignment problem for second-order systems. It is "direct," because the solutions are obtained directly in the second-order system without any types of reformulations; It is "partial modal," because only a part of spectral data is needed for the solution. This method do not, however, ensure the robustness of the closed-loop system. Qian [7], Qian and Xu [8] recently have proposed a direct method to solve the robust partial pole assignment problem for second-order systems which seems more efficient and reliable. Bai et al. ([9]) gave a new optimization approach for solving this problem.

According to new measures ([10, 11]), we suggest in this paper two numerical methods for solving the problem RQEA. Numerical results show that this problem can be solved effectively.

We begin by presenting the solutions to problem RQEA without linearization. In Section 3, we describe a new measure of robustness and formulate the problem as an optimization problem. Two numerical methods are developed in Section 4, and numerical results are given in Section 5.

2. Solution to Problem RQEA

Without loss of generality, We assume in this paper that the system (1.1) is completely controllable, and B is of full column rank. The next theorem provides necessary and sufficient conditions for the existence of solutions to problem RQEA.

Theorem 2.1 (see [5]). *Let $X \in \mathbf{C}^{\mathbf{n}\times\mathbf{2n}}$ be such that $\widetilde{X} = [X^T, (\Lambda X)^T]^T$ is nonsingular, where $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\}$, then there exist real matrices F, G, satisfying condition* (1.9) *of problem RQEA if and only if*

$$U_1^T \left(MX\Lambda^2 + CX\Lambda + KX \right) = 0, \tag{2.1}$$

where

$$B = [U_0, U_1] \begin{bmatrix} Z \\ 0 \end{bmatrix}, \tag{2.2}$$

with $U = [U_0, U_1]$ orthogonal and Z nonsingular. The matrices F, G are given by

$$\left[G^T, F^T\right] = Z^{-1} U_0^T \left[MX\Lambda^2 + CX\Lambda + KX \right] \widetilde{X}^{-1}. \tag{2.3}$$

An immediate consequence of Theorem 2.1 is the following.

Corollary 2.2. *Let x_j be the right eigenvector of $Q_c(\lambda)$ corresponding to the prescribed eigenvalue $\lambda_j \in \mathcal{L}$, then*

$$x_j \in \mathcal{N}\left\{ U_1^T \left(\lambda_j^2 M + \lambda_j C + K \right) \right\} = \mathcal{S}_j \quad (j = 1, 2, \ldots, 2n), \tag{2.4}$$

where $\mathcal{N}\{\cdot\}$ denotes right nullspace.

For every λ_j, find an orthogonal basis, comprised by columns of matrix $\mathcal{S}_j$ for the space $\mathcal{S}_j, j = 1, 2, \ldots, 2n$. Observe that the matrix $\mathcal{S}_j$ can be obtained by the QR decomposition of $(U_1^T(\lambda_j^2 M + \lambda_j C + K))^T$.

Theorem 2.3. *If the system* (1.1) *is completely controllable, then*

$$\dim(\mathcal{S}_j) = m, \quad (j = 1, 2, \ldots, 2n), \tag{2.5}$$

where dim *denotes the dimension of space $\mathcal{S}_j$.*

Proof. From Theorem 1.1, we can get

$$\operatorname{rank}\left[B, \lambda_j^2 M + \lambda_j C + K \right] = n, \tag{2.6}$$

then

$$\begin{aligned} n &= \operatorname{rank}\begin{bmatrix} U_0^T \\ U_1^T \end{bmatrix}\left[B, \lambda_j^2 M + \lambda_j C + K\right] \\ &= \operatorname{rank}\begin{bmatrix} Z & U_0^T\left(\lambda_j^2 M + \lambda_j C + K\right) \\ 0 & U_1^T\left(\lambda_j^2 M + \lambda_j C + K\right) \end{bmatrix} \\ &= m + \operatorname{rank}\left[U_1^T\left(\lambda_j^2 M + \lambda_j C + K\right)\right]. \end{aligned} \tag{2.7}$$

So we can have $\operatorname{rank}[U_1^T(\lambda_j^2 M + \lambda_j C + K)] = n - m$, from which (2.5) easily follows. □

According to the previous theory, we can write the following:

$$x_j = S_j w_j \in \mathcal{S}_j, \quad (j = 1, 2, \ldots, 2n), \tag{2.8}$$

where $w_j \in \mathbf{C}^m$, and if $\lambda_j = \overline{\lambda}_k$, then $x_j = \overline{x}_k$. Because in the case where λ_j is complex the associated eigenvector is a complex vector, in order to ensure that the computed feedback matrices are real, the eigenvector corresponding to the conjugate eigenvalue $\overline{\lambda}_j$ must be taken to be the conjugate vector $\overline{x}_j$.

3. The Measures of Robustness

In this section, we present the measures of robustness for the second-order system. The eigenvectors of the closed-loop system can be selected to minimize the measure of robustness.

Consider a matrix $\widetilde{A} \in R^{2n\times 2n}$, if $\widetilde{A}$ is nondefective, namely, then there exists a nonsingular matrix $\widetilde{Y} = [\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_{2n}] \in \mathbf{C}^{\mathbf{2n\times 2n}}$, such that

$$\widetilde{A}\widetilde{Y} = \widetilde{Y}\Lambda, \tag{3.1}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n})$, let

$$\widetilde{Z}^H = [\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{2n}]^H = \widetilde{Y}^{-1}, \tag{3.2}$$

then the sensitivity of the eigenvalues of λ_j of $\widetilde{A}$ to perturbations in the components of $\widetilde{A}$ depends upon the magnitude of the condition number c_j, where

$$c_j = \left\|\widetilde{y}_j\right\|_2 \left\|\widetilde{z}_j\right\|_2 \geq 1. \tag{3.3}$$

Hence, every reasonable measure of the magnitude of the vector $c = (c_1, c_2, \ldots, c_{2n})^T$ is a reflection of the robustness of the system.

If λ_1 is a multiple eigenvalue of $\widetilde{A}$ and

$$\lambda_1 = \lambda_2 \cdots = \lambda_r, \quad \lambda_1 \neq \lambda_j, \text{ for } j = r+1, \ r+2, \dots, 2n, \tag{3.4}$$

then the sensitivity of eigenvalue λ_1 depends upon the magnitude of the number

$$\widetilde{c}_1 = \max\{c_1, c_2, \dots, c_r\}. \tag{3.5}$$

Therefore, it is natural to take

$$\mho = (2n)^{-1} \left\| \widetilde{Y} \right\|_F \left\| \widetilde{Y}^{-1} \right\|_F := (2n)^{-1} \kappa_F \left(\widetilde{Y} \right) \tag{3.6}$$

as global measure of the sensitivity of eigenvalue.

In essence, the aim of the robust eigenvalue assignment problem is to select eigenvectors $\widetilde{y}_j$, such that $\|\widetilde{y}_j\|_2 = 1$ and the vectors $\widetilde{y}_j$ are as orthogonal as possible to each other. Therefore, in this paper we first consider the following measure $\mho_h(d)$:

$$\mho_h(d) = \left[\frac{\sum_{1 \le i < j \le 2n} d_{ij}^2 \left| \widetilde{y}_i^H \widetilde{y}_j \right|^2}{\sum_{1 \le i < j \le 2n} d_{ij}^2} \right]^{1/2}, \tag{3.7}$$

where

$$d = [d_{12}, d_{13}, \dots, d_{1,2n}, d_{23}, d_{24}, \dots, d_{2,2n}, \dots, d_{2n-1,2n}]^T, \quad d_{ij} > 0, \ \forall i, j. \tag{3.8}$$

Observe that if M is nonsingular, then the QEP (1.7) can be formulated as the following standard eigenvalue problem:

$$\widetilde{A} \begin{bmatrix} x \\ \lambda x \end{bmatrix} \equiv \begin{bmatrix} 0 & I \\ M^{-1}(K - BG^T) & M^{-1}(C - BF^T) \end{bmatrix} \begin{bmatrix} x \\ \lambda x \end{bmatrix} = \lambda \begin{bmatrix} x \\ \lambda x \end{bmatrix}. \tag{3.9}$$

Let $\widetilde{X}$ be the matrix comprised by the right eigenvectors of (3.9), then

$$\widetilde{X} = \begin{bmatrix} x_1 & x_2 & \dots & x_{2n} \\ \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_{2n} x_{2n} \end{bmatrix} := [\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_{2n}]. \tag{3.10}$$

By (2.8), we have

$$\widetilde{x}_j = \begin{bmatrix} I \\ \lambda_j I \end{bmatrix} x_j = \begin{bmatrix} I \\ \lambda_j I \end{bmatrix} S_j w_j. \tag{3.11}$$

From (3.7) it follows that

$$\begin{aligned}
[\mho_h(d)]^2 &= \frac{\sum_{1\le i<j\le 2n} d_{ij}^2 \left|\widetilde{x}_i^H \widetilde{x}_j\right|^2}{\sum_{1\le i<j\le 2n} d_{ij}^2} \\
&= \sum_{1\le i<j\le 2n} \delta_{ij} \left|\widetilde{x}_i^H \widetilde{x}_j\right|^2 \\
&= \sum_{1\le i<j\le 2n} \delta_{ij} \left| x_i^H \left[I, \overline{\lambda}_i I\right] \begin{bmatrix} I \\ \lambda j I \end{bmatrix} x_j \right|^2 \\
&= \sum_{1\le i<j\le 2n} \delta_{ij} \left| w_i^H S_i^H \left[I, \overline{\lambda}_i I\right] \begin{bmatrix} I \\ \lambda j I \end{bmatrix} S_j w_j \right|^2 \\
&\equiv f(w),
\end{aligned} \tag{3.12}$$

where

$$w = \left[w_1^H, w_2^H, \dots, w_{2n}^H\right]^H, \quad w_j = \left[w_{1j}, w_{2j}, \dots, w_{mj}\right]^H \in \mathbf{C}^{\mathbf{m}} \quad (j = 1, 2, \dots, 2n), \tag{3.13}$$

$$\delta_{ij} = \frac{d_{ij}^2}{\sum_{1\le i<j\le 2n} d_{ij}^2} > 0, \qquad \forall i, j. \tag{3.14}$$

Thus, we must solve an unconstrained optimization problem

$$\min f(w), \tag{3.15}$$

where $f(w)$ and w are defined by (3.12) and (3.13), $w \in \mathrm{C}^{\mathrm{N}}$ and $N = 2mn$.

4. Numerical Methods

Unfortunately, it is difficult to solve the optimization problem (3.15). Depending upon whether the prescribed eigenvalues are real or complex, we separate the discussion into two cases.

Case 1. Assume that the prescribed eigenvalues are real.

In this case, where the eigenvectors are real vectors, (3.12) becomes

$$f(w) = \sum_{1\le i<j\le 2n} \delta_{ij} \left(w_i^T S_i^T [I, \lambda_i I] \begin{bmatrix} I \\ \lambda j I \end{bmatrix} S_j w_j \right)^2. \tag{4.1}$$

Let

$$\mathcal{T} = \left\{ w = \left[w_1^T, \dots, w_{2n}^T \right]^T \in R^N : w_j \in R^m,\ \left\| w_j \right\|_2 = 1\ \forall j \right\},$$

$$A_{ij} = \gamma_{ij} S_i^T [I, \lambda_i I] \begin{bmatrix} I \\ \lambda j I \end{bmatrix} S_j, \qquad r_{ij}(w) = w_i^T A_{ij} w_j, \quad A_{ij}(w) = r_{ij}(w) A_{ij}, \quad \forall i \neq j,$$

$$A(w) = \begin{bmatrix} \alpha^2 I & -A_{12}(w) & \dots & -A_{1,2n}(w) \\ -A_{12}(w)^T & \alpha^2 I & \dots & -A_{2,2n}(w) \\ \vdots & \vdots & \vdots & \vdots \\ -A_{1,2n-1}(w)^T & \cdots & \alpha^2 I & -A_{2n-1,2n}(w) \\ -A_{1,2n}(w)^T & \cdots & -A_{2n-1,2n}(w)^T & \alpha^2 I \end{bmatrix}, \quad \alpha > 0, \tag{4.2}$$

where $\gamma_{ij} = \sqrt{\delta_{ij}}$, then minimizing the measure $\mathcal{V}_h(d)$ may be reduced to solving the following nonlinear programming problem with constrains:

$$\begin{aligned} &\max \quad w^T A(w) w, \\ &\text{subject to} \quad w \in \mathcal{T}. \end{aligned} \tag{4.3}$$

Let

$$\mathcal{B} = \left\{ w = \left[w_1^T, \dots, w_{2n}^T \right]^T \in R^N : w_j \in R^m, \left\| w_j \right\|_2 \leq 1\ \forall j \right\}, \tag{4.4}$$

we can prove easily that the programming problem (4.3) is equivalent to the following programming problem with inequality constrains:

$$\begin{aligned} &\max \quad w^T A(w) w, \\ &\text{subject to} \quad w \in \mathcal{B}. \end{aligned} \tag{4.5}$$

Let

$$\widetilde{D} = \left\{ D = \operatorname{diag}\left(\tau_1 I^{(m)}, \dots, \tau_{2n} I^{(m)} \right) : \tau_i \geq 0, \forall i \right\}. \tag{4.6}$$

We consider a multiparameter eigenvalue problem

$$A(w) w = D w, \qquad D \in \widetilde{D},\ w \in \mathcal{T}. \tag{4.7}$$

Using the Kuhn-Tucker optimality ([12, 13]), we can prove that every solution to problem (4.5) is necessarily a solution to problem (4.7).

In this section, we develop an algorithm, called Horst algorithm ([14]), to solve (4.7), for finding a global optimal solution w of problem (4.3).

Algorithm 1. Initialization Step

Let $\varepsilon > 0$ be the termination scalar. Given $M, C, K \in R^{n\times n}, B \in R^{n\times m}, \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}) \in \mathbf{C}^{2n\times 2n}$, choose an initial vector $w^{(0)} = [w_1^{(0)^T}, w_2^{(0)^T}, \ldots, w_{2n}^{(0)^T}]^T \in \mathcal{T}^N$.

Main Step

(1) Find the decomposition (2.2) of B and an orthonormal basis, comprised by the columns of the matrix S_j, for the subspaces $\mathcal{S}_j$, $j = 1, 2, \ldots, 2n$, defined by (2.4). Let $k = 0$.

(2) For $i = 1, 2, \ldots, 2n$, we compute

$$\begin{aligned} \hat{z}_i^{(k+1)} &= \alpha^2 w_i^{(k)} - \sum_{j=1}^{i-1} w_i^{(k)^T} A_{ij} w_j^{(k+1)} A_{ij} w_j^{(k+1)} - \sum_{j=i+1}^{2n} w_i^{(k)^T} A_{ij} w_j^{(k)} A_{ij} w_j^{(k)}, \\ \tau_i^{(k)} &= \left\| \hat{z}_i^{(k+1)} \right\|_2, \\ w_i^{(k+1)} &= \begin{cases} \dfrac{\hat{z}_i^{(k+1)}}{\tau_i^{(k)}} & \text{if } \tau_i^{(k)} > 0, \\ w_i^{(k)} & \text{if } \tau_i^{(k)} = 0. \end{cases} \end{aligned} \tag{4.8}$$

(3) Set $\Delta w^{(k)} = w^{(k+1)} - w^{(k)}$. If $\|\Delta w^{(k)}\|_2 \le \varepsilon$, then $w^* = w^{(k+1)}$ is an approximation optimal solution, go to (4); if $\|\Delta w^{(k)}\|_2 > \varepsilon$, then replace k by $k+1$ and repeat step (2).

(4) Let $X = [x_1, x_2, \ldots, x_{2n}]$, where x_j is defined by (2.8), and construct feedback matrices F, G by solving

$$\left[G^T, F^T\right] = Z^{-1} U_0^T \left[M X \Lambda^2 + C X \Lambda + K X\right] \widetilde{X}^{-1}. \tag{4.9}$$

Case 2. Assume that prescribed eigenvalues are complex.

In this case, it is almost impossible to solve optimization problem (3.15) without costing too much. Based on this ideal, we consider $\kappa_F(\widetilde{X}) = \|\widetilde{X}\|_F \|\widetilde{X}^{-1}\|_F$, where $\widetilde{X}$ is defined by (3.10), and use some similar techniques developed by Kautsky et al. in 1985 ([15]).

Obviously, $\kappa_F(\widetilde{X}) = \|\widetilde{X}\|_F \|\widetilde{X}^{-1}\|_F$ achieves the minimum if and only if $\widetilde{X}$ is unitary.

Let

$$\widetilde{W}_j = N\left(\begin{bmatrix} U_1^T\left(\lambda_j^2 M + \lambda_j C + K\right) \\ \lambda_j U_1^T\left(\lambda_j^2 M + \lambda_j C + K\right) \end{bmatrix}\right), \tag{4.10}$$

then $\tilde{x}_j \in \widetilde{W}_j$. Let the columns of matrix W_j comprise an orthonormal basis, for the subspace $\widetilde{W}_j$, and

$$X_j = \operatorname{span}\{\tilde{x}_1, \ldots, \tilde{x}_{j-1}, \tilde{x}_{j+1}, \ldots, \tilde{x}_{2n}\}, \tag{4.11}$$

and let y_j be a normalized vector orthogonal to X_j. The objective here is to choose vectors $\tilde{x}_j (j = 1, 2, \ldots, 2n)$, such that each vector is as orthogonal as possible to the space X_j. Observe that $\tilde{x}_j$ orthogonal to X_j is equivalent to choose $\tilde{x}_j \in \widetilde{W}_j$, such that the angle between $\tilde{x}_j$ and y_j is minimized.

Obviously, y_j must satisfy

$$\left[\tilde{x}_1, \ldots, \tilde{x}_{j-1}, \tilde{x}_{j+1}, \ldots, \tilde{x}_{2n}\right]^H y_j = 0. \tag{4.12}$$

The coefficient matrix of equation (4.12) is a $(2n - 1) \times 2n$ matrix, so we can easily solve y_j from equation (4.12), and let $\tilde{x}_j$ be the normalized projection of y_j onto the space $\widetilde{W}_j$, then

$$\tilde{x}_j = \frac{W_j W_j^H y_j}{\left\| W_j^H y_j \right\|_2}. \tag{4.13}$$

This method is then to sweep through the columns of $\widetilde{X}$ replacing the jth column in turn with the normalized projection $\tilde{x}_j$. Because the prescribed eigenvalue λ_j is complex, two columns need to be altered simultaneously. Repeat the previous procedure until satisfying the stopping criterion or reaching the maximal amount of iteration. We can choose

$$\left\| \widetilde{X}^H \widetilde{X} - I \right\|_F < \varepsilon, \tag{4.14}$$

as the stopping criterion.

It is worthwhile to point out that the procedure is not guaranteed to converge, that is, the criterion (4.14) may not be satisfied. To ensure the end of iteration, we need to set a maximal amount of iteration $k_{\max}$. Now we develop an algorithm, called orthogonal vector method, for solving the problem RQEA.

Algorithm 2. Initialization Step

Let $\varepsilon > 0$ be the termination scalar, $k_{\max}$ is the maximal amount of iteration. Given $M, C, K \in R^{n\times n}, B \in R^{n\times m}$, and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}) \in C^{2n\times 2n}$.

Main Step

(1) Find the decomposition (2.2) of B and an orthonormal basis, comprised by the columns of the matrix W_j, for the subspaces $\widetilde{W}_j$, $j = 1, 2, \ldots, 2n$, defined by (4.10).

(2) Select an initial matrix $\widetilde{X} = [\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{2n}]$, defined by (3.10), such that $\tilde{x}_j \in \widetilde{W}$ and $\widetilde{X}$ is nonsingular. Let $k = 0$.

(3) If $k < k_{\max}$, do

for $j = 1,\ 2, \dots, 2n$,

compute the solution y_j of equation (4.12) and normalize y_j;

compute $\tilde{x}_j = W_j W_j^H y_j / \|W_j^H y_j\|_2$;

compute $\text{res}_1 = \|\tilde{X}^H \tilde{X} - I\|_F$;

update the jth column of $\tilde{X}$ with $\tilde{x}_j$;

compute $\text{res}_2 = \|\tilde{X}^H \tilde{X} - I\|_F$.

If $|\text{res}_2 - \text{res}_1| < \varepsilon$, then go to (4), else continue;

(4) let $k = k + 1$; repeat (3) until convergence.

(5) let the first n rows of $\tilde{X}$ be X construct feedback matrices F, G by solving

$$\left[G^T, F^T\right] = Z^{-1} U_0^T \left[M X \Lambda^2 + C X \Lambda + K X\right] \tilde{X}^{-1}. \tag{4.15}$$

Although this method does not guarantee convergence, it is simple to implement, and numerical results show that it often leads to better conditioned closed-loop systems.

5. Numerical Results

To illustrate the performance of the present algorithms, in this section we give some numerical examples, which were carried out using MATLAB 6.1.

Example 5.1. In the first example we examine a case from [5], where the matrix M is very ill conditioned. The system matrices M, C, K, and B are defined by

$$M = \begin{bmatrix} 5000 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1.00001 \end{bmatrix}, \qquad C = 0, \qquad K = \begin{bmatrix} -40 & 40 & 0 \\ 40 & -80 & 40 \\ 0 & 40 & -40 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}. \tag{5.1}$$

The system is undamped and the open-loop eigenvalues are

$$\{\pm 48999.0i,\ \pm 4.4726i,\ \pm 0.051635i\}. \tag{5.2}$$

The desired closed-loop eigenvalues $\lambda_j, j = 1, 2, \dots, 6$ are given by

$$\mathcal{L} = \{-1, -2, -3, -4, -5, -6\}. \tag{5.3}$$

With Algorithm 1, we choose an initial vector $w^{(0)} = [w_1^{(0)^T}, w_2^{(0)^T}, \dots, w_{2n}^{(0)^T}]^T$ with $w_1^{(0)} = [1, 0]^T$, $w_2^{(0)} = [0, 1]^T$, $w_3^{(0)} = 1/\sqrt{2}[1, -1]^T$, $w_4^{(0)} = 1/\sqrt{2}[-1, 1]^T$, $w_5^{(0)} = [1, 0]^T$, and $w_6^{(0)} = [0, 1]^T$, and take $\gamma_{ij} = 1/\sqrt{6} (i, j = 1, 2, \dots, 6), \alpha^2 = 0.7$. The corresponding matrix

$\widetilde{X}$ has condition number $\kappa_F(\widetilde{X}) = 1.2626 \times 10^5$, and after three iterations, it is reduced to $\kappa_F(\widetilde{X}) = 3.8416 \times 10^4$. The computed feedback matrices are given by

$$F^T = \begin{bmatrix} -312.9824 & -7.3540 & -6.6748 \\ -0.0233 & -0.0003 & -0.0003 \end{bmatrix},$$
$$G^T = \begin{bmatrix} -194.9454 & 38.2092 & -79.2761 \\ 19.9877 & -60.0016 & 59.9985 \end{bmatrix}. \tag{5.4}$$

Example 5.2. In the second example, we examine a case from [16]. The system matrices M, C, K, and B are defined by

$$M = I_4, \qquad C = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, \qquad K = \begin{bmatrix} 5 & -5 & 0 & 0 \\ -5 & 10 & -5 & 0 \\ 0 & -5 & 10 & -5 \\ 0 & 0 & -5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.5}$$

The open-loop eigenvalues are

$$\{3.525, -3.559, -0.059 \pm 3.732i, -0.191 \pm 1.489i, -0.233 \pm 2.692i\}. \tag{5.6}$$

The desired closed-loop eigenvalues $\lambda_j, j = 1, 2, \ldots, 8$ are given by

$$\mathcal{L} = \{-1 \pm i, -2 \pm i, -3 \pm i, -4 \pm i\}. \tag{5.7}$$

With Algorithm 2, we choose $\varepsilon = 10^{-6}$, $k_{\max} = 8$, and the initial matrix $\widetilde{X}$ is generated by a random selection of vectors from each subspace. The corresponding matrix $\widetilde{X}$ has condition number $\kappa_F(\widetilde{X}) = 5.8818 \times 10^3$, and after eight iterations, it is reduced to $\kappa_F(\widetilde{X}) = 2.3927 \times 10^3$. The computed feedback matrices are given by

$$F^T = \begin{bmatrix} -5.5706 & -15.6774 & -92.7449 & 57.9760 \\ 0.0050 & -13.4294 & 2.4449 & 0.0238 \end{bmatrix},$$
$$G^T = \begin{bmatrix} -5.3794 & -173.3879 & 235.9619 & -93.4799 \\ -5.0455 & -46.2669 & 113.8836 & -70.6403 \end{bmatrix}. \tag{5.8}$$

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