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Research Article

Dynamic Tracking with Zero Variation and Disturbance Rejection Applied to Discrete-Time Systems

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The problem of signal tracking in discrete linear time invariant systems, in the presence of a disturbance signal in the plant, is solved using a new zero-variation methodology. A discrete-time dynamic output feedback controller is designed in order to minimize the $\mathscr{H}_\infty$ norm between the exogen input and the output signal of the system, such that the effect of the disturbance is attenuated. Then, the zeros modification is used to minimize the $\mathscr{H}_\infty$ norm from the reference input signal to the error signal. The error is taken as the difference between the reference and the output signal. The proposed design is formulated in linear matrix inequalities (LMIs) framework, such that the optimal solution of the stated problem is obtained. The method can be applied to plants with delay. The control of a delayed system illustrates the effectiveness of the proposed method.

1. Introduction

In a control systems theory, the design of controller using pole placement of closed loop discrete-time systems can be easily done. In [1] a controller using pole placement is used to obtain an exact plot of complementary root locus, of biproper open-loop transfer functions, using only well-known root locus rules. However, the problem of zero placement is not very much studied by the control researchers. In [2] a discrete-time pole placement is obtained by a control design technique that uses simple and multirate sample. The methodology proposed in [3] preserves the $\mathcal{H}_2$ state feedback controller optimality by pole placement in a Z plain region specified in design. In the field of discrete-time systems pole placement we find [4], where discrete adaptative controllers are designed considering arbitrary zero location. Also, in [5] a class of nonmodeled dynamics is controlled using a zero placement.

In [6] a methodology is proposed using zero and pole placement for discrete-time systems, to obtain the signal tracking and disturbance rejection, respectively. However, for the signal tracking problem, when a state feedback estimator is proposed, a modification occurs in $\mathcal{H}_\infty$-norm value obtained with the initial controller that provides the disturbance rejection. The methodology proposed in this paper has the advantage of maintaining the $\mathcal{H}_\infty$-norm value obtained with the initial controller for the signal tracking problem.

The problem of signal tracking, in the presence of disturbance signal for continuous-time plant, was solved in [7], using a zero variation methodology. A methodology with a simpler mathematic formulation is proposed in [8]. The signal tracking problem with disturbance rejection in discrete-time systems is solved by an analytic method in [9], however, the mathematic formulation is complex and a frequency selective tracking is not presented as proposed in this manuscript. In [10] the use of linear matrix inequalities (LMIs) is considered in design of controllers, filters and stability study. Also, in [11], LMIs are used for the design of a dynamic output feedback controller in order to guarantee the asymptotic stability of a continuous-time system and minimize the upper bound of a given quadratic cost function. Furthermore, the LMI formulation has been used in several engineering problems (see, e.g., [12–20]).

This manuscript proposes a formulation of a signal tracking with disturbance rejection optimization problem for discrete-time systems in the linear matrix inequalities framework, such that the optimal solution of the stated control problem is obtained. The proposed method is simpler than the other tracking techniques, and the main result is that when the problem is feasible the optimal solution is obtained with small computation effort, as the LMIs can be solved using linear programming algorithms, with polynomial convergence. The software MATLAB [21] is used to find the LMI solutions, when the problem is feasible. The control of a delayed system illustrates the effectiveness of the proposed method.

2. Statement of the Problems

Consider a controllable and observable linear time-invariant multi-input multi-output (MIMO) discrete-time system,

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) + B_w w(k), \\ y(k) &= C_1 x(k), \\ z(k) &= C_2 x(k), \quad x(0) = 0,\ k \in [0;\infty), \end{aligned} \tag{2.1}$$

where $A \in \Re^{n\times n}$, $B_u \in \Re^{n\times p}$, $B_w \in \Re^{n\times q}$, $C_1 \in \Re^{m\times n}$, $C_2 \in \Re^{m\times n}$, $x(k)$ is the state vector, $y(k)$ is the output vector, $u(k)$ is the control input and $w(k)$ is the disturbance input (exogenous input).

Problem 1. The disturbance rejection problem for discrete-time systems, using the dynamic output feedback of the system described in (2.1), is the following: minimize the upper bound

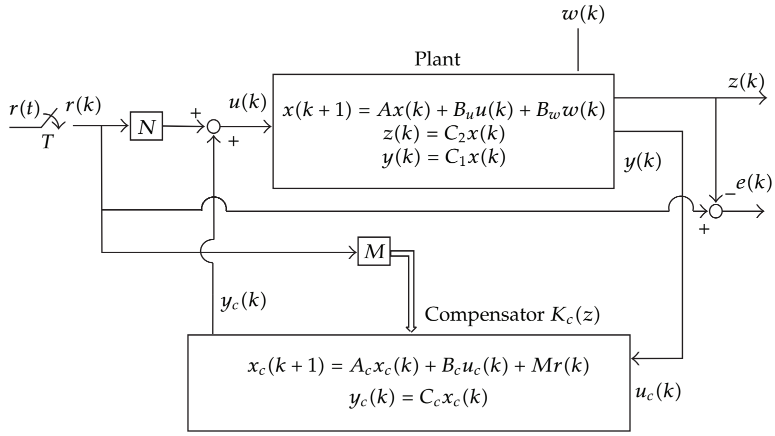

Figure 1: Discrete-time optimal control system with pole placement and zero modification.

of $\mathcal{H}_\infty$ norm from the exogenous input $w(k)$ to the output $z(k)$. In this context, the objective is to design a controller $\mathcal{H}_\infty$, $K_c(z)$, that attenuates the effect of disturbance signal in the output of the system. And in the tracking process, it is needful to design the controllers M and N that minimize the $\mathcal{H}_\infty$ norm between the reference input $r(k)$ and the tracking error $r(k) - z(k)$.

Remark 2.1. The problems of weighted disturbance reduction and weighted reference tracking are closely related to the above one and will be addressed in Section 5.

Remark 2.2. The block diagram of the control process used in this manuscript to solve Problem 1 is given in Figure 1, where $K_c(z) = C_c(zI - A_c)^{-1}B_c$ is a $\mathcal{H}_\infty$ dynamic output feedback controller, the controllers M and N were used to solve the zero variation problem in order to obtain a tracking system, $r(k)$ is the reference input signal.

The state space equation of the control system shown in Figure 1 can be written as:

$$\begin{gathered}
\begin{bmatrix} x(k+1) \\ x_c(k+1) \end{bmatrix} = \begin{bmatrix} A & B_u C_c \\ B_c C_1 & A_c \end{bmatrix} \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix} + \begin{bmatrix} B_u N \\ M \end{bmatrix} r(k) + \begin{bmatrix} B_w \\ 0 \end{bmatrix} w(k), \\
y(k) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}, \\
z(k) = \begin{bmatrix} C_2 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}, \\
e(k) = r(k) - z(k) = r(k) - \begin{bmatrix} C_2 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}.
\end{gathered} \tag{2.2}$$

Rewriting the system (2.2) in a compact form, it follows that:

$$\begin{aligned} \overline{x}(k+1) &= A_m\overline{x}_m(k) + B_m r(k) + B_n w(k), \\ e(k) &= -C_m\overline{x}(k) + D_m r(k), \\ z(k) &= C_m\overline{x}(k), \end{aligned} \tag{2.3}$$

where

$$\overline{x}_m(k) = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}, \qquad A_m = \begin{bmatrix} A & B_uC_c \\ B_cC_1 & A_c \end{bmatrix}, \qquad D_m = 1, \tag{2.4}$$

$$B_m = \begin{bmatrix} B_uN \\ M \end{bmatrix}, \qquad B_n = \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \qquad C_m = \begin{bmatrix} C_2 & 0 \end{bmatrix}. \tag{2.5}$$

Using the $\mathcal{Z}$-transform in order to solve the system (2.3), consider the initial conditions equal to zero. One obtains the transfer function between input signals (reference input and exogenous input) and the measured output of the system as showed in the following equation:

$$Z(z) = C_m(zI - A_m)^{-1}B_mR(z) + C_m(zI - A_m)^{-1}B_nW(z). \tag{2.6}$$

For the transfer function from $w(Z)$ to $z(Z)$ in (2.6), the minimization of $\mathcal{H}_\infty$ norm is obtained with the initial design of $\mathcal{H}_\infty$ controller, that implies in the minimization of the perturbation effect in to the system output.

Figure 1 shows the addition of the term $Mr(k)$ in the structure of the $K_c(z)$ controller. The purpose of the controller M is only to change the zeros of the transfer function from $r(k)$ to $u(k)$ and it does not change the poles obtained in the initial design of $K_c(z)$. The transfer function from $W(z)$ to $Z(z)$ is not changed by N or M, according to (2.4) and (2.5). In this way the performance of the $\mathcal{H}_\infty$ norm controller is not affected.

For the optimal tracking design, the relation between error signal and reference signal described in (2.7) is considered, making the perturbation signal $W(z)$ equal to zero in (2.6),

$$H_m(z) = \frac{E(z)}{R(z)} = -C_m(zI - A_m)^{-1}B_m + D_m. \tag{2.7}$$

In this case, using the zero modification one can design a tracking system that minimizes the $\mathcal{H}_\infty$ norm between the reference input $r(k)$ and the tracking error $r(k) - z(k)$. In Section 4, motivated by the work in [22], we show that M and N modify the zeros from $r(k)$ to $u(k)$. The process of the zeros modification does not interfere in the disturbance rejection. Therefore in agreement with (2.6), B_m has no influence on the transfer function from $W(z)$ to $Z(z)$. In (2.6) one uses the zeros location, by the specifications of the N and M in B_m, in the process of minimization of the $\mathcal{H}_\infty$ norm of the transfer function between the reference signal and the tracking error.

3. $\mathscr{H}_\infty$ Dynamic Output Feedback Controller Design

The following theorem leads to a new method to design the $K_c(z)$ in a LMI framework, and the goal is to attenuate the effects of exogenous signal in the output of discrete-time systems. By using [23], a pole placement constraint region with radius r and center in $(-q,0)$ is required and used in this work to provide the designer with an expedite way to keep the controller gains within appropriate bounds.

Theorem 3.1. *Consider the system* (2.1) *with dynamic output feedback by the* $\mathscr{H}_\infty$ *upper bound controller,* $K_c(z)$*. Then the optimal solution of the* $\mathscr{H}_\infty$ *norm between the input* $w(k)$ *and the output* $z(k)$*, with pole placement in a region of radius* r *and center in* $(-q,0)$ *can be obtained from the solution of the following LMI optimization problem,*

$$\|H\|_\infty^2 = \min \quad \mu$$

$$\text{s.t.} \begin{bmatrix} R & I & 0 & B_w & AR + B_uC_j & A \\ I & S & 0 & SB_w & A_j & SA + B_jC_2 \\ 0 & 0 & I & D_n & C_2R & C_2 \\ B_w' & B_w'S & D_n' & \mu I & 0 & 0 \\ RA' + C_j'B_u' & A_j' & RC_2' & 0 & R & I \\ A' & A'S + C_2'B_j' & C_2' & 0 & I & S \end{bmatrix} > 0, \tag{3.1}$$

$$\begin{bmatrix} -Rr & -Ir & AR + B_uC_j + Rq & A + Iq \\ -Ir & -Sr & A_j + Iq & SA + B_jC_1 + Sq \\ RA' + C_j'B_u' + Rq & A_j' + Iq & -Rr & -Ir \\ Iq + A' & A'S + C_1'B_j' + Sq & -Ir & -Sr \end{bmatrix} < 0 \tag{3.2}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{3.3}$$

where, $R = R' \in \Re^{n\times n}$, $S = S' \in \Re^{n\times n}$, A_j, B_j, C_j *and* D_j *in* (3.1), (3.2), *and* (3.3) *are the set of LMIs optimization variables. The radius* r *and center in* $(-q,0)$ *are the pole placement constraints illustred in Figure 2*

For solution of (3.4): $\psi E' = I - RS$, *where* ψ *and* E' *can be obtained by L-U decomposition of* $I - RS$ *[24].*

The $\mathscr{H}_\infty$ *compensator dinamic matrices* $K_c(z) = C_c(zI - A_c)^{-1}B_c + D_c$ *is obtained with the solution of equation* (3.4).

In (3.1), (3.2) *and* (3.3), *one has the following change of variables:*

$$\begin{aligned} C_j &= C_c\psi', \\ B_j &= EB_c, \\ A_j &= (SA + EB_cC_1)R + (SB_uC_c + EA_c)\psi'. \end{aligned} \tag{3.4}$$

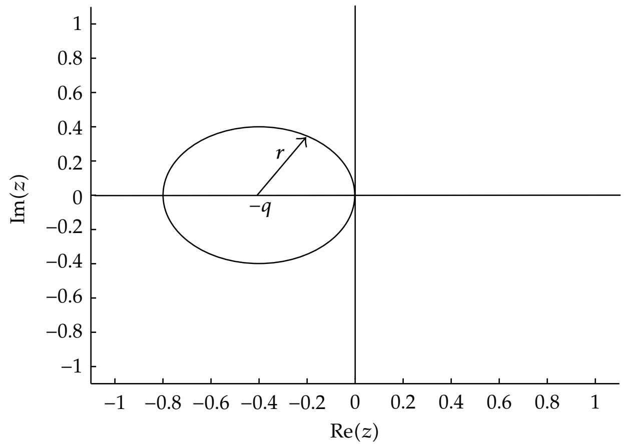

Figure 2: The pole placement region in the $\mathcal{Z}$ plane.

Proof. A realization of dynamic output feedback system R_{wz} is as followings:

$$R_{wz} = C_{\mathrm{cl}} + (zI - A_{\mathrm{cl}})^{-1} B_{\mathrm{cl}}, \tag{3.5}$$

where

$$A_{\mathrm{cl}} = \begin{bmatrix} A & B_u C_c \\ B_c C_1 & A_c \end{bmatrix}, \qquad B_{\mathrm{cl}} = \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \qquad C_{\mathrm{cl}} = \begin{bmatrix} C_2 & 0 \end{bmatrix}. \tag{3.6}$$

The optimization problem below described in LMI framework [25], is used to design the $\mathcal{H}_\infty$ compensator with pole placement constraints

$$\begin{aligned} \|H\|_\infty^2 &= \min \quad \mu \\ &\text{s.t.} \begin{bmatrix} \check{Q} & 0 & B_{\mathrm{cl}} & A_{\mathrm{cl}}\check{Q} \\ 0 & I & D & C_{\mathrm{cl}}\check{Q} \\ B'_{\mathrm{cl}} & D' & \mu I & 0 \\ \check{Q}A'_{\mathrm{cl}} & \check{Q}C'_{\mathrm{cl}} & 0 & \check{Q} \end{bmatrix} > 0, \end{aligned} \tag{3.7}$$

$$\begin{gathered} \begin{bmatrix} -r_p\check{Q} & A_{\mathrm{cl}}\check{Q} + q\check{Q} \\ \check{Q}q + \check{Q}A'_{\mathrm{cl}} & -r_p\check{Q} \end{bmatrix} < 0, \\ \check{Q} = \check{Q}' > 0, \\ \mu > 0. \end{gathered} \tag{3.8}$$

However, a high computational effort is needed to solve this problem, because the optimization problems (3.7) and (3.8) are described as a solution of BMIs. Then, using a linear transformation, the problem can be easily solved, based on LMI framework. First, the matrix $\check{Q}$ and its inverse are considered as follows

$$\check{Q} = \begin{bmatrix} R & \psi \\ \psi' & J \end{bmatrix}, \qquad \check{Q}^{-1} = \begin{bmatrix} S & E \\ E' & S \end{bmatrix}, \tag{3.9}$$

where $R = R' \in \Re^{n\times n}$, $S = S' \in \Re^{n\times n}$, and

$$\check{Q}\Gamma_2 = \Gamma_1, \quad \text{with } \Gamma_1 = \begin{bmatrix} R & I \\ \psi' & 0 \end{bmatrix}, \ \Gamma_2 = \begin{bmatrix} I & S \\ 0 & E' \end{bmatrix}. \tag{3.10}$$

The condition (3.3) is obtained by considering the Lyapunov matrix $\check{Q} > 0$ and premultiplying and postmultiplying $\check{Q}$ by Γ_2' and Γ_2, respectively. After, the condition (3.1) is obtained premultiplying and postmultiplying the inequation (3.7) by (3.11) and (3.12), respectively. Then, pre and postmultiplying (3.8) by (3.13) and (3.14), respectively, results in condition (3.2)

$$\begin{bmatrix} \Gamma_2' & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \Gamma_2' \end{bmatrix}, \tag{3.11}$$

$$\begin{bmatrix} \Gamma_2 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \Gamma_2 \end{bmatrix}, \tag{3.12}$$

$$\begin{bmatrix} \Gamma_2' & 0 \\ 0 & \Gamma_2' \end{bmatrix}, \tag{3.13}$$

$$\begin{bmatrix} \Gamma_2 & 0 \\ 0 & \Gamma_2 \end{bmatrix}. \tag{3.14}$$

The zero modification is showed in the next section. □

Remark 3.2. In this paper, the methodology adopted to solve the pole location problem affords the designer an expedite way to keep the controller gains within appropriate bounds, a key requisite for implementation purposes.

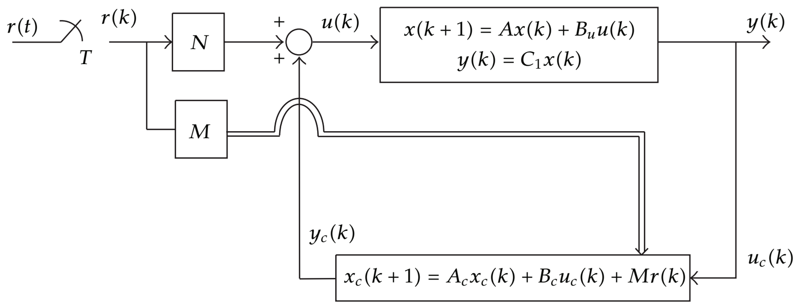

Figure 3: Zeros variation of the controlled system.

4. Zeros Variation

Motivated by the work in [22] the present paper uses the zero variation in order to obtain the global optimum of $\mathcal{H}_\infty$ norm to solve the tracking problem.

A controller that modifies the zeros from $r(k)$ to $u(k)$ is designed considering the system (A, B_u, C_1) as shown in Figure 3. The zeros of closed-loop system are allocated at arbitrary places according to the M and N values, where $M \in \Re^{n\times 1}$ and $N \in \Re^{1\times 1}$.

The plant is described by

$$\begin{aligned} x(k+1) &= Ax(k) + B_u(k), \\ y(k) &= C_1 x(k). \end{aligned} \tag{4.1}$$

The $\mathcal{Z}$ transform of (4.1) with zero initial condition, is

$$\begin{aligned} [zI - A]X(z) &= B_u U(z), \\ Y(z) &= C_1 X(z). \end{aligned} \tag{4.2}$$

A zero of the system is a value of z such that the system output is zero even with a nonzero state-and-input combination. Thus if we are able to find a nontrivial solution for $X(z_0)$ and $U(z_0)$ such that $Y(z_0)$ is null, then z_0 is a zero of the system [22]. Combining the two parts of (4.2) we must satisfy the following requirement:

$$\begin{bmatrix} z_i I - A & -B_u \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} X(z) \\ U(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.3}$$

Also, the compensator can be described as follows

$$x_c(k+1) = A_c x_c(k) + B_c u_c(k), \tag{4.4}$$

where $u_c(k) = C_1 x(k)$.

A more general method to introduce $r(k)$ is to add a term $Mr(k)$ to $x_c(k+1)$ and also a term $Nr(k)$ to the control equation $u(k) = C_c x_c(k)$, as shown in Figure 3. The controller, with these additions, becomes equal to:

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c u_c(k) + Mr(k), \\ u(k) &= C_c x_c(k) + Nr(k). \end{aligned} \tag{4.5}$$

Now, considering the controller (4.5), if there exists a transmission zero from $r(k)$ to $u(k)$, then necessarily there exists a transmission zero from $r(k)$ to $y(k)$, unless a pole of the plant cancels the zero. The equation to obtain z_i from $r(k)$ to $u(k)$ (we let $y(k) = 0$ because we are considering only the effects of $r(k)$, then $u_c(k) = 0$) in (4.5), is the following:

$$\begin{bmatrix} z_i I - A_c & -M \\ C_c & N \end{bmatrix} \begin{bmatrix} x_{co} \\ r_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.6}$$

Because the coefficient matrix in (4.6) is square, the condition for a nontrivial solution is that the determinant of this matrix must be zero. Thus we have

$$\det \begin{bmatrix} z_i I - A_c & -M \\ C_c & N \end{bmatrix} = 0. \tag{4.7}$$

Multiplying the second column of the matrix described in (4.7) at right by a nonzero matrix N^{-1} and then adding to the first column of (4.7) the product of $-C_c$ by the last column, we have:

$$\det \begin{bmatrix} z_i I - A_c + MN^{-1} C_c & -MN^{-1} \\ 0 & 1 \end{bmatrix} = 0. \tag{4.8}$$

And so, considering $z_i = z$,

$$\det\left(zI - A_c + MN^{-1} C_c\right) = 0, \tag{4.9}$$

where the modified zeros from $r(k)$ to $u(k)$ are the solutions $z = z_i$. It is important to notice that the gain N and the vector M do not only modify the system zeros but also are used to obtain the optimal solution of the tracking problem.

5. Tracking Design

The solution of the tracking problem is based on the design of the matrices of the controller M and N, that minimize the $\mathscr{H}_\infty$ norm of $(A_m, B_m, -C_m, 1)$. Weighted frequency is added in the tracking system in order to track signals in a frequency band specified in the design.

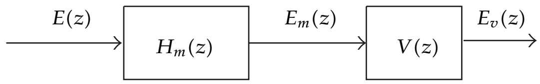

Figure 4: System structure to the tracking with frequency wheighted.

In tracking design with wheigthed frequency, the goal is to find a global solution that optimize the problem described as follows:

$$\min \|H_m(z)V(z)\|_\infty, \tag{5.1}$$

where $V(z) = (A_v, B_v, C_v, D_v)$ is a dynamic system designed to specify wheighted frequency in the output. A stable, linear and time invariant system realization $H_m = (A_m, B_m, -C_m, D_m)$ is considered as indicated in (2.7). Figure 4 illustrates the structure of inclusion of frequency wheighted in the design of tracking system.

The system (2.4) can be represented by state variables in function of $x_m(k)$ and $x_v(k)$, as follows:

$$\begin{gathered}\begin{bmatrix} x_m(k+1) \\ x_v(k+1) \end{bmatrix} = \begin{bmatrix} A_m & 0 \\ -B_v C_m & A_v \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_v(k) \end{bmatrix} + \begin{bmatrix} B_m \\ B_v D_m \end{bmatrix} r(k), \\ y_v(k) = \begin{bmatrix} 0 & C_v \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_v(k) \end{bmatrix}. \end{gathered} \tag{5.2}$$

In addition, a possible state space realization of $\check{H}_f = H_m(z)V(z)$ is:

$$\left[\begin{array}{c|c} \check{A}_f & \check{B}_f \\ \hline \check{C}_f & \check{D}_f \end{array}\right] = \left[\begin{array}{cc|c} A_m & 0 & B_m \\ -B_v C_m & A_v & B_v D_m \\ \hline 0 & C_v & 0 \end{array}\right]. \tag{5.3}$$

A metodology for the tracking design problem solution with wheigthed band is proposed in Theorem 5.1 considering the elements of the compensator matrix already fixed.

Theorem 5.1. *Considering the system with output filter* (5.3)*, if there exist a solution for the LMI described in* (5.4) *and* (5.5)*, then the gain N and vector M that minimize the $\mathcal{H}_\infty$ norm from $r(k)$ to $e(k)$ can be obtained*

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}' & Q_{22} & Q_{23} \\ Q_{13}' & Q_{23}' & Q_{33} \end{bmatrix} > 0, \tag{5.4}$$

$$\mu > 0. \tag{5.5}$$

Proof. For the tracking design with band weighted, substitute $\check{A}_f$, $\check{B}_f$, $\check{C}_f$, and $\check{D}_f$ in (5.3) by (3.7). It results in the optimization problem described in (5.4) and (5.5). The gain N and vector M are obtained from this process. The matrix Q is partitioned in the form $Q_{ij} = Q'_{ij}$, $i, j = 1, 2, 3$. $\square$

The theorem proposes that one can obtain an optimal zero location that guarantee the global optimization of the tracking error $\mathscr{H}_\infty$ norm. The proposed design is formulated in linear matrix inequalities (LMIs) framework, such that the optimal solution is obtained.

The controller M and N are the optimal solution of (5.4) and (5.5) and minimize the $\mathscr{H}_\infty$ norm between the reference input signal $r(k)$ and the tracking error signal $r(k) - e(k)$.

$$
\begin{aligned}
\|H\|_\infty^2 = \min \quad & \mu \\
\text{s.t.} \quad & \left[\begin{array}{cccccccc}
Q_{11} & Q_{12} & Q_{13} & 0 & B_u N & AQ_{11} + B_u C_c Q'_{12} & AQ_{12} + B_u C_c Q_{22} & AQ_{13} + B_u C_c Q_{23} \\
Q'_{12} & Q_{22} & Q_{23} & 0 & M & B_c C_2 Q_{11} + A_c Q'_{12} & B_c C_2 Q_{12} + A_c Q_{22} & B_c C_2 Q_{13} + A_c Q_{23} \\
Q'_{13} & Q'_{23} & Q_{33} & 0 & B_v & -B_v C_2 Q_{11} + A_v Q'_{13} & -B_v C_2 Q_{12} + A_v Q'_{23} & -B_v C_2 Q_{13} + A_v Q_{33} \\
0 & 0 & 0 & I & I & -C_v Q'_{13} & -C_v Q'_{23} & -C_v Q_{33} \\
N' B'_u & M' & B'_v & I & \mu I & 0 & 0 & 0 \\
Q_{12} B_u C'_c + Q_{11} A' & Q_{12} A'_c + Q_{11} B_c C'_2 & Q_{13} A'_v - Q_{11} C'_2 B'_v & -Q_{13} C'_v & 0 & Q_{11} & Q_{12} & Q_{13} \\
Q_{22} B_u C'_c + Q'_{12} A' & Q_{22} A'_c + Q'_{12} B_c C'_2 & Q_{23} A'_v - Q'_{12} C'_2 B'_v & -Q_{23} C'_v & 0 & Q'_{12} & Q_{22} & Q_{23} \\
Q'_{23} B_u C'_c + Q'_{13} A' & Q'_{23} A'_c + Q'_{13} B_c C'_2 & Q_{33} A'_v - Q'_{13} C'_2 B'_v & -Q_{33} C'_v & 0 & Q'_{13} & Q'_{23} & Q_{33}
\end{array}\right] > 0.
\end{aligned}
\tag{5.6}
$$

The filter has the goal of adjusting the controllers M and N for the selected frequency band. Then the zero variation described in LMI framework considers the filter dynamics to adjust the operation of the system tracking in this frequency band. However, in the implementation or simulation of the tracking process control, the filter is discarded.

The paper also proposes the statement of weighted disturbance minimization problem that is achieved based on Theorems 3.1 and 5.1 demonstrations and uses the following formulation:

$$
\begin{gathered}
\|H\|_{\infty}^{2} = \min \quad \mu \\
\text{s.t.} \quad \begin{bmatrix} R & I & 0 & B_w & AR + B_u C_j + A_{fr} + B_{fr} C_1 R & A + B_{fr} C_1 \\ I & S & 0 & S B_w & A_j & SA + B_j C_2 \\ 0 & 0 & I & 0 & C_{fr} & 0 \\ B'_w & B'_w S & 0 & \mu I & 0 & 0 \\ RA' + C'_j B'_u + A'_{fr} + R' C'_1 B'_{fr} & A'_j & C'_{fr} & 0 & R & I \\ A' + C'_1 B'_{fr} & A'S + C'_2 B'_j & C'_2 & 0 & I & S \end{bmatrix} > 0, \\
\begin{bmatrix} -Rr & -Ir & AR + B_u C_j + Rq + A_{fr} + B_{fr} C_1 R & A + Iq + B_{fr} C_1 \\ -Ir & -Sr & A_j + Iq & SA + B_j C_1 + Sq \\ RA' + C'_j B'_u + Rq + A'_{fr} + RC'_1 B'_{fr} & A'_j + Iq & -Rr & -Ir \\ Iq + A' + C'_1 B'_{fr} & A'S + C'_1 B'_j + Sq & -Ir & -Sr \end{bmatrix} < 0, \\
\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0.
\end{gathered}
\tag{5.7}
$$

The following examples illustrate the effectiveness of the proposed method.

6. Example 1

Consider a tank temperature control system described in [22], with a zero-order hold. The goal is to design a tracking system for flow control with disturbance attenuation. The model of the delayed system [22] is,

$$H(s) = \frac{e(-\lambda * s)}{\dfrac{s}{a} + 1}. \tag{6.1}$$

A sampling period of 0.01 seconds is used in the design. The parameters $a = 1$ and $\lambda = 0.005\,\mathrm{s}$ are adopted in the design.

We found $\lambda = 1 * 0.01 - 0.5 * 0.01$, and therefore, $l = 1$ and $m = 0.5$.

Using the process to find the $\mathcal{Z}$ transform of a delayed continuous-time function (6.1), the parameters are substituted and we obtain

$$H(z) = \left(1 - e^{-0.005}\right) \frac{z + \left(e^{-0.005} - e^{-0.01}\right) / \left(1 - e^{-0.005}\right)}{z\left(z - e^{-0.01}\right)}, \tag{6.2}$$

or,

$$H(z) = \frac{0.0050z + 0.0050}{z^2 - 0.9900z}. \tag{6.3}$$

The state-space description of the system is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.99 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k),$$
$$y(k) = \begin{bmatrix} 0.005 & 0.005 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{6.4}$$

where $x(k)$ is the state vector, $u(k)$ is the control signal and $w(k)$ is a disturbance signal in the system.

The design of the tracking system must include operation for reference signals of low frequencies (smaller than 0.1 rad/s). In such a case, the following filter $J(z)$ was considered

$$J(z) = \frac{(0.4500z + 0.4500) \times 10^{-7}}{z^2 - 1.9999z - 0.9999}. \tag{6.5}$$

Using Theorem 3.1 the controller $K_c(z)$ is designed for the system described in (6.4) and showed in (6.6). This controller minimizes the $\mathcal{H}_\infty$-norm of $w(k)$ to $z(k)$. In this design we obtain a disk of radius $r = 0.5$ and center in $q = 0.5$ is used as a pole placement constraint:

$$K_c(z) = \frac{-24.1395z - 4.6956}{z^2 - 0.2981z + 0.1392}. \tag{6.6}$$

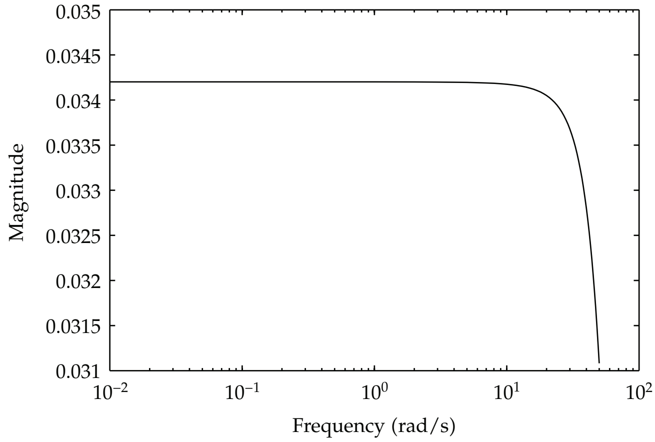

Figure 5: Frequency response of $Z(z)/W(z)$.

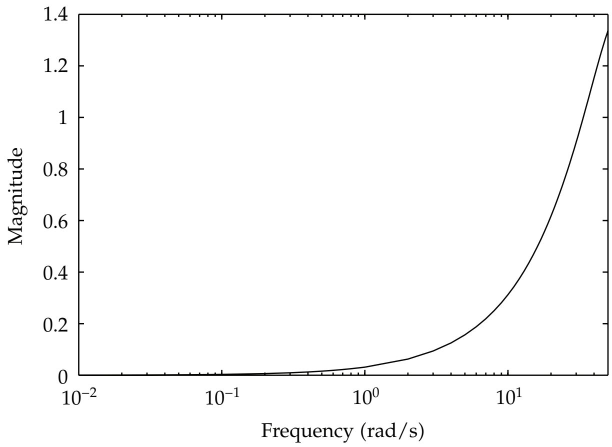

Figure 6: Frequency response of $E(z)/R(z)$.

The $\mathscr{H}_\infty$-norm of $w(k)$ to $z(k)$ for the closed-loop system is 0.0342, implying the attenuation of the effect of the disturbance signal in the system. Figure 5 illustrates the frequency response of $Z(z)/W(z)$.

To design a tracking system, the proposed zero-variation methodology given by (5.4) and (5.5) were used and the $\mathscr{H}_\infty$ norm of $r(k)$ to $e(k)$ was minimized considering the signal $r(k)$ with low frequencies (smaller than 0.1 rad/s). The obtained $\mathscr{H}_\infty$-norm for all frequency spectra was equal to 1.32, while for frequency band specified in the problem, the largest magnitude of frequency response was 3.13×10^{-3}). This implies that the tracking system operated adequately in the frequency band specified in the problem.

Figure 6 illustrates the frequency response of $E(z)/R(z)$ and one can verify that the $\mathscr{H}_\infty$-norm in frequency band follows the characteristcs of a tracking system. The M and N optimum controller values were

$$M = \begin{bmatrix} 0.0088 \\ -0.0030 \end{bmatrix}, \qquad N = 31.4617. \tag{6.7}$$

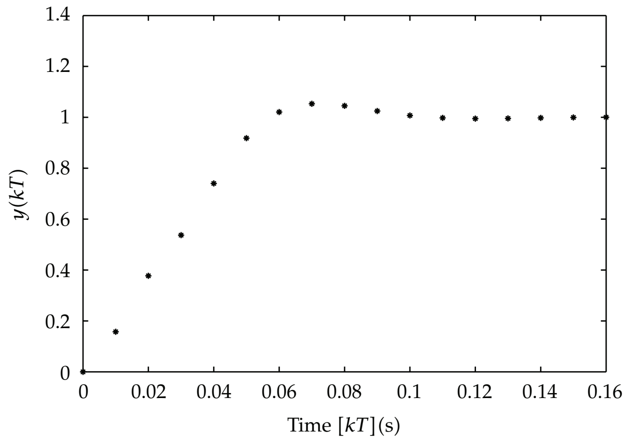

Figure 7: Response of the output $z(k)$ to a unit step input $r(k)$.

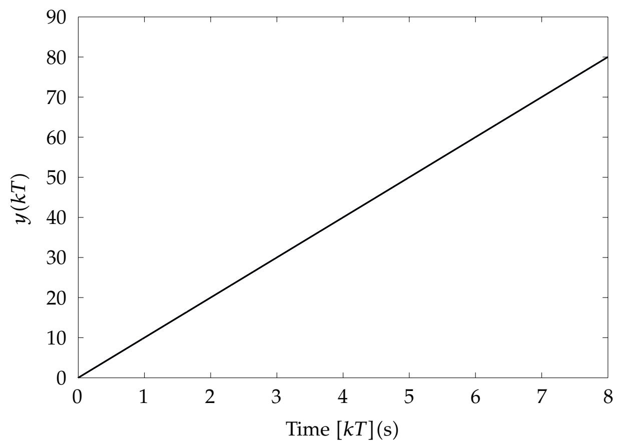

Figure 8: Output of the system for ramp input.

In the simulation, an unit step input is considered. The simulation result is illustrated in Figure 7.

In the second simulation we use the following ramp signal: $r(kT) = 0.7kT$ with $T = 0.01$ s. The simulation results are illustrated in Figure 8.

Finally, an input signal $r(kT) = \text{sen}(0.1kT)$ and a disturbance signal $w(k)$ with random amplitudes were simulated. The maximum amplitude of the random signal was equal to 1. The simulation results are illustrated in Figure 9.

For this example, the zeros of the system where -1; $0.4446+0.8004i$ and $0.4446-0.8004i$. The poles of the system with a feedback controller where $0.5004 + 0.3694i$; $0.5004 - 0.3694i$; $0.1436 + 0.2002i$ and $0.1436 - 0.2002i$. It is possible to see that the poles of the system with a feedback controller were allocated according to the circle constraint specification. The pole-zero map is shown in Figure 10.

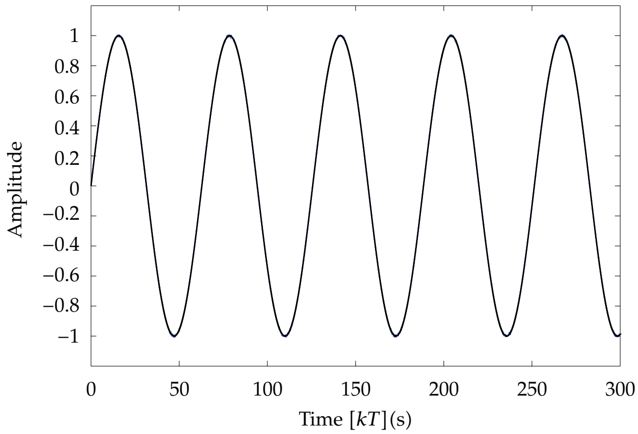

Figure 9: Output signal $z(k)$ and input signal $r(k)$ are almost overlapped.

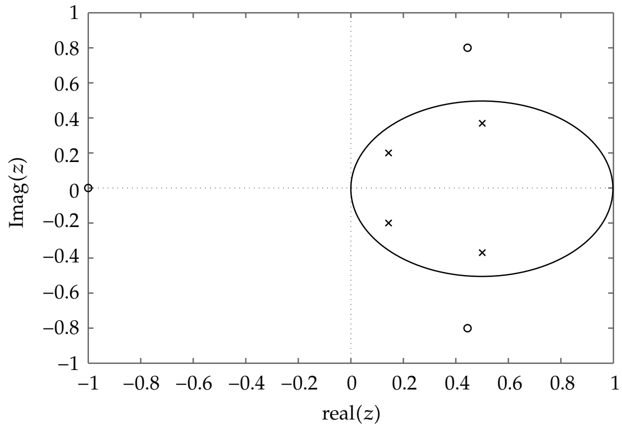

Figure 10: Pole-zero map of the closed-loop system obtained with $K_c(z)$, M and N.

The example above shows the methodology effectiveness. The disturbance rejection and the minimization of the tracking error for specified frequency band were reached. It was showed that the methodology works properly for ramp, unit step and sinusoidal signals for any frequency in the specified frequency band.

7. Example 2

Consider a discrete form of a continuous model plant that has a zero in the right-half plane. A sampling period of 0.01 seconds is used in design. The state-space description for the system is as follows:

Continuous Model:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -14 & -28 & -48 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 & -90 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}. \tag{7.1}$$

Discrete Model:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.868 & -0.263 & -0.448 \\ 0.009 & 0.999 & -0.002 \\ 0 & 0.01 & 0.899 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.0093 \\ 0 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0.0093 \\ 0 \\ 0 \end{bmatrix} w(k), \tag{7.2}$$

$$y(k) = \begin{bmatrix} 0 & 1 & -90 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix},$$

where $x(k)$ is the state vector, $u(k)$ is the control signal and $w(k)$ is a disturbance signal in the system.

The design of the tracking system must include operation for reference signals of low frequencies (down to 5 rad/s). In such a case, the filter $F(z)$ was considered

$$F(z) = \frac{\left(0.033z^2 + 0.127z + 0.031\right) \times 10^{-4}}{z^3 - 2.999z^2 + 2.805z - 0.905}. \tag{7.3}$$

Using Theorem 3.1 the controller $K_c(z)$ is designed for the system described in (7.2) and shown in (7.4). This controller minimizes the $\mathscr{H}_\infty$-norm of $w(k)$ to $z(k)$. In this design a disk of radius 0.85 and center in -0.1 is used as a pole placement constraint

$$K_c(z) = \frac{\left(2.38z^2 - 3.63z + 1.44\right) \times 10^4}{z^3 + 3.75z^2 + 15.77z + 9.79}. \tag{7.4}$$

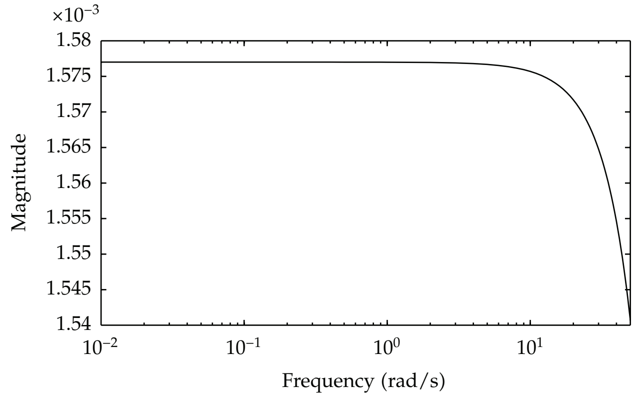

Figure 11: Frequency response of $Z(z)/W(z)$.

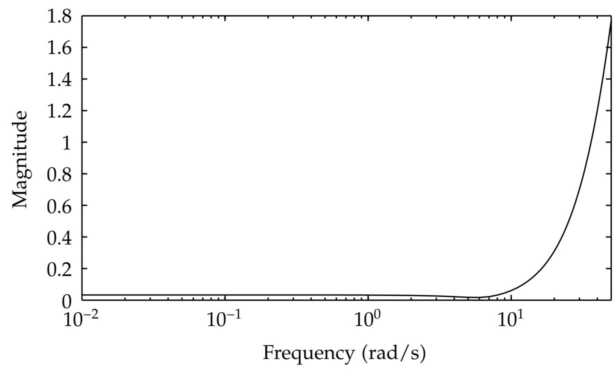

Figure 12: Frequency response of $E(z)/R(z)$.

The $\mathscr{H}_\infty$-norm of $w(k)$ to $z(k)$ for the closed-loop system was $1.577 * 10^{-3}$, implying the attenuation of the effect of the disturbance signal in the system. Figure 11 illustrates the frequency response of $Z(z)/W(z)$.

To design a tracking system, the proposed zero-variation methodology (5.4) was used in which the $\mathscr{H}_\infty$ norm of $r(k)$ to $e(k)$ is minimized considering the signal $r(k)$ with low frequencies (down to 5 rad/s). The obtained $\mathscr{H}_\infty$-norm for all frequency spectra was equal to 1.8, while for frequency band specified in the problem, the largest magnitude of frequency response was 0.031. This implies that the tracking system operated adequately in the frequency band specified in the problem.

Figure 12 illustrates the frequency response of $E(z)/R(z)$ and one can verify that the $\mathscr{H}_\infty$-norm in frequency band follows the characteristcs of a tracking system. The M and N optimum parameters values were

$$M = \begin{bmatrix} 8.5 * 10^{-7} \\ 1.87 \\ 194.6 \end{bmatrix}, \qquad N = 2746. \tag{7.5}$$

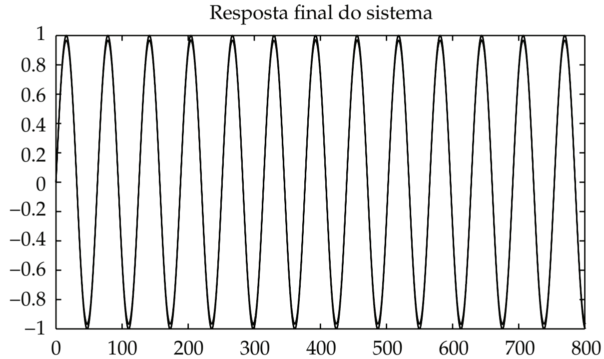

Figure 13: Output signal $y(k)$ and input signal $r(k)$ are almost overlapped.

Then, an input signal $r(kT) = \text{sen}(0, 1kT)$ and a disturbance signal $w(k)$ with random amplitudes were simulated, and it was found that the maximum amplitude of the random signal was equal to 1. The simulation results are illustrated in Figure 13.

8. Conclusion

In this manuscript, it is proposed a methodology to solve the tracking and disturbance rejection problem applied to discrete-time systems. Considering Figure 1 the disturbance signal acting in the plant can be attenuated by minimizing the $\mathscr{H}_\infty$-norm from $w(k)$ to $z(k)$, by using a dynamic feedback compensation. In the tracking process, a zero-variation methodology is used in order to minimize the $\mathscr{H}_\infty$-norm between the reference signal and tracking error signal, where the tracking error is the diference between the reference signal $r(k)$ and system output signal $z(k)$. In the tracking design with disturbance rejection, the pole placement is used to attenuate the disturbance signal effect, while the zero variation allows the tracking. The zero modification do not interfere in the design of the disturbance rejection. In the tracking process, the frequency band wheighted allows to choose the frequency band on the reference input signal. The tracking method and disturbance rejection are based on LMI framework. Then, when there exists a feasible solution the design can be obtained by convergence polynomial algorithms [23, 25] available in the literature.

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References

[1] M. C. M. Teixeira, E. Assunção, R. Cardim, N. A. P. da Silva, and E. R. M. D. Machado, "On complementary root locus of biproper transfer functions," *Mathematical Problems in Engineering*, vol. 2009, Article ID 727908, 14 pages, 2009.

[2] M. De la Sen, "Pole-placement in discrete systems by using single and multirate sampling," *Journal of the Franklin Institute B*, vol. 333, no. 5, pp. 721–746, 1996.

[3] A. Saberi, P. Sannuti, and A. A. Stoorvogel, "H_2 optimal controllers with measurement feedback for discrete-time systems: flexibility in closed-loop pole placement," *Automatica*, vol. 33, no. 3, pp. 289–304, 1997.

[4] M. M'Saad, R. Ortega, and I. D. Landau, "Adaptive controllers for discrete-time systems with arbitrary zeros: an overview," *Automatica*, vol. 21, no. 4, pp. 413–423, 1985.

[5] W. C. Messner and C. J. Kempf, "Zero placement for designing discrete-time repetitive controllers," *Control Engineering Practice*, vol. 4, no. 4, pp. 563–569, 1996.

[6] R. A. T. Mendes, *Controle Ótimo $\mathscr{H}_\infty$ com Modificação de Zeros para o Problema do Rastreamento em Sistemas Discretos usando LMI*, M.S. thesis, Unesp, São Paulo, Brazil, 2007.

[7] C. Q. Andrea, *Controle Ótimo $\mathscr{H}_2$ e $\mathscr{H}_\infty$ com Alocação de Zeros para o Problema de Rastreamento usando LMI*, M.S. thesis, UNESP, São Paulo, Brazil, 2002.

[8] E. Assunção, C. Q. Andrea, and M. C. M. Teixeira, "$\mathscr{H}_2$ and $\mathscr{H}_\infty$-optimal control for the tracking problem with zero variation," *IET Control Theory and Applications*, vol. 1, no. 3, pp. 682–688, 2007.

[9] B. M. Chen, Z. Lin, and K. Liu, "Robust and perfect tracking of discrete-time systems," *Automatica*, vol. 38, no. 2, pp. 293–299, 2002.

[10] M. C. Oliveira, *Controle de Sistemas Lineares Baseado nas Desigualdades Matriciais Lineares*, Ph.D. thesis, Unicamp, Campinas, Brazil, 1999.

[11] J. H. Park, "On design of dynamic output feedback controller for GCS of large-scale systems with delays in interconnections: LMI optimization approach," *Applied Mathematics and Computation*, vol. 161, no. 2, pp. 423–432, 2005.

[12] E. Assunção and P. L. D. Peres, "A global optimization approach for the $\mathscr{H}_2$-norm model reduction problem," in *Proceedings of the 38th IEEE Conference on Decision and Control (CDC '99)*, pp. 1857–1862, Phoenix, Ariz, USA, December 1999.

[13] E. Assunção, M. C. M. Teixeira, F. A. Faria, N. A. P. da Silva, and R. Cardim, "Robust state-derivative feedback LMI-based designs for multivariable linear systems," *International Journal of Control*, vol. 80, no. 8, pp. 1260–1270, 2007.

[14] M. C. M. Teixeira, E. Assunção, and R. G. Avellar, "On relaxed LMI-based designs for fuzzy regulators and fuzzy observers," *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 5, pp. 613–623, 2003.

[15] M. C. M. Teixeira, E. Assunção, and R. M. Palhares, "Discussion on: "$\mathscr{H}_\infty$ output feedback control design for uncertain fuzzy systems with multiple time scales: an LMI approach"," *European Journal of Control*, vol. 11, no. 2, pp. 167–169, 2005.

[16] M. C. M. Teixeira, E. Assunção, and H. C. Pietrobom, "On relaxed LMI-based design for fuzzy regulators and fuzzy observers," in *Proceedings of the 6th European Control Conference*, pp. 120–125, Porto, Portugal, 2001.

[17] E. Teixeira, E. Assunção, and R. G. Avelar, "Design of SPR systems with dynamic compensators and output variable structure control," in *Proceedings of the International Workshop on Variable Structure Systems*, vol. 1, pp. 328–333, Alghero, Italy, 2006.

[18] F. A. Faria, E. Assunção, M. C. M. Teixeira, R. Cardim, and N. A. P. da Silva, "Robust state-derivative pole placement LMI-based designs for linear systems," *International Journal of Control*, vol. 82, no. 1, pp. 1–12, 2009.

[19] R. Cardim, M. C. M. Teixeira, E. Assunção, and M. R. Covacic, "Variable-structure control design of switched systems with an application to a DC-DC power converter," *IEEE Transactions on Industrial Electronics*, vol. 56, no. 9, pp. 3505–3513, 2009.

[20] F. A. Faria, E. Assunção, M. C. M. Teixeira, and R. Cardim, "Robust state-derivative feedback LMI-based designs for linear descriptor systems," *Mathematical Problems in Engineering*, vol. 2010, Article ID 927362, 15 pages, 2010.

[21] P. Gahinet, A. Nemirovsk, A. J. Laub, and M. Chiali, *LMI Control Toolbox User's Guide*, The Mathworks Inc., Natick, Mass, USA, 1995.

[22] G. F. Franklin, J. D. Powell, and M. L. Workman, *Digital Control of Dynamic Systems*, Addison Wesley, New York, NY, USA, 2nd edition, 1990.

[23] M. Chilali and P. Gahinet, "$\mathscr{H}_\infty$ design with pole placement constraints: an LMI approach," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 358–367, 1996.

[24] E. L. Lima, *Albebra Linear*, Coleção Matteméatica Universitéaria, IMPA, Rio de Janeiro, Brazil, 4th edition, 2000.

[25] R. M. Palhares, R. H. C. Takahashi, and P. L. D. Peres, "$\mathscr{H}_\infty$ and Ótimo $\mathscr{H}_2$ guaranteed costs computation for uncertain linear systems," *International Journal of Systems Science*, vol. 28, no. 2, pp. 183–188, 1997.