Research Article

Signal Processing and Sampling Method for Obtaining Time Series Corresponding to Higher Order Derivatives

Andreea Sterian[1] and Alexandru Toma[2]

[1] *Department of Physics, Politehnica University, 313 Spl. Independentei, 060042 Bucharest, Romania*
[2] *Department of Informatics, Titu Maiorescu University, 187 Cl. Vacaresti, 040051 Bucharest, Romania*

Correspondence should be addressed to Alexandru Toma, vimarelectro@yahoo.com

For modeling and controlling dynamic phenomena it is important to establish with higher accuracy some significant quantities corresponding to the dynamic system. For fast phenomena, such significant quantities are represented by the derivatives of the received signals. In case of advanced computer modeling, the received signal should be filtered and converted into a time series corresponding to the estimated values for the dynamic system through a sampling procedure. This paper will show that present-day methods for computing in a robust manner the first derivative of a received signal (using an oscillating system working on a limited time interval and a supplementary differentiation method) can be extended to the robust computation of higher order derivatives of the received signal by using a specific set of second-order oscillating systems (working also on limited time intervals) so as estimative values for higher-order derivatives are to be directly generated (avoiding the necessity of additional differentiation or amplifying procedures, which represent a source of supplementary errors in present-day methods).

1. Introduction

For modeling and controlling dynamic phenomena it is important to establish with higher accuracy some physical quantities corresponding to the dynamic system. Usually this procedure is based on signal processing method applied upon the signal received from the dynamic system, implying some filtering methods (for noise rejection). In case of advanced computer modeling, the filtered signal should be converted further into a time series corresponding to the estimated values for the dynamic system through a sampling procedure. Many times these filtering and sampling devices consist of lowpass filters represented by asymptotically stable systems, the sampling moment of time being set after the transient regime of the filtering device has passed.

However, for fast phenomena, significant quantities are represented by the derivatives of the received signals. Usually the derivatives of a received signal $y(t) = z(t) + n(t)$ (where $z(t)$ represents the useful part of the signal, and $n(t)$ represents the noise) are computed by filtering the received signal and dividing the difference between the filtered values $u(t_k)$ of the signal at two consecutive sampling moments of time by the time difference between these time moments (for first order derivative), by dividing the difference between the values of first-order derivative by the same time difference (for the second-order derivative), and so on. Yet this method requires very good filtering properties, while any difference in sampled values can drastically affect the estimation for the derivative.

The average value of the first derivative can be approximated by

$$\frac{z(t_k) - z(t_{k-1})}{t_k - t_{k-1}} = \frac{z(t_k) - z(t_{k-1})}{\Delta t} \tag{1.1}$$

and can be estimated by the mathematical operation

$$\frac{u(t_k) - u(t_{k-1})}{t_k - t_{k-1}} = \frac{u(t_k) - u(t_{k-1})}{\Delta t}, \tag{1.2}$$

where $u(t_k)$ represents the filtered values of the received signal $y(t)$ (as has been shown). In the ideal case, $u(t_k)$ should be equal to $z(t_k)$. This cannot be achieved. For avoiding significant errors the difference $z(t_k) - z(t_{k-1})$ should be estimated with higher accuracy. This implies that the filtered values $u(t_k)$ should be close to the values of the useful part of the received signal $z(t_k)$.

As a consequence, the filtering device should reject the noise (supposed to present fast variation as compared to the variations of the useful part $z(t)$) in a significant manner. For this purpose, the filtering and sampling devices based on asymptotically stable systems can be improved. They have the transfer function

$$H(s) = \frac{1}{T_0 s + 1} \tag{1.3}$$

(for a first-order system) and

$$H(s) = \frac{1}{T_0^2 s^2 + 2bT_0 s + 1} \tag{1.4}$$

(for a second-order system). They attenuate an alternating signal of angular frequency $\omega \gg \omega_0 = 1/T_0$ (corresponding to noise) about ω/ω_0 times (for a first-order system) or about $(\omega/\omega_0)^2$ times (for a second-order system). The response time of such systems at a c useful signal is about $5T_0$ ($5T_0$ for the first-order system and $4T_0/b$ for the second-order system). If the signal generated by the first- or second-order system is integrated over such a period, a supplementary attenuation for the alternating signal of about $5\omega/\omega_0$ can be obtained.

However, such structures are very sensitive at the random variations of the integration period (for unity-step input, the signal, which is integrated, is equal to unity at the sampling moment of time). Even if we use oscillators with a very high accuracy, such random variations will appear due to the fact that the integration is performed by an electric current charging a capacitor. This capacitor must be charged at a certain electric charge Q necessary for further conversions; this electric charge cannot be smaller than a certain value Q_{lim}, while it has to supply a minimum value I_{min} for the electric current necessary for conversions on the time period t_{conv} required by these conversions, the relation

$$Q_{\text{lim}} = I_{\text{min}} t_{\text{conv}} \tag{1.5}$$

being valid. So the minimum value $I_{\text{int}}(\text{min})$ for the electric current charging the capacitor in the integrator system is determined by the relation

$$I_{\text{int}}(\text{min}) = \frac{Q_{\text{lim}}}{t_{\text{int}}}, \tag{1.6}$$

where t_{int} is the integration period required by the application (knowing the sampling frequency f_s, we can approximately establish t_{int} using the relation $t_{\text{int}} = 1/f_s$). So the current charging the capacitor cannot be less than a certain value; thus random variations of the integration period will appear due to the fact that the random phenomena are generated when a nonzero electric current is switched off.

2. Specific Aspects of Using Oscillating Systems for Filtering the Received Signal

The disadvantage of using asymptotically stable systems (previously mentioned) can be avoided by using an oscillating second-order system having the transfer function

$$H_{\text{osc}} = \frac{1}{T_0^2 s^2 + 1} \tag{2.1}$$

working on the time interval $[0, 2\pi T_0]$ (see [1] for more details). For initial conditions equal to zero, the response of the oscillating system at a step input with amplitude A will have the form

$$w(t) = A\left(1 - \cos\left(\frac{t}{T_0}\right)\right). \tag{2.2}$$

By integrating this result on the time interval $[0, 2\pi T_0]$, we obtain the result $u = 2\pi A T_0$, and we can also notice that the quantity which is integrated and its slope are equal to zero at the end of the integration period. Thus the influence of the random variations of the integration period (generated by the switching phenomena) is practically rejected. Analyzing the influence of the oscillating system upon an alternating input, we can observe that the oscillating system attenuates about $(\omega/\omega_0)^2$ times such an input. The use of the integrator

leads to a supplementary attenuation of about $(2\pi)(\omega/\omega_0)$ times. The oscillations having the form

$$w_{\text{osc}} = a\sin(\omega_0 t) + b\cos(\omega_0 t) \tag{2.3}$$

generated by the input alternating component have a lower amplitude and generate a null result after an integration over the time interval $[0, 2\pi T_0]$. As a conclusion, such a structure provides practically the same performances as a structure consisting of an asymptotically stable second-order system and an integrator (response time of about $6T_0$, an attenuation of about $6(\omega/\omega_0)^3$ times for an alternating component having frequency ω) moreover being less sensitive at the random variations of the integration period. It is the most suitable for the operation

$$\frac{u(t_k) - u(t_{k-1})}{\Delta t}, \tag{2.4}$$

where $\Delta t = t_k - t_{k-1}$. The difference $u(t_k) - u(t_{k-1})$ can be further divided by a constant value (corresponding to Δt) so as to estimate the first-order derivative $z(t_k) - z(t_{k-1})$. For restoring the initial null conditions after the sampling procedure (at the end of the working period) some electronic devices must be added (see [2] for more details).

However, this method presents a major disadvantage: the filtering devices can generate an electronic voltage corresponding to $u(t_k)$ within a certain range (less than 10 Volts, usually). This means that the difference $u(t_k) - u(t_{k-1})$ would correspond to a small voltage, implying the necessity of amplifying this voltage so as to achieve a result in a certain range (suitable for modeling, controlling, and data acquisition). This represents a supplementary source of errors (the resolution being limited by the resolution of the operation $u(t_k) - u(t_{k-1})$). It implies the necessity of using an oscillating second-order system so as the result of the integration on a working interval corresponds to the derivative of the useful part $z(t)$ of the received signal $y(t)$ (if possible).

3. Analog Signal Processing Methods Suitable for Derivative Procedures

A general mathematical method for obtaining the derivatives of the useful part $z(t)$ of a received signal $y(t)$ in a robust manner (with good filtering properties and also with a good resolution, avoiding a supplementary amplification of the difference between two previously sampled values) consists in using a signal processing device with the transfer function $H(s)$

$$H(s) = \frac{s^N}{\left(T_1^2 s^2 + 1\right)\left(T_2^2 s^2 + 1\right) \cdots \left(T_k^2 s^2 + 1\right) \cdots \left(T_{n+1}^2 s^2 + 1\right)}, \tag{3.1}$$

which can be also written as

$$H(s) = s^N \prod_{k=1}^{n+1} \frac{1}{\left(T_k^2 s^2 + 1\right)}. \tag{3.2}$$

The degree of the denominator polynomial must be greater or equal to the degree of numerator polynomial so as the transfer function is to be implemented using electronic devices; this means that $N \leq 2(n+1)$ (as can be easily noticed).

The output of this signal processing system for an input corresponding to $y(s) = 1/s^{N+1}$ (the Laplace transformation of the time function $y(t) = (1/N!)t^N$) is represented by

$$w(s) = \frac{1}{s^{N+1}} s^N \frac{1}{(T_1^2 s^2 + 1)} \frac{1}{(T_2^2 s^2 + 1)} \cdots \frac{1}{(T_n^2 s^2 + 1)}, \tag{3.3}$$

which can be also written as

$$w(s) = \frac{1}{s} \frac{1}{(T_1^2 s^2 + 1)} \frac{1}{(T_2^2 s^2 + 1)} \cdots \frac{1}{(T_n^2 s^2 + 1)} \tag{3.4}$$

or

$$w(s) = \frac{1}{s} \prod_{k=1}^{n+1} \frac{1}{(T_k^2 s^2 + 1)}, \tag{3.5}$$

which represents a set of multiplication of a unity step input $1/s$ by transfer functions $1/(T_k^2 s^2 + 1)$.

This means that it can be written as

$$w_1(s) = \frac{1}{(T_1^2 s^2 + 1)} \frac{1}{s},$$

$$w_2(s) = \frac{1}{(T_2^2 s^2 + 1)} w_1(s), \tag{3.6}$$

$$w_3(s) = \frac{1}{(T_3^2 s^2 + 1)} w_2(s)$$

or (in a general form)

$$w_{k+1}(s) = \frac{1}{(T_{k+1}^2 s^2 + 1)} w_k(s) \tag{3.7}$$

until

$$w_n(s) = \frac{1}{(T_{n-1}^2 s^2 + 1)} w_{n-1}(s). \tag{3.8}$$

As was shown in previous paragraph, the function $w_1(t)$ will be represented by a unity step function and by an alternating function $-\cos(t/T_1)$ (which can be also written as $\sin(t/T_1 + \phi_1) = \sin(\omega_1 t + \phi_1)$, where the angular frequency $\omega_1 = 1/T_1$). This represents the input for

the transfer function $1/(T_2^2 s^2 + 1)$. Its output w_2 will be represented by the sum of the output of this transfer function for a unity step input $1/s$ and the output of this transfer function for the input $\sin(t/T_1 + \phi_1) = \sin(\omega_1 t + \phi_1)$. The output generated by the unity step input will be represented once again by an unity step function and by an alternating function with angular frequency $\omega_1 = 1/T_1$; the output generated by this transfer function for the alternating input with angular frequency ω_1 will be represented by a sum of two alternating functions with angular frequencies ω_1 and ω_2, respectively. It results that w_2 can be written as

$$w_2(t) = 1 + a_1^{(2)} \sin\left(\omega_1 t + \phi_1^{(2)}\right) + a_2^{(2)} \sin\left(\omega_2 t + \phi_2^{(2)}\right) \tag{3.9}$$

(there are just three terms because alternating functions with the same angular frequency ω_1 were grouped together in $a_1^{(2)} \sin(\omega_1 t + \phi_1)$).

The whole procedure can continue by analyzing the output w_3 of the transfer function $1/(T_3^2 s^2 + 1)$ for the input represented by $w_2(t)$. It results that a unity step output $1/s$ will appear once again, together with three alternating components with angular frequencies ω_1, ω_2, and ω_3, respectively. This means that $w_3(t)$ can be written as

$$w_3(t) = 1 + a_1^{(3)} \sin\left(\omega_1 t + \phi_1^{(3)}\right) + a_2^{(3)} \sin\left(\omega_2 t + \phi_2^{(3)}\right) + a_3^{(3)} \sin\left(\omega_3 t + \phi_3^{(3)}\right), \tag{3.10}$$

where $\omega_3 = 1/T_3$. In the general form, the output $w_k(t)$ can be written as

$$w_k(t) = 1 + a_1^{(k)} \sin\left(\omega_1 t + \phi_1^{(k)}\right) + \cdots + a_k^{(k)} \sin\left(\omega_k t + \phi_k^{(k)}\right) \tag{3.11}$$

or

$$w_k(t) = 1 + \sum_{i=1}^{k} a_i^{(k)} \sin\left(\omega_i t + \phi_i^{(k)}\right) \tag{3.12}$$

(it can be noticed that the coefficient and phase corresponding to a certain angular frequency ω_i are changed from $a_i^{(k-1)}, \phi_i^{(k-1)}$ to $a_i^{(k)}, \phi_i^{(k)}$ at each step due to the mixture of alternating functions with the same angular frequency ω_i generated by the transfer function $1/(T_k^2 + 1)$ for input represented by a sum of the unity step function and by alternating functions with angular frequencies ω_j, $j < i$ (as was shown for $w_2(t)$, where certain functions with the same angular frequency were grouped together)). Finally, the output of the signal processing system $w(t)$ (at step $n + 1$) will be represented by function w_n which can be written as

$$w_{n+1}(t) = 1 + a_1^{(n+1)} \sin\left(\omega_1 t + \phi_1^{(n+1)}\right) + \cdots + a_{n+1}^{(n+1)} \sin\left(\omega_{n+1} t + \phi_k^{(n+1)}\right) \tag{3.13}$$

or

$$w(t) = w_{n+1} = 1 + \sum_{i=1}^{n+1} a_i^{(n+1)} \sin\left(\omega_i t + \phi_i^{(n+1)}\right). \tag{3.14}$$

By integrating this function $w(t)$ on a time interval T represented by a multiple of all time periods $2\pi T_1, 2\pi T_2, \ldots, 2\pi T_n$ (this means that any ratio T_i/T_j should be expressed by a rational number) the influence of all alternating components vanishes. As a consequence, the result $u(T)$ of the integration will be

$$u(T) = 2\pi T \tag{3.15}$$

for a received signal corresponding to $y(t) = (1/N!)t^N$ which has been processed by $H(s)$. This means that the result $u(T)$ is proportional to the derivative of order N of the received signal (supposed to have the form $y(t) = (1/N!)t^N$).

The analysis of the action of transfer function $H(s)$ upon an input represented by

$$y(t) = \frac{1}{s^M}, \quad M < N \tag{3.16}$$

(corresponding to the time function $y(t) = (1/(N-M)!)t^{N-M}$) can be considered as

$$w(s) = H(s)\frac{1}{s^M} = H(s)s^{N-M}\frac{1}{s^N} = s^{N-M}H(s)\frac{1}{s^N}. \tag{3.17}$$

According to Laplace transformation properties, the operator s^{N-M} corresponds to a derivative procedure applied $(N-M)$ times upon a certain function. Since $H(s)1/s^N$ corresponds to the sum of a step function and a set of alternating components previously presented, it results that the output $w(t)$ corresponding to the input $1/s^{N-M}$ can be represented as

$$w(t) = \frac{d^{N-M}}{dt^{N-M}}\left(1 + \sum_{i=1}^{n+1} a_i^{(n+1)} \sin\left(\omega_i t + \phi_i^{(n+1)}\right)\right). \tag{3.18}$$

But the derivatives of a constant function equal zero, and the derivatives of alternating functions of certain angular frequency are represented also by alternating functions with the same angular frequency. This means that the integration of this function on the time interval T represented by a multiple of all time periods will generate a null result. As a consequence, if the received signal $y(t)$ can be written as a sum

$$y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_N t^N \tag{3.19}$$

(according to properties of Taylor series), the action of a transfer function $H(s)$ is represented by

$$H(s) = s^N \prod_{k=1}^{n+1} \frac{1}{(T_k^2 s^2 + 1)}. \tag{3.20}$$

The integration of the output $w(t)$ of this filtering device on a time interval T represented by a multiple of all time periods T_i will generate a result proportional to $N!c_N$, being proportional

to the derivative of order N of the received signal. Thus this signal processing method based on a set of oscillating second-order systems and an integrator can generate a sequence of sampling values corresponding to the Nth derivative of the received signal at the end of each working interval, the derivative of order N being transformed into a time series suitable for modeling, control, and/or data acquisition. Filtering properties are still good, as long as any angular frequency corresponding to noise ($n(t)$ is several times greater than any angular frequencies ω_i from the set $\omega_1, \ldots, \omega_{n+1}$.

We must point the fact that certain limitations appear as the order N of the derivative to be estimated increases.

(i) For a great number of alternating functions, it is quite possible for the maximum value of the sum of alternating functions of angular frequencies ω_i (part of $w(t)$) to become several times greater than the constant part of $w(t)$ (corresponding to the derivative to be estimated); so the resolution of the method decreases (the voltage range of estimated derivative decreases since it represents a small part of the maximum voltage allowed by electronic devices).

(ii) Taylor series for $y(t)$ was restricted to $(N + 1)$ terms. This means that the influence of derivatives of order $K > N$ on the integration period T was neglected. This approximation should be carefully checked in any signal analysis.

4. Conclusions

This paper has presented a possibility of obtaining the derivatives of the received electrical signal using a filtering device consisting of a sequence of certain oscillating second-order systems and an integrator. The oscillating systems are working on a time period for filtering a received electrical signal, with initial null conditions. The output of this system is integrated over a time period corresponding to a multiple of all time periods of the second-order systems which are part of the signal processing device (at the end of this period the integrated signal being sampled). The influence of all alternating components is rejected due to the integration performed on a multiple of all time periods, and thus the final result corresponds to the integration of a constant function which is proportional to the derivative having to be estimated. The proposed method has shown that present-day methods for computing in a robust manner the first derivative of a received signal (using an oscillating system working on a limited time interval and a supplementary differentiation method) can be extended to the robust computation of higher-order derivatives of the received signal by using a specific set of second-order oscillating systems (working also on limited time intervals) so as estimative values for higher order derivatives are to be directly generated (avoiding the necessity of additional differentiation or amplifying procedures, which represent a source of supplementary errors in present-day methods). It can be used for decreasing the phase delay for signal processing methods, but without using a weighted sum of real and filtered derivatives of the received signal, as in [3]. The proposed method is similar to other attempts for computing the derivative without using additional procedures (see also [4] where the need of sampled and digitised data is avoided).

In future studies, the analysis will continue by trying to use nonlinear dynamical equations able to generate practical test functions for estimating in robust manner and with greater accuracy the derivatives of the signal transmitted by dynamic systems (see [5] for general properties of nonlinear differential equations able to generate practical test functions). The results presented in this paper can be extended for modeling phenomena described by partial differential equations as traveling waves and wavelets inside certain

materials [6, 7] or as general transformations of waves when the material reference system is changed (see [8] for classical field of interaction and [9] for quantum field of interaction) due to advantages presented by the accurate estimation of higher-order derivatives. It could be also used for integrating derivative procedures in procedures of analyzing time series (as presented in [10–12]) or directly into machine learning algorithms or emergent dynamic routing by establishing certain sampling moments (see [13, 14]).

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