Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 796398, 5 pages
doi:10.1155/2010/796398

Research Article

Exact Solutions of the Generalized Benjamin-Bona-Mahony Equation

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Received 5 February 2010; Revised 30 June 2010; Accepted 26 September 2010

Academic Editor: Yuri Vladimirovich Mikhlin

We apply the theory of Weierstrass elliptic function to study exact solutions of the generalized Benjamin-Bona-Mahony equation. By using the theory of Weierstrass elliptic integration, we get some traveling wave solutions, which are expressed by the hyperbolic functions and trigonometric functions. This method is effective to find exact solutions of many other similar equations which have arbitrary-order nonlinearity.

1. Introduction

The nonlinear phenomena in the scientific work or engineering fields are more and more attractive to scientists. To depict and analyze such nonlinear phenomena, the nonlinear evolutionary equations are playing an important role and their solitary wave solutions are the main interests of mathematicians and physicists.

To obtain the traveling wave solutions of these nonlinear evolution equations, many methods were attempted, such as the inverse scattering method, Hirota's bilinear transformation, the tanh-sech method, extended tanh method, sine-cosine method, homogeneous balance method, and exp-function method. With the aid of symbolic computation system, many explicit solutions are easily obtained, and many interesting works deeply promote the research of nonlinear phenomena.

The present work is interested in generalized Benjamin-Bona-Mahony (BBM) equation:

$$u_t + au_x + \left(bu^n + cu^{2n}\right)u_x + ku_{xxx} = 0. \tag{1.1}$$

In the above equation, the first term of left side represents the evolution term while parameters b and c represent the coefficients of dual-power law nonlinearity, a and k are the coefficients of dispersion terms, n is the power law parameter, and variable u is the wave profile. In [1], Biswas used the solitary wave ansatz and obtained an exact 1-soliton solution of (1.1). In order to find more exact solutions of some nonlinear evolutionary equations, the Weierstrass elliptic function was introduced. For example, Kuru [2, 3] discussed the BBM-like equation, and Estévez et al. [4] analyzed another type of generalized BBM equations. In [5], Deng et al. also applied the similar method to the study of a nonlinear variant of the PHI-four equation. In this paper, we will apply the method to the generalized Benjamin-Bona-Mahony equation.

The rest of this paper is organized as follows. In Section 2, we first outline the Weierstrass elliptic function method. In Section 3, we give exact expression of some traveling wave solutions of generalized Benjamin-Bona-Mahony (BBM) equation (1.1) by using the Weierstrass elliptic function method. Finally, some conclusions are given in Section 4.

2. Description of the Weierstrass Elliptic Function Method

When we search for the solutions of some evolutionary equations, we will meet the following ordinary differential equation:

$$\left(\frac{d\varphi}{d\theta}\right)^2 = P_4(\varphi) = a_0\varphi^4 + 4a_1\varphi^3 + 6a_2\varphi^2 + 4a_3\varphi + a_4. \tag{2.1}$$

In [6], Whittaker and Watson introduced two invariants:

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \qquad g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4, \tag{2.2}$$

and a discriminant $\Delta = g_2^3 - 27g_3^2$; then the solutions of differential (2.1) have the following form:

$$\varphi(\theta) = \varphi_0 + \frac{1}{4}P_\varphi(\varphi_0)\left(\wp(\theta; g_2, g_3) - \frac{1}{24}P_{\varphi\varphi}(\varphi_0)\right)^{-1}, \tag{2.3}$$

where φ_0 is one of the roots of the polynomial $P_4(\varphi)$, and $P_\varphi(\varphi_0)$, $P_{\varphi\varphi}(\varphi_0)$, respectively, denote the first and second derivative of $P_4(\varphi)$ with respect to φ at the φ_0. Particularly, if $\Delta = g_2^3 - 27g_3^2 = 0$, then the Weierstrass elliptic function $\wp(\theta; g_2, g_3)$ satisfies these conditions:

$$\begin{aligned}\wp\left(\theta; 12b^2, -8b^3\right) &= b + 3b\sinh^{-2}\left(\sqrt{3b}\theta\right),\\ \wp\left(\theta; 12b^2, 8b^3\right) &= -b + 3b\sin^{-2}\left(\sqrt{3b}\theta\right).\end{aligned} \tag{2.4}$$

Once we get the solution of (2.1) with the form of (2.3), we will get the exact expressions of solutions of many partial differential equations. Suppose that $\Delta = g_2^3 - 27g_3^2 = 0$; it is obvious that if $g_2 > 0$, $g_3 > 0$, then there exist period solutions of original evolutionary equation, otherwise if $g_2 > 0$, $g_3 < 0$, then there exist solitary solutions.

3. Discussion on Generalized BBM Equation (1.1)

Let $u(x,t) = u(\xi)$, $\xi = x - vt$, so (1.1) is carried to

$$(a-v)u + \frac{b}{n+1}u^{n+1} + \frac{c}{2n+1}u^{2n+1} + ku'' = D, \tag{3.1}$$

where D is an integral constant. We introduce the transformation that

$$u(\xi) = \left(\frac{k(n+1)}{b}\right)^{1/n} W(\xi), \tag{3.2}$$

so that (3.1) is changed to

$$W'' - \frac{v-a}{k}W + W^{n+1} + \frac{ck(n+1)^2}{b^2(2n+1)}W^{2n+1} = \frac{D}{k}\left(\frac{b}{k(n+1)}\right)^{1/n}. \tag{3.3}$$

Let $B = (v-a)/k$, $C = -ck(n+1)^2/b^2(2n+1)$, and $R_1 = (D/k)(b/k(n+1))^{1/n}$; then (3.3) is transformed to the following form:

$$W'' = BW - W^{n+1} + CW^{2n+1} + R_1. \tag{3.4}$$

By multiplying each side of (3.4) by W', integrating once again, we get that

$$(W')^2 = BW^2 - \frac{2}{n+2}W^{n+2} + \frac{C}{n+1}W^{2n+2} + 2R_1W + R_2, \tag{3.5}$$

where R_2 is the other integration constant.

Let $W = \varphi^p$; then the above equation is changed to

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{B}{p^2}\varphi^2 - \frac{2}{(n+2)p^2}\varphi^{np+2} + \frac{C}{(n+1)p^2}\varphi^{2np+2} + \frac{2R_1}{p^2}\varphi^{2-p} + \frac{R_2}{p^2}\varphi^{2(1-p)}. \tag{3.6}$$

Only if all the power of φ appearing in the right side of (3.6) are the integers and are between 0 and 4, can we guarantee the integrability of (3.6). Therefore, if $R_1 = R_2 = 0$, then $p = \pm 1/n$. We choose the only case that if $p = -1/n$, then

$$\left(\frac{d\varphi}{d\xi}\right)^2 = n^2B\varphi^2 - \frac{2n^2}{n+2}\varphi + \frac{n^2C}{n+1}. \tag{3.7}$$

The polynomial $P(\varphi)$ has two roots: $\varphi_0 = ((n+1)\pm\sqrt{(n+1)[BC(n+2)^2 + (n+1)]})/B(n+1)(n+2)$. For simplicity, let $\lambda = \sqrt{(n+1)[BC(n+2)^2 + (n+1)]}$, then $\varphi_0 = ((n+1)\pm\lambda)/B(n+1)(n+2)$

$$g_2 = \frac{n^4B^2}{12}, \qquad g_3 = -\frac{n^6B^3}{216}, \qquad \Delta = 0. \tag{3.8}$$

Then at the point $\varphi_0 = ((n+1) \pm \lambda)/B(n+1)(n+2)$, the nonzero solutions of (3.7) are

$$\begin{aligned} \wp(\theta; g_2, g_3) &= \frac{n^2 B}{12} + \frac{1}{4} n^2 B \csc h^2\left(\frac{\sqrt{B}}{2} n\theta\right), \quad (\text{if } B > 0), \\ \wp(\theta; g_2, g_3) &= \frac{n^2 B}{12} - \frac{1}{4} n^2 B \csc^2\left(\frac{\sqrt{-B}}{2} n\theta\right), \quad (\text{if } B < 0). \end{aligned} \tag{3.9}$$

The solutions of the ordinary differential equation are

$$\begin{aligned} \varphi(\xi) &= \frac{(n+1) \pm \left[\lambda - 2\lambda \cosh^2\left(\left(\sqrt{B}/2\right) n\theta\right)\right]}{B(n+1)(n+2)}, \quad (\text{if } B > 0), \\ \varphi(\xi) &= \frac{(n+1) \pm \left[\lambda - 2\lambda \cos^2\left(\left(\sqrt{-B}/2\right) n\theta\right)\right]}{B(n+1)(n+2)}, \quad (\text{if } B < 0), \end{aligned} \tag{3.10}$$

where $B = (v-a)/k$, $C = -ck(n+1)^2/b^2(2n+1)$, $R_1 = (D/k)(b/k(n+1))^{1/n}$. Therefore, the solutions of partial differential equation are

$$\begin{aligned} u(x,t) &= \left(\frac{k(n+1)}{b} \frac{B(n+1)(n+2)}{(n+1) \pm \left[\lambda - 2\lambda \cosh^2\left(\left(\sqrt{B}/2\right) n\theta\right)\right]}\right)^{1/n}, \quad (\text{if } B > 0), \\ u(x,t) &= \left(\frac{k(n+1)}{b} \frac{B(n+1)(n+2)}{(n+1) \pm \left[\lambda - 2\lambda \cos^2\left(\left(\sqrt{-B}/2\right) n\theta\right)\right]}\right)^{1/n}, \quad (\text{if } B < 0), \end{aligned} \tag{3.11}$$

where $B = (v-a)/k, C = -ck(n+1)^2/b^2(2n+1)$, and $R_1 = (D/k)(b/k(n+1))^{1/n}$.

Remark 3.1. We have checked that (3.11) is the solution of (1.1). From the above discussion, we find more solutions than the work of [1].

4. Conclusions

From the above discussion, we find the traveling wave solutions of the generalized BBM equation, which are expressed by the hyperbolic functions and trigonometric functions, without the aid of symbolic computations. In addition, the method is effective to find exact solutions of many other similar equations which have arbitrary-order nonlinearity.

Acknowledgments

The authors would like to express their sincere thanks to referees for the valuable advice. The present work was supported by the National Nature Science Foundation of China (no. 10771088) and Foundation for Younger Personnel of Jiangsu University (no. 1241190006).

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