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Research Article

Matrix Bounds for the Solution of the Continuous Algebraic Riccati Equation

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We propose new upper and lower matrix bounds for the solution of the continuous algebraic Riccati equation (CARE). In certain cases, these lower bounds improve and extend the previous results. Finally, we give a corresponding numerical example to illustrate the effectiveness of our results.

1. Introduction

In many areas of optimal control, filter design, and stability analysis, the continuous algebraic Riccati equation plays an important role (see [1–5]). For example, consider the following linear system (see [5]):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $x_0 \in R^n$ is the initial state. The state feedback control and the performance index of the system (1.1), respectively, are

$$\begin{aligned} u(t) &= -Kx(t), \qquad K = B^T P, \\ J &= \int_0^\infty \left(x^T Q x + u^T u\right) dt, \end{aligned} \tag{1.2}$$

where P is the symmetric positive semidefinite solution of the continuous algebraic Riccati equation (CARE)

$$AP + PA^T - PRP = -Q, \tag{1.3}$$

with $R = BB^T \in R^{n\times n}$ and $Q \in R^{n\times n}$ are symmetric positive semidefinite matrices. Assume that the pair $(A^T, R^{1/2})$ is stabilizable. Then the above CARE has a unique symmetric positive semidefinite stabilizing solution if the pair $(A^T, Q^{1/2})$ is observable (detectable).

Besides, from [1, 6], we know that in the optimal regulator problem, the optimal cost can be written as

$$J^* = x_0^T P x_0, \tag{1.4}$$

where x_0 is the initial state of the system (1.1) and P is the symmetric positive semidefinite solution of CARE (1.3). An interpretation of $\mathrm{tr}(P)$ is that $\mathrm{tr}(P)/n$ is the average value of the cost J^* as x_0 varies over the surface of a unit sphere.

Considering these applications, deriving the solution of the CARE has become a heated topic in the recent years. However, as we all know, for one thing, the analytical solution of this equation is often computational difficult and time-consuming as the dimensions of the system matrices increase, and we can only solve some special Riccati matrix equations and design corresponding algorithms (see [7, 8]). For another, in practice, the solution bounds can also be used as approximations of the exact solution or initial guesses in the numerical algorithms for the exact solution (Barnett and Storey 1970 [9]; Patel and Toda 1984 [10]; Mori and Derese 1984 [11]; Kwon et al. 1996 [12]). Therefore, during the past two and three decades, many scholars payed attention to estimate the bounds for the solution of the continuous algebraic Riccati equation (Kwon and Pearson 1977 [13]; Patel and Toda 1978 [14]; Yasuda and Hirai 1979 [15]; Karanam 1983 [16]; Kwon et al. 1985 [17]; Wang et al. 1986 [6]; Saniuk and Rhodes 1987 [18]; Kwon et al. 1996 [12]; Lee, 1997 [19]; Choi and Kuc, 2002 [20]; Chen and Lee, 2009 [21]). The previous results during 1974-1994 have been summarized in Kwon et al. 1996 [12].

In this paper, we propose new upper and lower matrix bounds for the solution of the continuous algebraic Riccati equation. And, using the upper and lower matrix bounds we obtain the trace, the eigenvalue, and the determinant bounds. In certain cases, these lower bounds improve and extend the previous results. Finally, we give a numerical example to illustrate the effectiveness of our results.

In the following, let $R^{n\times n}$ and R^n denote the set of $n\times n$ real matrices and n-dimensional column vector. Let $X \in R^{n\times n}$ be an arbitrary symmetric matrix, then we assume that the eigenvalues of X are arranged so that $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$. For $X \in R^{m\times n}$, we assume that the singular values of X are arranged so that $\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_{\min\{m,n\}}(X)$. If $X \in R^{n\times n}$, let $\mathrm{tr}(X), X^T, X^{-1}, \det(X), \|X\|$ denote the trace, the transpose, the inverse, the determinant and the spectral norm of X, respectively. The notation $X > 0$ $(X \geq 0)$ is used to denote that X is symmetric positive definite (semidefinite). For any symmetric matrices $X, Y \in R^{n\times n}$, $X > (\geq) Y$) means that $X - Y$ is positive definite (semidefinite).

The following lemmas are used to prove the main results.

Lemma 1.1 (see [15]). *The symmetric positive semidefinite solution P to CARE* (1.3) *has the following lower bound on its minimum eigenvalue:*

$$\lambda_n(P) \geq f\left(-\lambda_n\left(\overline{A}Q^{-1}\right), \lambda_1\left(RQ^{-1}\right), 1\right) \equiv \eta, \tag{1.5}$$

with $f(a,b,c) = (-a + (a^2 + bc)^{1/2})/b$, $\overline{A} = (A + A^T)/2$.

Lemma 1.2 (see [22]). *For any symmetric matrix* $\mathrm{X} \in R^{n\times n}$, *the following inequality holds:*

$$\lambda_n(\mathrm{X})I \leq \mathrm{X} \leq \lambda_1(\mathrm{X})I. \tag{1.6}$$

Lemma 1.3 (see [23, Chapter 9, A.1.a]). *Let* $B, C \in R^{n\times n}$, *for* $i = 1, 2, \ldots, n$, *then*

$$\lambda_i(BC) = \lambda_i(CB). \tag{1.7}$$

Lemma 1.4 (see [24, page 48]). *Let* $A, B \in R^{n\times n}$ *be symmetric positive semidefinite matrices and there exist an integer* k *such that* $1 \leq k \leq n$. *Then for any index sequences* $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$, *we have*

$$\sum_{t=1}^{k} \lambda_{i_t}(A)\lambda_{n-t+1}(B) \leq \sum_{t=1}^{k} \lambda_{i_t}(AB) \leq \sum_{t=1}^{k} \lambda_{i_t}(A)\lambda_t(B). \tag{1.8}$$

Lemma 1.5 (see [24, page 49]). *Let* $A, B \in R^{n\times n}$ *be symmetric matrices and there exist an integer* k *such that* $1 \leq k \leq n$. *Then for any index sequences* $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$, *we have*

$$\sum_{t=1}^{k} \lambda_{i_t}(A) + \sum_{t=1}^{k} \lambda_{n-t+1}(B) \leq \sum_{t=1}^{k} \lambda_{i_t}(A+B) \leq \sum_{t=1}^{k} \lambda_{i_t}(A) + \sum_{t=1}^{k} \lambda_t(B). \tag{1.9}$$

Lemma 1.6 (see [25]). *Let* $A, B \in R^{n\times n}$, *for* $i = 1, 2, \ldots, n$, *then*

$$\sigma_i(A+B) \leq \sigma_i(A) + \sigma_1(B). \tag{1.10}$$

Remark 1.7. From Lemma 1.6, for $i = 1, 2, \ldots, n$, we have

$$\sigma_i(A) = \sigma_i[(A \pm B) + (\mp B)] \leq \sigma_i(A \pm B) + \sigma_1(\mp B) = \sigma_i(A \pm B) + \sigma_1(\mathrm{B}), \tag{1.11}$$

which is equivalent to

$$\sigma_i(A \pm B) \geq \sigma_i(A) - \sigma_1(B). \tag{1.12}$$

Lemma 1.8 (see [26]). *The following matrix inequality:*

$$\begin{pmatrix} W & S \\ S^T & V \end{pmatrix} > 0, \tag{1.13}$$

where $W = W^T$ and $V = V^T$, is equivalent to either

$$V > 0, \quad W - SV^{-1}S^T > 0, \tag{1.14}$$

or

$$W > 0, \quad V - S^T W^{-1} S > 0. \tag{1.15}$$

2. Lower Matrix Bounds for the Continuous Algebraic Riccati Equation

Choi and Kuc in [20] obtained the following. Assume that Q is symmetric positive definite and there exists a unique symmetric positive semidefinite solution P to CARE (1.3). Then P satisfies the following inequality:

$$P \geq \left[\varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T\right]^{1/2} = \Psi(\varepsilon, A, Q, R), \tag{2.1}$$

where ε is any positive constant such that

$$0 < \varepsilon < \left\| R + A^T Q^{-1} A \right\|^{-1}. \tag{2.2}$$

In this section, we will give new lower matrix bounds for the solution of the continuous algebraic Riccati equation which improve (2.1).

Theorem 2.1. *Assume that Q is symmetric positive definite and there exists a unique symmetric positive semidefinite solution P to CARE* (1.3). *Then P satisfies the following inequality:*

$$P \geq \left\{\varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T + \frac{1}{4}\eta^2 \lambda_n (I - \varepsilon R) I\right\}^{1/2} = \Phi_l(\varepsilon, A, Q, R), \tag{2.3}$$

where ε is any positive constant such that

$$0 < \varepsilon \leq \min\left\{\frac{\eta}{2\sigma_1(A) + \eta\lambda_1(R)}, \left\| R + A^T Q^{-1} A \right\|^{-1}\right\}, \tag{2.4}$$

and η is defined by Lemma 1.1.

Proof. By adding and subtracting $(1/\varepsilon)PP + A((1/\varepsilon)\ (I - R))^{-1}A^T$ from (1.3), we can get

$$\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\left(\frac{1}{\varepsilon}I - R\right)\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T - \frac{1}{\varepsilon}PP - A\left(\frac{1}{\varepsilon}I - R\right)^{-1}A^T + Q = 0, \tag{2.5}$$

which implies that

$$\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\left(\frac{1}{\varepsilon}I - R\right)\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T = \frac{1}{\varepsilon}PP + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}A^T - Q. \tag{2.6}$$

Applying Lemmas 1.2, 1.3, and 1.4 to (2.6) gives

$$\begin{aligned}
&\frac{1}{\varepsilon}PP + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}A^T - Q \\
&\quad = \left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\left(\frac{1}{\varepsilon}I - R\right)\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T \\
&\quad \geq \lambda_n\left\{\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\left(\frac{1}{\varepsilon}I - R\right)\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T\right\}I \\
&\quad = \lambda_n\left\{\left(\frac{1}{\varepsilon}I - R\right)\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\right\}\ I \\
&\quad \geq \lambda_n\left(\frac{1}{\varepsilon}I - R\right)\lambda_n\left\{\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\right\}I \\
&\quad = \lambda_n\left(\frac{1}{\varepsilon}I - R\right)\sigma_n^2\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]I.
\end{aligned} \tag{2.7}$$

If $Q > 0$ and ε satisfies (2.4), then

$$I - \varepsilon R - \varepsilon A^T Q^{-1} A > 0, \quad \varepsilon Q > 0. \tag{2.8}$$

Using the Schur complement formula of Lemma 1.8 we can see that the above inequalities are satisfied if and only if

$$\begin{pmatrix} \varepsilon Q & \varepsilon A \\ \varepsilon A^T & I - \varepsilon R \end{pmatrix} > 0, \tag{2.9}$$

that is,

$$\varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T > 0, \quad I - \varepsilon R > 0. \tag{2.10}$$

On the other hand, if ε satisfies (2.4), then

$$\varepsilon \leq \frac{\eta}{2\sigma_1(A) + \eta\lambda_1(R)}. \tag{2.11}$$

Further,

$$\varepsilon\left[2\sigma_1(A) + \eta\lambda_1(R)\right] \leq \eta. \tag{2.12}$$

Hence,

$$2\varepsilon\sigma_1(A) \leq \eta(1 - \varepsilon\lambda_1(R)) = \eta\lambda_n(I - \varepsilon R). \tag{2.13}$$

In terms of (1.5), we have

$$\begin{aligned} \varepsilon &\leq \frac{\eta\lambda_n(I - \varepsilon R)}{2\sigma_1(A)} = \frac{\eta}{2\sigma_1(A)\lambda_1\left[(I - \varepsilon R)^{-1}\right]} \\ &\leq \frac{\eta}{2\sigma_1\left[A(I - \varepsilon R)^{-1}\right]} \leq \frac{\lambda_n(P)}{2\sigma_1\left[A(I - \varepsilon R)^{-1}\right]}, \end{aligned} \tag{2.14}$$

which means that

$$\sigma_n(P) - \sigma_1\left[A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right] = \lambda_n(P) - \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right] \geq \frac{1}{2}\eta \geq 0. \tag{2.15}$$

Applying (1.12) and (2.15) to (2.7) gives

$$\begin{aligned} \frac{1}{\varepsilon}PP + \varepsilon A(I - \varepsilon R)^{-1}A^T - Q &\geq \frac{1}{\varepsilon}\lambda_n(I - \varepsilon R)\left\{\sigma_n(P) - A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right\}^2 I \\ &= \frac{1}{\varepsilon}\lambda_n(I - \varepsilon R)\left\{\sigma_n(P) - \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2 I \\ &= \frac{1}{\varepsilon}\lambda_n(I - \varepsilon R)\left\{\lambda_n(P) - \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2 I \\ &\geq \frac{1}{4\varepsilon}\eta^2\lambda_n(I - \varepsilon R)I. \end{aligned} \tag{2.16}$$

Thus, by (2.10), we can easily get that

$$P^2 \geq \varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T + \frac{1}{4}\eta^2 \lambda_n (I - \varepsilon R) I > 0. \tag{2.17}$$

After all, we obtain

$$P \geq \left\{ \varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T + \frac{1}{4}\eta^2 \lambda_n (I - \varepsilon R) I \right\}^{1/2} > 0. \tag{2.18}$$

This completes the proof. □

By using Theorem 2.1, we can derive the following result immediately.

Corollary 2.2. *Assume that Q is symmetric positive definite and there exists a unique symmetric positive semidefinite solution P to CARE* (1.3)*. Then P satisfies the following lower eigenvalue bounds for any ε satisfying* (2.4)*:*

$$\begin{gathered}
\lambda_i(P) \geq \max_{\varepsilon} \lambda_i(\Phi_l(\varepsilon, A, Q, R)) = \Phi_{il}^* \geq \lambda_i(\Phi_l(\varepsilon, A, Q, R)), \\
\operatorname{tr}(P) \geq \sum_{i=1}^{n} \Phi_{il}^* \geq \max_{\varepsilon} \operatorname{tr}(\Phi_l(\varepsilon, A, Q, R)) \geq \operatorname{tr}(\Phi_l(\varepsilon, A, Q, R)), \\
\det(P) \geq \prod_{i=1}^{n} \Phi_{il}^* \geq \max_{\varepsilon} \det(\Phi_l(\varepsilon, A, Q, R)) \geq \det(\Phi_l(\varepsilon, A, Q, R)).
\end{gathered} \tag{2.19}$$

2.1. Remarks and Comparisons to Results

Remark 2.3. For CARE (1.3), the condition that $n/(2\sigma_1(A) + \eta\lambda_1(R)) \geq \|R + A^T Q^{-1} A\|^{-1}$ often appears in the theory and practice. Then we can obtain simple choices of the tuning parameter ε to be ε_i^* $(i = 1, \ldots, 4)$ as in [20]. That is,

$$\begin{aligned}
\varepsilon_1^* &= \frac{1}{2}\left\| R + A^T Q^{-1} A \right\|^{-1}, \\
\varepsilon_2^* &= \arg\max_{\varepsilon} \lambda_n(\Psi(\varepsilon, A, Q, R)), \\
\varepsilon_3^* &= \frac{\lambda_{\mathrm{n}}(Q)}{\lambda_n(Q)\|R\| + \|A\|^2 + \|A\|\sqrt{\|A\|^2 + \lambda_n(Q)\|R\|}}, \\
\varepsilon_4^* &= \arg\max_{\varepsilon} \operatorname{tr}\left(\Psi^2(\varepsilon, A, Q, R)\right) \\
&= \frac{\operatorname{tr}(Q)}{\operatorname{tr}(Q)\|R\| + \operatorname{tr}(AA^T) + \sqrt{\operatorname{tr}^2(AA^T) + \operatorname{tr}(AA^T)\operatorname{tr}(Q)\|R\|}}.
\end{aligned} \tag{2.20}$$

The authors in [20] point out that, usually, ε_1^* as well as ε_4^* yields good bounds.

Remark 2.4. From Remark 2.3, since

$$L = \frac{\eta^2}{4}\lambda_n(I - \varepsilon R)I \geq 0, \tag{2.21}$$

then

$$\Phi_l^2 = \Psi^2 + L \geq \Psi^2. \tag{2.22}$$

This implies that (2.3) improves (2.1). And the additional computational effort required for (2.3) in comparison to (2.1) is the calculation of $(\eta^2/4)\lambda_n(I - \varepsilon R)I$.

Remark 2.5. It is known to us that in most of the previous results much attention had been payed to derive the bounds for the maximum eigenvalue; the minimum eigenvalue; the trace; the determinant for the exact solution of the CARE, while there have been little work focusing on the matrix solution bounds. However, matrix bounds are the most general and desirable as they can infer all other types of bounds mentioned above. The matrix bounds yields less conservative results than eigenvalue bounds in the practical applications of the solution bounds (Mori and Derese 1984 [11], Lee 1997 [19]; Kwon et al. 1996 [12]).

Remark 2.6. From Section 1, it is easy to see that even though $\lambda_n(Q) = 0$ CARE (1.3) has a unique positive definite solution. However, if $\lambda_n(Q) = 0$, most of the previous results cannot be applied or satisfy the trivial lower bound 0 which is not so significant. And, if $\lambda_n(Q) = 0$, as long as A is in the range space of $Q^{1/2}$, Lee 1997 [19], Choi and Kuc 2002 [20], and our method satisfy positive semidefinite matrix bounds of CARE (1.3). The lower matrix bounds for the CARE given in Lee 1997 [19], Choi and Kuc 2002 [20], and ours involve searching the optimal parameter values, which require much more computational efforts than the other methods.

Mori and Derese 1984 [11], Kwon et al. 1996 [12], and Chen and Lee 2009 [21] pointed out that a general comparison between any parallel bounds for the same measure is either difficult or actually impossible. However, we can make definite illustrations about the tightness in some cases as follows.

Comparison

Viewing the literatures, we know that lower matrix bounds for the solution of CARE (1.3) have been presented only in Kwon and Pearson 1977 [13], Lee 1997 [19], Choi and Kuc 2002 [20], and Chen and Lee 2009 [21]. Firstly, in Choi and Kuc 2002 [20] it has been shown that if we choose the parameter ε as $\varepsilon = (\|R\| + \alpha)^{-1}$ which yields the constraint

$$0 < \varepsilon < \left\| \|R\| + A^T Q^{-1} A \right\|^{-1} \tag{2.23}$$

such that $\alpha Q > AA^T$, then the method of Choi and Kuc 2002 [20] can always obtain a sharper lower matrix bound than that of Lee 1997 [19],

$$\Psi(\varepsilon, A, Q, R) \geq \left[\frac{1}{\|R\| + \alpha}\left(Q - \frac{1}{\alpha}AA^T\right)\right]^{1/2} = E(\alpha). \tag{2.24}$$

Secondly, in Lee 1997 [19] it has been shown that if we take the parameter α as

$$\alpha = f\left(-\|A\|^2, \lambda_n(Q), \|A\|^2\lambda_1(R)\right) \tag{2.25}$$

such that $\alpha Q > AA^T$, then the method of Lee 1997 [19] can always get a sharper lower matrix bound than that of Kwon and Pearson 1977 [13],

$$P \geq f(\|A\|, \|R\|, \lambda_n(Q))I = G, \tag{2.26}$$

where $f(a, b, c) = (-a + \sqrt{a^2 + bc})/b$, $b \neq 0$. Thirdly, as Chen and Lee 2009 [21] pointed out, it is hard to compare the sharpness of bounds of Lee 1997 [19] and Choi and Kuc 2002 [20]. Further, we will give a numerical example (Example 4.1) in Section 4 to illustrate our lower bound is tighter than Chen and Lee 2009 [21] under certain cases. Consequently, considering Remark 2.4, it is simple to see that our lower matrix bound are tighter than Chen and Lee 2009 [21] in certain cases.

3. Upper Matrix Bounds for the Continuous Algebraic Riccati Equation

In this section, we will give new upper matrix bounds for the solution of the continuous algebraic Riccati equation.

Theorem 3.1. *Assume that Q is symmetric positive definite and there exists a unique symmetric positive semidefinite solution P to CARE* (1.3)*. Then P satisfies the following inequality:*

$$P \leq \left\{\varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1}A^T + \lambda_1(I - \varepsilon R)\left\{\tau + \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2 I\right\}^{1/2} = \Phi_u(\varepsilon, A, Q, R), \tag{3.1}$$

where ε is any positive constant such that

$$0 < \varepsilon < \left\| R + A^T Q^{-1} A \right\|^{-1}, \tag{3.2}$$

$$\begin{aligned}
\tau \equiv \frac{1}{1 - \lambda_1(I - \varepsilon R)} \\
&\times \left\{ \varepsilon\lambda_1(I - \varepsilon R)\sigma_1\left[A(I - \varepsilon R)^{-1}\right] \right. \\
&\quad + \left\langle \varepsilon^2\lambda_1^2(I - \varepsilon R)\sigma_1^2\left[A(I - \varepsilon R)^{-1}\right] - [1 - \lambda_1(I - \varepsilon R)] \right. \\
&\quad\quad \left.\left. \times\left\{ \varepsilon^2\lambda_n\left[A(I - \varepsilon R)^{-1}A^T\right] - \varepsilon\lambda_1(Q) - \varepsilon^2\lambda_1(I - \varepsilon R)\sigma_1^2\left[A(I - \varepsilon R)^{-1}\right] \right\} \right\rangle^{1/2} \right\}.
\end{aligned} \tag{3.3}$$

Proof. Applying Lemmas 1.2, 1.3, 1.4, and 1.6 to (2.6) gives

$$\begin{aligned}
&\frac{1}{\varepsilon}PP + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}A^T - Q \\
&\quad = \left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\left(\frac{1}{\varepsilon}I - R\right)\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T \\
&\quad \le \lambda_1\left\{\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\left(\frac{1}{\varepsilon}I - R\right)\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T\right\}I \\
&\quad = \lambda_1\left\{\left(\frac{1}{\varepsilon}I - R\right)\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\right\}I \\
&\quad \le \lambda_1\left(\frac{1}{\varepsilon}I - R\right)\lambda_1\left\{\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]^T\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]\right\}I \\
&\quad = \lambda_1\left(\frac{1}{\varepsilon}I - R\right)\sigma_1^2\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1}\right]I \\
&\quad \le \lambda_1\left(\frac{1}{\varepsilon}I - R\right)\left\{\sigma_1(P) + \sigma_1\left[A\left(\frac{1}{\varepsilon}\mathrm{I} - R\right)^{-1}\right]\right\}^2 I \\
&\quad = \frac{1}{\varepsilon}\lambda_1(I - \varepsilon R)\left\{\lambda_1(P) + \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2 I,
\end{aligned} \tag{3.4}$$

which is equivalent to

$$P^2 \le \varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1}A^T + \lambda_1(I - \varepsilon R)\left\{\lambda_1(P) + \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2 I. \tag{3.5}$$

If $Q > 0$ and ε satisfies (3.2), from (2.10), then $I - \varepsilon R > 0$ and

$$\varepsilon Q - \varepsilon^2 A(\mathrm{I} - \varepsilon R)^{-1} A^T + \lambda_1(I - \varepsilon R)\left\{\lambda_1(P) + \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2 I \geq 0. \tag{3.6}$$

Hence, (3.5) changes to

$$P \leq \left\{\varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T + \lambda_1(I - \varepsilon R)\left\{\lambda_1(P) + \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2 I\right\}^{1/2}. \tag{3.7}$$

Introducing Lemma 1.2 to (3.5) gives

$$P^2 \leq \lambda_1\left\{\varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T + \lambda_1(I - \varepsilon R)\left[\lambda_1(P) + \varepsilon\sigma_1\left(A(I - \varepsilon R)^{-1}\right)\right]^2 I\right\} I. \tag{3.8}$$

Then

$$\lambda_1\left(P^2\right) \leq \lambda_1\left[\lambda_1\left\{\varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T + \lambda_1(I - \varepsilon R)\left[\lambda_1(P) + \varepsilon\sigma_1\left(A(I - \varepsilon R)^{-1}\right)\right]^2 I\right\} I\right]. \tag{3.9}$$

Applying Lemmas 1.4 and 1.5 to (3.9) gives

$$\lambda_1^2(P) \leq \varepsilon\lambda_1(Q) - \varepsilon^2\lambda_n\left[A(I - \varepsilon R)^{-1} A^T\right] + \lambda_1(I - \varepsilon R)\left\{\lambda_1(P) + \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2. \tag{3.10}$$

Solving (3.10) for $\lambda_1(P)$ gives

$$\begin{aligned}\lambda_1(P) \leq{} & \frac{1}{1 - \lambda_1(I - \varepsilon R)} \\ & \times\left\{\varepsilon\lambda_1(I - \varepsilon R)\sigma_1\left[A(I - \varepsilon R)^{-1}\right] + \left\langle\varepsilon^2\lambda_1^2(I - \varepsilon R)\sigma_1^2\left[A(I - \varepsilon R)^{-1}\right] - [1 - \lambda_1(I - \varepsilon R)]\right.\right. \\ & \left.\left.\times\left\{\varepsilon^2\lambda_n\left[A(I - \varepsilon R)^{-1} A^T\right] - \varepsilon\lambda_1(Q) - \varepsilon^2\lambda_1(I - \varepsilon R)\sigma_1^2\left[A(I - \varepsilon R)^{-1}\right]\right\}\right\rangle^{1/2}\right\} \\ \equiv{} & \tau.\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.7) gives

$$P \leq \left\{\varepsilon Q - \varepsilon^2 A(I - \varepsilon R)^{-1} A^T + \lambda_1(I - \varepsilon R)\left\{\tau + \varepsilon\sigma_1\left[A(I - \varepsilon R)^{-1}\right]\right\}^2 I\right\}^{1/2}. \tag{3.12}$$

This completes the proof. □

By using Theorem 3.1, we can derive the following result immediately.

Corollary 3.2. *Assume that Q is symmetric positive definite and there exists a unique symmetric positive semidefinite solution P to CARE* (1.3). *Then P satisfies the following upper eigenvalue bounds for any ε satisfying* (3.2)*:*

$$\begin{aligned}
\lambda_i(P) &\leq \min_{\varepsilon} \lambda_i(\Phi_u(\varepsilon, A, Q, R)) = \Phi_{iu}^* \\
&\leq \lambda_i(\Phi_u(\varepsilon, A, Q, R)), \\
\operatorname{tr}(P) &\leq \sum_{i=1}^{n} \Phi_{iu}^* \leq \min_{\varepsilon} \operatorname{tr}(\Phi_u(\varepsilon, A, Q, R)) \\
&\leq \operatorname{tr}(\Phi_u(\varepsilon, A, Q, R)), \\
\det(P) &\leq \prod_{i=1}^{n} \Phi_{iu}^* \leq \min_{\varepsilon} \det(\Phi_u(\varepsilon, A, Q, R)) \\
&\leq \det(\Phi_u(\varepsilon, A, Q, R)).
\end{aligned} \tag{3.13}$$

Remark 3.3. As Chen and Lee 2009 [21] pointed out, to give a general comparison between any parallel upper bounds for the same measure is either difficult or actually impossible. We also find that it is hard to compare the sharpness of our upper bounds to the parallel results.

4. A Numerical Example

Consider the following example.

Example 4.1. Let

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, \qquad R = \begin{pmatrix} 9 & 4 \\ 4 & 16 \end{pmatrix}. \tag{4.1}$$

Choose $\varepsilon = 0.0417$, then using (2.3) shows the following lower matrix bound:

$$P \geq \Phi_l(\varepsilon = 0.0417 = \varepsilon_1^*, A, Q, R) = \begin{pmatrix} 0.2279 & 0.0567 \\ 0.0567 & 0.5864 \end{pmatrix}, \tag{4.2}$$

leading to the following eigenvalue bounds:

$$\lambda_n(P) \geq 0.2192, \qquad \lambda_1(P) \geq 0.5951, \qquad \operatorname{tr}(P) \geq 0.8143, \qquad \det(P) \geq 0.1304. \tag{4.3}$$

Using Theorem 4 of Choi and Kuc 2002 [20] we obtain the following lower matrix bound:

$$P \geq \Psi(\varepsilon = 0.0417, A, Q, R) = \begin{pmatrix} 0.1962 & 0.0567 \\ 0.0567 & 0.5748 \end{pmatrix}, \tag{4.4}$$

leading to the following eigenvalue bounds:

$$\lambda_n(P) \geq 0.1879, \qquad \lambda_1(P) \geq 0.5831, \qquad \operatorname{tr}(P) \geq 0.771, \qquad \det(P) \geq 0.1095. \tag{4.5}$$

Using Theorem 2.1 of Chen and Lee 2009 [21], we have

$$P \geq P_l = \begin{pmatrix} 0.1418 & 0.1479 \\ 0.1479 & 0.4385 \end{pmatrix}, \tag{4.6}$$

leading to the following eigenvalue bounds:

$$\lambda_n(P) \geq 0.0807, \qquad \lambda_1(P) \geq 0.4996, \qquad \operatorname{tr}(P) \geq 0.5803, \qquad \det(P) \geq 0.1226. \tag{4.7}$$

Using Theorem 4 of Lee 1997 [19], we have

$$P \geq E(\alpha = 6.5) = \begin{pmatrix} 0.1865 & 0.0795 \\ 0.0795 & 0.5681 \end{pmatrix}, \tag{4.8}$$

leading to the following eigenvalue bounds:

$$\lambda_n(P) \geq 0.1706, \qquad \lambda_1(P) \geq 0.5840, \qquad \operatorname{tr}(P) \geq 0.7546, \qquad \det(P) \geq 0.0996. \tag{4.9}$$

By using the method of Kwon and Pearson 1977 [13], we can also obtain the following lower matrix bound:

$$P \geq G = \begin{pmatrix} 0.1705 & 0 \\ 0 & 0.1705 \end{pmatrix}. \tag{4.10}$$

By computation, it is obvious that

$$\Phi_l - \Psi \geq 0, \qquad \Phi_l - P_l \geq 0, \qquad \Phi_l - E \geq 0, \qquad \Phi_l - G \geq 0, \tag{4.11}$$

that is,

$$\Phi_l \geq \Psi, \qquad \Phi_l \geq P_l, \qquad \Phi_l \geq E, \qquad \Phi_l \geq G, \tag{4.12}$$

which implies that our lower bound is tighter than the parallel results for this case.

In Table 1, we summarize the above lower bounds together with the numerical results that can be obtained by other methods. From Table 1, we can see that our lower eigenvalue bounds of CARE (1.3) can be tighter than the previous results.

Using (3.1) yields to the following upper matrix bound:

$$P \leq \Phi_u(\varepsilon = 0.0176, A, Q, R) = \begin{pmatrix} 0.6493 & 0.0193 \\ 0.0193 & 0.7381 \end{pmatrix}. \tag{4.13}$$

Table 1: Numerical results of the lower eigenvalue bounds.

Method	Eigenvalue bounds
Ours ($\varepsilon = 0.0417$)	$\lambda_n(P) \geq 0.2192,\ \lambda_1(P) \geq 0.5951,\ \mathrm{tr}(P) \geq 0.8143,\ \det(P) \geq 0.1304$
Choi and Kuc 2002 [20] ($\varepsilon = 0.0417$)	$\lambda_n(P) \geq 0.1879,\ \lambda_1(P) \geq 0.5831,\ \mathrm{tr}(P) \geq 0.771,\ \det(P) \geq 0.1095$
Chen and Lee 2009 [21]	$\lambda_n(P) \geq 0.0807,\ \lambda_1(P) \geq 0.4996,\ \mathrm{tr}(P) \geq 0.5803,\ \det(P) \geq 0.1226$
Lee 1997 [19]	$\lambda_n(P) \geq 0.1706,\ \lambda_1(P) \geq 0.5840,\ \mathrm{tr}(P) \geq 0.7546,\ \det(P) \geq 0.0996$
Kwon and Pearson 1977 [13]	$\lambda_n(P) \geq 0.1705$
Yasuda and Hirai 1979 [15]	$\lambda_1(P) \geq 0.2369$
Karanam 1983 [16]	$\lambda_1(P) \geq 0.225$
Patel and Toda 1978 [14]	$\lambda_1(P) \geq 0.5613$
Kwon et al. 1985 [17]	$\mathrm{tr}(P) \geq 0.4210$
Wang et al. 1986 [6]	$\mathrm{tr}(P) \geq 0.6568$

5. Conclusion

In this paper, we have proposed new lower and upper bounds for the solution of the continuous algebraic Riccati equation (CARE). The numerical example has illustrated that in certain cases our lower bounds are tighter than the previous results.

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