Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 934714, 32 pages
doi:10.1155/2010/934714

Review Article

Modeling of Ship Roll Dynamics and Its Coupling with Heave and Pitch

R. A. Ibrahim and I. M. Grace

Department of Mechanical Engineering, Wayne State University, Detroit, MI 48202, USA

Correspondence should be addressed to R. A. Ibrahim, ibrahim@eng.wayne.edu

Received 14 May 2009; Accepted 11 June 2009

Academic Editor: Jose Balthazar

In order to study the dynamic behavior of ships navigating in severe environmental conditions it is imperative to develop their governing equations of motion taking into account the inherent nonlinearity of large-amplitude ship motion. The purpose of this paper is to present the coupled nonlinear equations of motion in heave, roll, and pitch based on physical grounds. The ingredients of the formulation are comprised of three main components. These are the inertia forces and moments, restoring forces and moments, and damping forces and moments with an emphasis to the roll damping moment. In the formulation of the restoring forces and moments, the influence of large-amplitude ship motions will be considered together with ocean wave loads. The special cases of coupled roll-pitch and purely roll equations of motion are obtained from the general formulation. The paper includes an assessment of roll stochastic stability and probabilistic approaches used to estimate the probability of capsizing and parameter identification.

1. Introduction

Generally, ships can experience three types of displacement motions (heave, sway or drift, and surge) and three angular motions (yaw, pitch, and roll) as shown in Figure 1. The general equations of motion have been developed either by using Lagrange's equation (see, e.g., [1–4]) or by using Newton's second law (see, e.g., [5–7]). In order to derive the hydrostatic and hydrodynamic forces and moments acting on the ship, two approaches have been used in the literature. The first approach utilizes a mathematical development based on a Taylor expansion of the force function (see, e.g., [8–12]). The second group employs the integration of hydrodynamic pressure acting on the ship's wetted surface to derive the external forces and moments (see, e.g., [13–18]). Stability against capsizing in heavy seas is one of the fundamental requirements in ship design. Capsizing is related to the extreme motion of the ship and waves. Of the six motions of the ship, the roll oscillation is the most critical motion that can lead to the ship capsizing. For small angles of roll motions, the response of ships can be described by a linear equation. However, as the amplitude of oscillation increases,

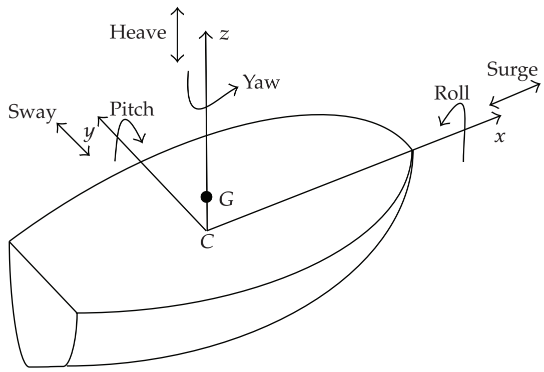

Figure 1: Ship schematic diagram showing the six degrees of freedom.

nonlinear effects come into play. Nonlinearity can magnify small variations in excitation to the point where the restoring force contributes to capsizing. The nonlinearity is due to the nature of restoring moment and damping. The environmental loadings are nonlinear and beyond the control of the designer. The nonlinearity of the restoring moment depends on the shape of the righting arm diagram.

Abkowitz [19] presented a significant development of the forces acting on a ship in surge, sway, and yaw motions. He used Taylor series expansions of the hydrodynamic forces about a forward cruising speed. The formulation resulted in an unlimited number of parameters and can model forces to an arbitrary degree of accuracy. Thus, it can be reduced to linear and extended to nonlinear equations of motion. Later, Abkowitz [20, 21], Hwang [22], and Källström and Åström [23] provided different approaches to estimate the coefficients of these models. Son and Nomoto [24] extended the work of Abkowitz [19] to include ship roll motion in deriving the forces and moments acting on the ship. Ross [25] developed the nonlinear equations of motion of a ship maneuvering through waves using Kirchhoff's [26] convolution integral formulation of the added mass. Kirchhoff's [26] equations are a set of relations used to obtain the equations of motion from the derivatives of the system kinetic energy. They are special cases of the Euler-Lagrange equations. The derived equations also give the Coriolis and centripetal forces [27, 28].

Rong [29] considered some problems of weak and strong nonlinear sea loads on floating marine structures. The weak nonlinear problem considers hydrodynamic loads on marine structures due to wave-current-body interaction. The strong nonlinear problem considers slamming loads acting on conventional and high-speed vessels. Theoretical and numerical methods to analyze wave-current interaction effects on large-volume structure were developed. The theory is based on matching a local solution to a far-field solution. It is known that large-amplitude ship motions result in strongly nonlinear, even chaotic behavior [30]. The current trends toward high-speed and unique hull-form vessels in commercial and military applications have broadened the need for robust mathematical approaches to study the dynamics of these innovative ships.

Various models of roll motion containing nonlinear terms in damping and restoring moments have been studied by many researchers [31–33]. Bass and Haddara [34, 35] considered various forms for the roll damping moment and introduced two techniques to identify the parameters of the various models together with a methodology for their evaluation. Taylan [36] demonstrated that different nonlinear damping and restoring moment formulations reported in the literature have resulted in completely different roll amplitudes, and further yielded different ship stability characteristics. Since ship capsizing is strongly dependent on the magnitude of roll motions, an accurate estimation of roll damping is crucial to the prediction of the ship motion responses. Moreover, the designer should consider the influence of waves on roll damping, especially nonlinear roll damping of large-amplitude roll motion, and subsequently on ship stability.

Different models for the damping moment introduced in the equation of roll motion were proposed by Dalzell [37], Cardo et al. [38], and Mathisen and Price [39]. They contain linear-quadratic or linear-cubic terms in the angular roll velocity. El-Bassiouny [40] studied the dynamic behavior of ships roll motion by considering different forms of damping moments consisting of the linear term associated with radiation, viscous damping, and a cubic term due to frictional resistance and eddies behind bilge keels and hard bilge corners.

This paper presents the derivation of the equations of motion based on physical grounds. The equations of motion will then be simplified to consider the roll-pitch coupling, which is very critical in studying the problem of ship capsizing. It begins with a basic background and terminology commonly used in Marine Engineering. This is followed by considering the hydrostatics of ships in calm water and the corresponding contribution due to sea waves. An account of nonlinear damping in ship roll oscillation will be made based on the main results reported in the literature. The paper includes an overview of ship roll dynamic stability and its stochastic modes, probability methods used in estimating ship capsizing and parameter identification.

2. Background and Terminology

One needs to be familiar with naval architecture terminology. This includes key stability terms that are used in the design and analysis of navigation vessels and their structure components. A list of the main terms is provided in the appendix. Those terms described in this section are written in italics. The purpose of this section is to introduce the fundamental concept of ship roll hydrostatic stability.

A floating ship displaces a volume of water whose weight is equal to the weight of the ship. The ship will be buoyed up by a force equal to the weight of the displaced water. The *metacenter* M shown in Figure 2(a) is the point through which the *buoyant* forces act at small angles of *list*. At these small angles the center of buoyancy tends to follow an arc subtended by the *metacentric* radius $\overline{BM}$, which is the distance between the *metacenter* and the *center of buoyancy* B. As the vessels' *draft* changes so does the *metacenter* moving up with the center of *buoyancy* when the draft increases and vice versa when the draft decreases. For small angle stability it is assumed that the *metacenter* does not move.

The *center of Buoyancy* B is the point through which the buoyant forces act on the wetted surface of the *hull*. The position of the *center of buoyancy* changes depending on the attitude of the vessel in the water. As the vessel increases or reduces its *draft* (drawing or pulling), its *center of buoyancy* moves up or down, respectively, caused by a change in the water displaced. As the vessel *lists* the *center of buoyancy* moves in a direction governed by the changing shape of the submerged part of the *hull* as demonstrated in Figure 2(b). For small angles, the *center of buoyancy* moves towards the side of the ship that is becoming more submerged. This is true for small angle stability and for vessels with sufficient *freeboard*. When the water line reaches and moves above the main deck level a relatively smaller volume of

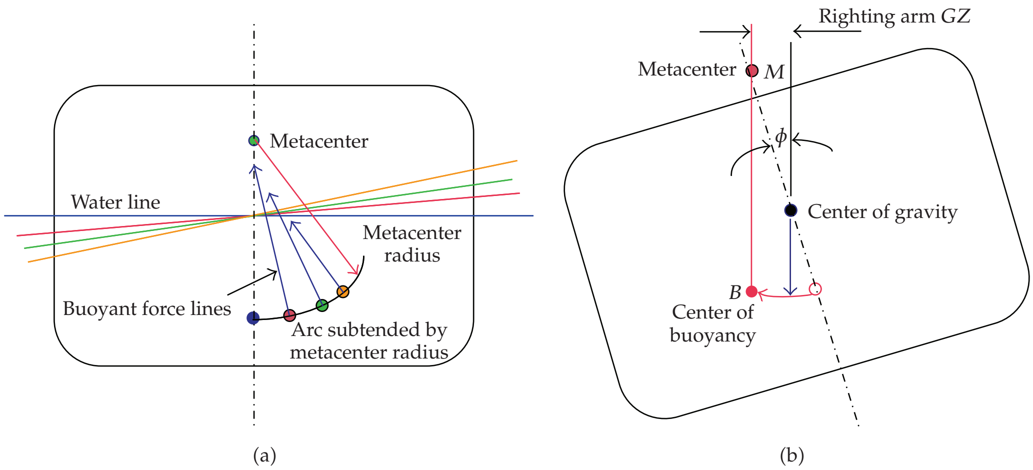

Figure 2: (a) Possible locations of the metacenter and (b) the righting arm.

the *hull* is submerged on the lower side for every centimeter movement as the water moves up the deck. The *center buoyancy* will now begin to move back towards the centerline.

As a vessel rolls its *center of buoyancy* moves off the centerline. The center of gravity, however, remains on the centerline. For small roll angles up to 10°, depending on *hull* geometry, the *righting arm* $\overline{GZ}$ is

$$\overline{GZ} = \overline{GM} \sin \phi. \tag{2.1}$$

It can be seen that the greater the *metacenter* height the greater the *righting arm* and therefore the greater the force restoring the vessel (*righting moment*) to the upright position one. When the *metacenter* is at or very near the centre of gravity then it is possible for the vessel to have a permanent *list* due to the lack of an adequate righting arm. Note that this may occur during loading operations. A worst case occurs when the *metacenter* is located substantially below the center of gravity as shown in Figure 3. This situation will lead to the ship *capsizing*. As long as the *metacenter* is located above the center of gravity, the righting arm has a stabilizing effect to bring the ship back to its normal position. If, on the other hand, the righting arm is displaced below the center of gravity, the ship will lose its roll stability and *capsize*.

Hydrostatic and hydrodynamic characteristics of ships undergo changes because of the varying underwater volume, *centers of buoyancy* and gravity and pressure distribution. Another factor is the effect of forward speed on ship stability and motions, particularly on rolling motion in synchronous *beam* waves. Taylan [41] examined the influence of forward speed by incrementing its value and determining the roll responses at each speed interval. Various characteristics of the $\overline{GZ}$ curve for a selected test vessel were found to change systematically.

The roll stability of a ship is usually measured by the stability diagram shown in Figure 4. The diagram shows the dependence of the *righting arm* on the roll angle (*list*) and is an important design guide for roll stability.

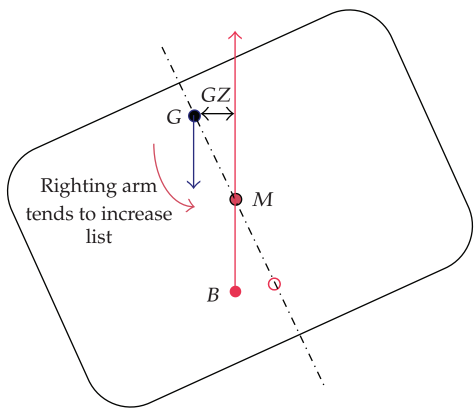

Figure 3: Negative ship stability.

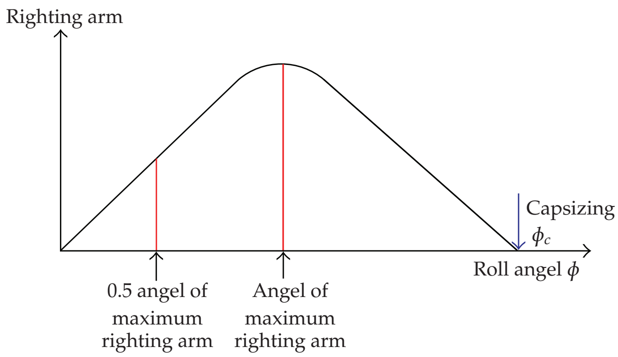

Figure 4: Dependence of the righting arm on the roll angle.

The roll oscillation of a ship is associated with a restoring moment to stabilize the ship about the x-axis given by the expression

$$M_x = W\overline{GM}\sin\phi. \tag{2.2}$$

where is the weight of water of displaced volume of the ship which is equal to the weight of the ship. If the ship experiences pitching motion of angle θ the righting arm will be raised by an increment $\overline{GM}\sin\phi\sin\theta$. In the case the net roll moment becomes

$$M_x = W\overline{GM}\sin\phi(1+\sin\theta) \approx W\overline{GM}\phi(1+\theta). \tag{2.3}$$

Note that the static stability is governed by the minimum value that the metacenter height, $\overline{GM}$, should have and the shape of the static stability curve with respect to the roll angle. This approach is still being applied in the assessment of stability criterion. The dynamic stability approach, on the other hand, is based on the equation of rolling motion. This involves constructing a model for a ship rolling in a realistic sea. The linear restoring parameters can be easily obtained from ship hydrostatics.

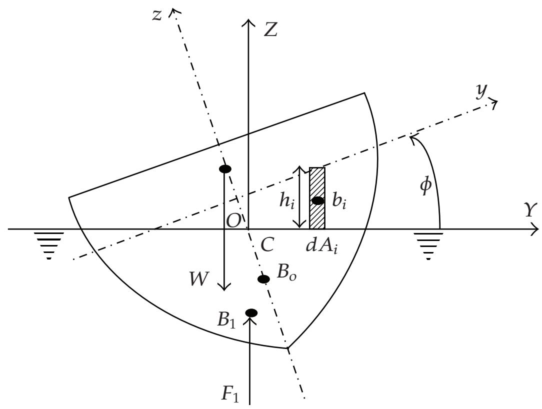

Figure 5: Ship schematic diagrams showing hydrostatic forces in a displaced position.

The curve for righting arm, known also as the restoring lever, has been represented by an odd-order polynomial up to different degrees [42–45]. Different representations of the restoring moment have been proposed in the literature. For example, Roberts [46, 47], Falzarano and Zhang [48], Huang et al. [49], and Senjanović et al. [50] represented $M_x(\phi)$ by the polynomial

$$M_x(\phi) = k_1\phi + k_3\phi^3 + k_5\phi^5 + k_7\phi^7 + \cdots, \tag{2.4}$$

where $k_1 > 0$, $k_3 < 0$, $k_5 > 0$, and $k_7 < 0$ for a damaged vessel, but $k_7 = 0$ for an intact vessel. Moshchuk et al. [51] proposed the following representation:

$$M_x(\phi) = k_0 \sin\left(\frac{\pi\phi}{\phi_s}\right) + \tilde{\gamma}\left(\frac{\pi\phi}{\phi_s}\right), \tag{2.5}$$

where ϕ_s is the capsizing angle, and the function $\tilde{\gamma}(\pi\phi/\phi_s)$ accounts for the difference between the exact function $M_x(\phi)$ and $k_0 \sin(\pi\phi/\phi_s)$.

3. Heave-Pitch-Roll Equations of Motion

Consider a ship sitting in its static equilibrium position with a submerged volume v_0. During its motion, its instantaneous submerged volume is v_1, and the difference in the submerged volume is $\delta v = v_1 - v_0$. The inertial frame of axes is XYZ with unit vectors $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ along X-, Y-, and Z-axes, respectively. On the other hand, the body frame that moves with the ship is xyz with unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ along x-, y-, and z-axes, respectively. Figure 5 shows the instantaneous buoyant center located at point B_1 and the corresponding instantaneous force is $\mathbf{F}_1 = \rho g v_1 \mathbf{K} = \rho g (v_0 + \delta v)\mathbf{K}$. The weight of the ship is $\mathbf{W} = -\rho g v_0 \mathbf{K}$. In this case the instantaneous restoring hydrostatic force is

$$\mathbf{F}_H = \rho g \delta v \mathbf{K}. \tag{3.1}$$

The restoring moment is the resultant between the moments of weight and instantaneous buoyancy

$$\mathbf{M}_H = \rho g\left[v_0\mathbf{K}\times\left(\overrightarrow{OG}-\overrightarrow{OB}_o\right)+\int\overrightarrow{Ob}_i\times dv_i\mathbf{K}\right], \tag{3.2}$$

where dv_i is the volume of the infinitesimal prism of height h_i, $\overrightarrow{OG}=\overline{z}_G\mathbf{k}$, $\overline{z}_G$ is the center of mass location from O, $\overrightarrow{OB}_o=\overline{z}_{BO}\mathbf{k}$, $\overrightarrow{Ob}_i\approx x_{Ai}\mathbf{I}+y_{Ai}\mathbf{J}+(h_i/2)\mathbf{K}$, $\mathbf{k}=\sin\theta\mathbf{I}-\sin\phi\mathbf{J}+\mathbf{K}$, $\overline{z}_G$ is the vertical coordinate of the center of gravity, $\delta v=\int h_i dA_i=\int h dA$, and $\overline{z}_{BO}$ is the vertical coordinate of the center of gravity of the submerged volume, and (x_A, y_A) are the coordinates of the elemental prism in the instantaneous plane with respect to the inertial frame $CXYZ$. Substituting these parameters in (3.2) gives

$$\mathbf{M}_H = -\mathbf{I}\rho g\left[v_0(\overline{z}_{BO}-\overline{z}_G)\sin\phi-\int y_A h dA\right]-\mathbf{J}\rho g\left[v_0(\overline{z}_{BO}-\overline{z}_G)\sin\theta+\int x_A h dA\right]. \tag{3.3}$$

The elemental prism height $h_i = h$ can be written in terms of the heave displacement z of the origin O above the water level, the pitch, θ, and roll, ϕ, angles as

$$h_i = -z - y_{Ai}\sin\phi + x_{Ai}\sin\theta. \tag{3.4}$$

The volume variation δv is

$$\begin{aligned}\delta v &= \int dv \\ &= \int(-z - y_A\sin\phi + x_A\sin\theta)dA \\ &= -z\int dA - \sin\phi\int y_A dA + \sin\theta\int x_A dA.\end{aligned} \tag{3.5}$$

The above summations are dependent on z, ϕ, and θ. They represent the following geometric properties:

$$\begin{gathered}\int dA = A(z,\phi,\theta)\ \text{area of instantaneous plane of floatation,} \\ \int y_A dA = A_x(z,\phi,\theta)\ \text{first static moment of the area about } x\text{-axis,} \\ \int x_A dA = A_y(z,\phi,\theta)\ \text{first static moment of area about } y\text{-axis.}\end{gathered} \tag{3.6}$$

In this case, one may write the volume variation in the form

$$\delta v = -zA(z,\phi,\theta) - \sin\phi A_x(z,\phi,\theta) + \sin\theta A_y(z,\phi,\theta). \tag{3.7}$$

Note the above summations could have been replaced by integrals. The instantaneous restoring hydrostatic force given by (3.1) takes the form

$$\mathbf{F}_H = -\rho g\left[zA(z,\phi,\theta) + \sin\phi A_x(z,\phi,\theta) - \sin\theta A_y(z,\phi,\theta)\right]\mathbf{K}. \tag{3.8}$$

In scalar form, the absolute value of the restoring force is

$$F_H = \rho g\left[zA(z,\phi,\theta) + \sin\phi A_x(z,\phi,\theta) - \sin\theta A_y(z,\phi,\theta)\right]. \tag{3.9}$$

The summations in (3.3) can also be written in terms of (3.4) as

$$\begin{aligned}
\int y_A h dA &= \int y_A(-z - y_A\sin\phi + x_A\sin\theta)dA \\
&= -zA_x(z,\phi,\theta) - \sin\phi I_{xx}(z,\phi,\theta) + \sin\theta I_{xy}(z,\phi,\theta), \\
\int x_A h dA &= \int x_A(-z - y_A\sin\phi + x_A\sin\theta)dA \\
&= -zA_y(z,\phi,\theta) - \sin\phi I_{xy}(z,\phi,\theta) + \sin\theta I_{yy}(z,\phi,\theta),
\end{aligned} \tag{3.10}$$

where

$$I_{xx}(z,\phi,\theta) = \int y_A^2 dA, \qquad I_{yy}(z,\phi,\theta) = \int x_A^2 dA, \qquad I_{xy}(z,\phi,\theta) = \int x_A y_A dA. \tag{3.11}$$

Introducing (3.10) into (3.3) and writing the result in the absolute and scalar form give

$$\begin{aligned}
M_{xH} &= \rho g\left[A_x z + v_0(\overline{z}_{BO} - \overline{z}_G)\sin\phi + I_{xx}\sin\phi - I_{xy}\sin\theta\right], \\
M_{yH} &= \rho g\left[-A_y z - I_{xy}\sin\phi + v_0(\overline{z}_{BO} - \overline{z}_G)\sin\theta + I_{yy}\sin\theta\right].
\end{aligned} \tag{3.12}$$

Note that the geometrical parameters (3.6) and (3.11) depend on the instantaneous displacements of the ship (z, ϕ, θ). These properties may be expanded in multivariable Taylor series around the average position, that is,

$$
\begin{aligned}
A(z,\phi,\theta) &= A_0 + \left.\frac{\partial A}{\partial z}\right|_0 z + \left.\frac{\partial A}{\partial \theta}\right|_0 \theta + \frac{1}{2}\left.\frac{\partial^2 A}{\partial z^2}\right|_0 z^2 + \left.\frac{\partial^2 A}{\partial z \partial \theta}\right|_0 z\theta + \frac{1}{2}\left.\frac{\partial^2 A}{\partial \phi^2}\right|_0 \phi^2 + \frac{1}{2}\left.\frac{\partial^2 A}{\partial \theta^2}\right|_0 \theta^2, \\
A_x(z,\phi,\theta) &= \left.\frac{\partial A_x}{\partial \phi}\right|_0 \phi + \left.\frac{\partial^2 A_x}{\partial z \partial \phi}\right|_0 z\phi + \left.\frac{\partial^2 A_x}{\partial \phi \partial \theta}\right|_0 \phi\theta, \\
A_y(z,\phi,\theta) &= A_y\big|_0 + \left.\frac{\partial A_y}{\partial z}\right|_0 z + \left.\frac{\partial A_y}{\partial \theta}\right|_0 \theta + \frac{1}{2}\left.\frac{\partial^2 A_y}{\partial z^2}\right|_0 z^2 + \left.\frac{\partial^2 A_y}{\partial z \partial \theta}\right|_0 z\theta + \frac{1}{2}\left.\frac{\partial^2 A_y}{\partial \phi^2}\right|_0 \phi^2 + \frac{1}{2}\left.\frac{\partial^2 A_y}{\partial \theta^2}\right|_0 \theta^2, \\
I_{xx}(z,\phi,\theta) &= I_{xx}\big|_0 + \left.\frac{\partial I_{xx}}{\partial z}\right|_0 z + \left.\frac{\partial I_{xx}}{\partial \theta}\right|_0 \theta + \frac{1}{2}\left.\frac{\partial^2 I_{xx}}{\partial z^2}\right|_0 z^2 + \left.\frac{\partial^2 I_{xx}}{\partial z \partial \theta}\right|_0 z\theta + \frac{1}{2}\left.\frac{\partial^2 I_{xx}}{\partial \phi^2}\right|_0 \phi^2 + \frac{1}{2}\left.\frac{\partial^2 I_{xx}}{\partial \theta^2}\right|_0 \theta^2, \\
I_{yy}(z,\phi,\theta) &= I_{yy}\big|_0 + \left.\frac{\partial I_{yy}}{\partial z}\right|_0 z + \left.\frac{\partial I_{yy}}{\partial \theta}\right|_0 \theta + \frac{1}{2}\left.\frac{\partial^2 I_{yy}}{\partial z^2}\right|_0 z^2 + \left.\frac{\partial^2 I_{yy}}{\partial z \partial \theta}\right|_0 z\theta + \frac{1}{2}\left.\frac{\partial^2 I_{yy}}{\partial \phi^2}\right|_0 \phi^2 + \frac{1}{2}\left.\frac{\partial^2 I_{yy}}{\partial \theta^2}\right|_0 \theta^2, \\
I_{xy}(z,\phi,\theta) &= \left.\frac{\partial I_{xy}}{\partial \phi}\right|_0 \phi + \left.\frac{\partial^2 I_{xy}}{\partial z \partial \phi}\right|_0 z\phi + \left.\frac{\partial^2 I_{xy}}{\partial \phi \partial \theta}\right|_0 \phi\theta.
\end{aligned}
\tag{3.13}
$$

A_0, $A_y|_0$, $I_{xx}|_0$, and $I_{yy}|_0$ are the geometric properties evaluated at the average plane of floatation. Note that the variation of first moment of area about the x-axis is dependent on an odd order of roll angle. That dependence does not exist in variations of other geometrical parameters. Paulling and Rosenberg [8] showed that the dependencies of the heave and pitch coefficients on roll are of even order, while the coefficients in roll due to heave and pitch are odd.

The restoring hydrodynamic force and moments given by (3.8) and (3.3) take the form

$$
\begin{aligned}
F_H = \rho g \Bigg\{ & zA_0 - A_y\big|_0 \sin\theta + \left.\frac{\partial A}{\partial z}\right|_0 z^2 + \left.\frac{\partial A}{\partial \theta}\right|_0 z(\theta + \sin\theta) + \left.\frac{\partial A_x}{\partial \phi}\right|_0 \phi \sin\phi \\
& - \left.\frac{\partial A_y}{\partial \theta}\right|_0 \theta \sin\theta + \frac{1}{6}\left.\frac{\partial^2 A}{\partial z^2}\right|_0 z^3 + \left.\frac{\partial^2 A}{\partial z \partial \theta}\right|_0 z^2(\theta + \sin\theta) + \frac{1}{2}\left.\frac{\partial^2 A}{\partial \phi^2}\right|_0 z\phi\left(\frac{\phi}{2} + \sin\phi\right) \\
& + \left.\frac{\partial^2 A_x}{\partial \phi \partial \theta}\right|_0 \phi\left(\theta \sin\phi + \frac{\phi}{2}\sin\theta\right) + \left.\frac{\partial^2 A}{\partial \theta^2}\right|_0 z\theta\left(\frac{\theta}{2} + \sin\theta\right) - \frac{1}{2}\left.\frac{\partial^2 A_y}{\partial \theta^2}\right|_0 \theta^2 \sin\theta \Bigg\},
\end{aligned}
$$

$$M_{xH} = \rho g \left\{ v_0(\overline{z}_{BO} - \overline{z}_G)\sin\phi + I_{xx}|_0 z\sin\phi + \left.\frac{\partial I_{xx}}{\partial z}\right|_0 z(\phi + \sin\phi) + \left.\frac{\partial I_{xx}}{\partial \theta}\right|_0 (\theta\sin\phi + \phi\sin\theta) \right.$$
$$+ \left.\frac{\partial^2 I_{xx}}{\partial z^2}\right|_0 z^2\left(\phi + \frac{1}{2}\sin\phi\right) + \frac{1}{2}\left.\frac{\partial^2 I_{xx}}{\partial \theta^2}\right|_0 \theta(\theta\sin\phi + \phi\sin\theta)$$
$$\left. + \left.\frac{\partial^2 I_{xx}}{\partial z\partial\theta}\right|_0 z(\phi\theta + \theta\sin\phi + \phi\sin\phi) + \frac{1}{2}\left.\frac{\partial^2 I_{xx}}{\partial \phi^2}\right|_0 \phi^2\sin\phi \right\},$$

$$M_{yH} = \rho g \left\{ v_0(\overline{z}_{BO} - \overline{z}_G)\sin\theta - A_y\big|_0 z + I_{yy}\big|_0\sin\theta - \left.\frac{\partial A_y}{\partial z}\right|_0 z^2 + \left.\frac{\partial I_{yy}}{\partial z}\right|_0 z(\theta + \sin\theta) \right.$$
$$- \left.\frac{\partial I_{xy}}{\partial \phi}\right|_0 \phi\sin\phi + \left.\frac{\partial I_{yy}}{\partial \theta}\right|_0 \theta\sin\theta - \frac{1}{6}\left.\frac{\partial^2 A_y}{\partial z^2}\right|_0 z^3 + \left.\frac{\partial^2 I_{yy}}{\partial z^2}\right|_0 z^2\left(\theta + \frac{1}{2}\sin\theta\right)$$
$$- \left.\frac{\partial^2 I_{xy}}{\partial z\partial\phi}\right|_0 z\phi\left(\frac{\phi}{2} + \sin\phi\right) + \left.\frac{\partial^2 I_{yy}}{\partial z\partial\theta}\right|_0 z\theta\left(\frac{\theta}{2} + \sin\theta\right) - \left.\frac{\partial^2 I_{xy}}{\partial\phi\partial\theta}\right|_0 \phi\theta\sin\phi$$
$$\left. + \frac{1}{2}\left.\frac{\partial^2 I_{yy}}{\partial \phi^2}\right|_0 \phi^2\sin\theta + \frac{1}{2}\left.\frac{\partial^2 I_{yy}}{\partial \theta^2}\right|_0 \theta^2\sin\theta \right\}. \tag{3.14}$$

In achieving the above equations use has been made of the following equalities verified by Neves and Rodríguez [11]:

$$\left.\frac{\partial A}{\partial\theta}\right|_0 = -\left.\frac{\partial A_y}{\partial z}\right|_0, \qquad \left.\frac{\partial A_x}{\partial\phi}\right|_0 = \left.\frac{\partial I_{xx}}{\partial z}\right|_0, \qquad \left.\frac{\partial I_{xx}}{\partial\theta}\right|_0 = -\left.\frac{\partial I_{xy}}{\partial\phi}\right|_0,$$
$$\left.\frac{\partial I_{yy}}{\partial z}\right|_0 = -\left.\frac{\partial A_y}{\partial\theta}\right|_0, \qquad \left.\frac{\partial^2 A}{\partial\phi^2}\right|_0 = \left.\frac{\partial^2 A_x}{\partial z\partial\phi}\right|_0 = \left.\frac{\partial^2 I_{xx}}{\partial z^2}\right|_0,$$
$$\left.\frac{\partial^2 A}{\partial\theta^2}\right|_0 = -\left.\frac{\partial^2 A_y}{\partial z\partial\theta}\right|_0 = \left.\frac{\partial^2 I_{yy}}{\partial z^2}\right|_0, \qquad \left.\frac{\partial^2 A}{\partial z\partial\theta}\right|_0 = -\left.\frac{\partial^2 A_y}{\partial z^2}\right|_0,$$
$$-\left.\frac{\partial^2 I_{xx}}{\partial z\partial\theta}\right|_0 = -\left.\frac{\partial^2 I_{xy}}{\partial z\partial\phi}\right|_0 = \left.\frac{\partial^2 A_x}{\partial\phi\partial\theta}\right|_0 = -\left.\frac{\partial^2 A_y}{\partial\phi^2}\right|_0, \qquad \left.\frac{\partial^2 I_{xx}}{\partial\theta^2}\right|_0 = -\left.\frac{\partial^2 I_{xy}}{\partial\phi\partial\theta}\right|_0 = \left.\frac{\partial^2 I_{yy}}{\partial\phi^2}\right|_0,$$
$$\left.\frac{\partial^2 I_{yy}}{\partial z\partial\theta}\right|_0 = -\left.\frac{\partial^2 A_y}{\partial\theta^2}\right|_0. \tag{3.15}$$

3.1. Wave Motion Effects

The influence of incident sea waves of arbitrary direction along the hull is to change the average submerged shape defined by the instantaneous position of the wave. These waves exert external forces and moments in heave, roll, and pitch in addition they introduce an additional restoring forces and moments. For the case of head sea, Neves and Rodríguez [11, 12] considered the Airy linear theory in representing longitudinal waves (along x-axis) defined by the expression [14]

$$\eta(x, y, t; \chi) = \eta_0 \cos(kx + \omega_e t), \tag{3.16}$$

where η_0 is the wave amplitude, $k = \omega_w^2/g = 2\pi/\lambda$ is the wave number, ω_w is the wave frequency, λ is the wave length, g is the gravitational acceleration, χ is the wave incidence, and $\omega_e = (\omega_w - kU\cos\chi)$ is the encounter frequency of the wave by the ship when the ship advances with speed U.

Note that h_i expressed by (3.4) should read

$$h_i = -[z - \eta(x_{Ai}, y_{Ai}, t)] - y_{Ai}\sin\phi + x_{Ai}\sin\theta. \tag{3.17}$$

The contributions of longitudinal waves to the restoring force, F_η, and the restoring moments, $M_{x\eta}$ and $M_{y\eta}$, obtained using Taylor series expansion about the average position up to third-order terms are given by the expressions [11, 12]

$$\begin{aligned}
F_\eta &= \left.\frac{\partial^2 F_\eta}{\partial\eta\partial z}\right|_0 z + \left.\frac{\partial^2 F_\eta}{\partial\eta\partial\theta}\right|_0 \theta + \left.\frac{\partial^3 F_\eta}{\partial\eta\partial z^2}\right|_0 z^2 + \left.\frac{\partial^3 F_\eta}{\partial\eta\partial z\partial\theta}\right|_0 z\theta + \left.\frac{\partial^3 F_\eta}{\partial\phi^2\partial\eta}\right|_0 \phi^2 + \left.\frac{\partial^3 F_\eta}{\partial\eta^2\partial\theta}\right|_0 \theta \\
&\quad + \left.\frac{\partial^3 F_\eta}{\partial\eta\partial\theta^2}\right|_0 \theta^2, \\
M_{x\eta} &= \left.\frac{\partial^2 M_{x\eta}}{\partial\eta\partial\phi}\right|_0 \phi + \left.\frac{\partial^3 M_{x\eta}}{\partial\eta\partial z\partial\phi}\right|_0 z\phi + \left.\frac{\partial^3 M_{x\eta}}{\partial\eta^2\partial\phi}\right|_0 \phi + \left.\frac{\partial^3 M_{x\eta}}{\partial\eta\partial\phi\partial\theta}\right|_0 \phi\theta, \\
M_{y\eta} &= \left.\frac{\partial^2 M_{y\eta}}{\partial\eta\partial z}\right|_0 z + \left.\frac{\partial^2 M_{y\eta}}{\partial\eta\partial\theta}\right|_0 \theta + \left.\frac{\partial^3 M_{y\eta}}{\partial\eta^2\partial z}\right|_0 z + \left.\frac{\partial^3 M_{y\eta}}{\partial\eta\partial z^2}\right|_0 z\theta + \left.\frac{\partial^3 M_{y\eta}}{\partial\phi^2\partial\eta}\right|_0 \phi^3 + \left.\frac{\partial^3 M_{y\eta}}{\partial\eta^2\partial\theta}\right|_0 \theta \\
&\quad + \left.\frac{\partial^3 M_{y\eta}}{\partial\theta^2\partial\eta}\right|_0 \theta^2,
\end{aligned} \tag{3.18}$$

where the derivatives of the above equations are given by the following expressions:

$$\begin{aligned}
\left.\frac{\partial^2 F_\eta}{\partial\eta\partial z}\right|_0 &= 2\rho g\int_L \frac{\partial y}{\partial z}\eta\, dx,\\
\left.\frac{\partial^2 F_\eta}{\partial\eta\partial\theta}\right|_0 &= -2\rho g\int_L x\frac{\partial y}{\partial z}\eta\, dx,\\
\left.\frac{\partial^3 F_\eta}{\partial\phi^2\partial\eta}\right|_0 &= -\rho g\int_L \left[2y\left(\frac{\partial y}{\partial z}\right)^2 + y\right]\eta\, dx,\\
\left.\frac{\partial^3 F_\eta}{\partial\eta\partial z^2}\right|_0 = \left.\frac{\partial^3 F_\eta}{\partial\eta\partial z\partial\theta}\right|_0 &= \left.\frac{\partial^3 F_\eta}{\partial\eta^2\partial\theta}\right|_0 = \left.\frac{\partial^3 F_\eta}{\partial\eta\partial\theta^2}\right|_0 = 0,\\
\left.\frac{\partial^2 M_{x\eta}}{\partial\eta\partial\phi}\right|_0 &= 2\rho g\int_L y^2\frac{\partial y}{\partial z}\eta\, dx,\\
\left.\frac{\partial^3 M_{x\eta}}{\partial\eta\partial z\partial\phi}\right|_0 &= -2\rho g\int_L \left[2y\left(\frac{\partial y}{\partial z}\right)^2 + y\right]\eta\, dx,\\
\left.\frac{\partial^3 M_{x\eta}}{\partial\eta^2\partial\phi}\right|_0 &= \rho g\int_L \left[2y\left(\frac{\partial y}{\partial z}\right)^2 + y\right]\eta^2\, dx,\\
\left.\frac{\partial^3 M_{x\eta}}{\partial\eta\partial\phi\partial\theta}\right|_0 &= 2\rho g\int_L \left[2xy\left(\frac{\partial y}{\partial z}\right)^2 + xy\right]\eta\, dx,\\
\left.\frac{\partial^2 M_{y\eta}}{\partial\eta\partial z}\right|_0 &= -2\rho g\int_L x\frac{\partial y}{\partial z}\eta\, dx,\\
\left.\frac{\partial^2 M_{y\eta}}{\partial\eta\partial\theta}\right|_0 &= 2\rho g\int_L x^2\frac{\partial y}{\partial z}\eta\, dx,\\
\left.\frac{\partial^3 M_{y\eta}}{\partial\phi^2\partial\eta}\right|_0 &= \rho g\int_L \left[2xy\left(\frac{\partial y}{\partial z}\right)^2 + xy\right]\eta dx,\\
\left.\frac{\partial^3 M_{y\eta}}{\partial\eta^2\partial z}\right|_0 = \left.\frac{\partial^3 M_{y\eta}}{\partial\eta\partial z^2}\right|_0 &= \left.\frac{\partial^3 M_{y\eta}}{\partial\theta^2\partial\eta}\right|_0 = \left.\frac{\partial^3 M_{y\eta}}{\partial\eta^2\partial\theta}\right|_0 = 0.
\end{aligned} \tag{3.19}$$

3.2. Ships Roll Damping

The surface waves introduce inertia and drag hydrodynamic forces. The inertia force is the sum of two components. The first is a buoyancy force acting on the structure in the fluid due to a pressure gradient generated from the flow acceleration. The buoyancy force is equal to the mass of the fluid displaced by the structure multiplied by the acceleration of the flow. The second inertia component is due to the added mass, which is proportional to the relative acceleration between the structure and the fluid. This component accounts for the flow entrained by the structure. The drag force is the sum of the viscous and pressure drags produced by the relative velocity between the structure and the flow. This type of hydrodynamic drag is proportional to the square of the relative velocity.

Viscosity plays an important role in ship responses especially at large-amplitude roll motions in which the wave radiation damping is relatively low. The effect of the *bilge keel* on the roll damping was first discussed by Bryan [52]. Hishida [53–55] proposed an analytical approach to roll damping for ship hulls in simple oscillatory waves. The regressive curve of the roll damping obtained from the experiments by Kato [56] has been widely used in the prediction of ship roll motions. Since amplitudes and frequencies are varying in random waves, the hydrodynamic coefficients are time-dependent and irregular. Several experimental investigations have been conducted to measure the effect of *bilge keels* on the roll damping (see, e.g., [57–66]).

It was indicated by Bishop and Price [67] that existing information on the structural damping of ships is far from satisfactory. It cannot be calculated and it can only be measured in the presence of hydrodynamic damping, whose nature and magnitude are also somewhat obscure. Yet it is very important. Much less is known about antisymmetric responses to waves, either as regards the means of estimating them or the appropriate levels of hull damping. Vibration at higher frequencies, due to excitation by machinery (notably propellers), is limited by structural damping to a much greater extent than it is by the fluid actions of the sea. Damping measurements at these frequencies therefore give more accurate estimates of hull damping. The damping moment of ships is related to multiplicity of factors such as hull shape, loading condition, *bilge keel*, rolling frequency, and range of rolling angle. For small roll angles, the damping moment is directly proportional to the angular roll velocity. But with increasing roll angle, nonlinear damping will become significant. Due to the occurrence of strong viscous effects, the roll damping moment cannot be computed by means of potential theory. Himeno [68] provided a detailed description of the equivalent damping coefficient and expressed it in terms of various contributions due to hull skin friction damping, hull eddy shedding damping, free surface wave damping, lift force damping, and bilge keel damping. The viscous damping is due to the following sources.

(i) Wave-making moment, B_W.

(ii) Skin-friction damping moment, B_F.

(iii) The moment resulting from the bare hull arising from separation and eddies mostly near the bilge keels, B_E.

(iv) Lift damping moment due to an apparent angle of attack as the ship rolls, B_L.

(v) Bilge-keel damping moment, B_{BK}.

Damping due to bilge keels can be decomposed into the following components.

(i) Bilge keels moment due to normal force, B_{BKN}.

(ii) Moment due to interaction between hull and bilge keel, B_{BKH}.

(iii) Modification to wave making due to the presence of bilge keels, B_{BKW}.

The damping components B_F, B_L, B_W, and B_{BKW} are linear, while B_E, B_{BKN}, and B_{BKH} are nonlinear. The linear and nonlinear damping moments can be expressed as follows:

$$B_{\text{lin}} = B_F + B_L + B_W + B_{BKW},$$
$$B_{\text{nonlin}} = B_E + B_{BKN} + B_{BKH}. \tag{3.20}$$

A pseudospectral model for nonlinear ship-surface wave interactions was developed by Lin et al. [69]. The algorithm is a combination of spectral and boundary element methods. All possible ship-wave interactions were included in the model. The nonlinear bow waves at high Froude numbers from the pseudospectral model are much closer to the experimental results than those from linear ship wave models. One of the main problems in modeling ship-wave hydrodynamics is solving for the forcing (pressure) at the ship boundary. With an arbitrary ship, singularities occur in evaluating the velocity potential and the velocities on the hull. Inaccuracies in the evaluation of the singular terms in the velocity potential result in discretization errors, numerical errors, and excessive computational costs. Lin and Kuang [70, 71] presented a new approach to evaluating the pressure on a ship. They used the digital, self-consistent, ship experimental laboratory (DiSSEL) ship motion model to test its effectiveness in predicting ship roll motion. It was shown that the implementation of this roll damping component improves significantly the accuracy of numerical model results. Salvesen [72] reported some results pertaining numerical methods such as large amplitude motion program (LAMP) used to evaluate hydrodynamic performance characteristics. These methods were developed for solving fully three-dimensional ship-motions, ship-wave-resistance and local-flow problems using linearized free-surface boundary conditions. Lin et al. [73] examined the capabilities of the 3D nonlinear time-domain Large Amplitude Motion Program (LAMP) for the evaluation of fishing vessels operating in extreme waves. They extended their previous work to the modeling of maritime casualties, including a time-domain simulation of a ship capsizing in stern quartering seas.

The damping characteristics of a variety of ship shapes and offshore structures undergoing roll oscillation in the presence of ocean waves have been assessed by Chakrabarti [74]. Chakrabarti [74] relied on empirical formulas derived from a series of model experiments reported by Ikeda [75] and Ikeda et al. [76–78]. These experiments were performed on two-dimensional shapes. The damping roll moment $B(\dot{\phi})$ is nonlinear and may be expressed by the expression [40, 74]

$$B(\dot{\phi}) = c_1\dot{\phi} + c_2\dot{\phi}|\dot{\phi}| + c_3\dot{\phi}^3 + \cdots = \sum_{k=1}^{K} c_k\dot{\phi}|\dot{\phi}|^{k-1}. \tag{3.21}$$

The first term is the usual linear viscous damping, the second is the quadratic damping term originally developed by Morison et al. [79]. It is in phase with the velocity but it is quadratic because the flow is separated and the drag is primarily due to pressure rather than the skin friction. Sarpkaya and Isaacson [80] provided a critical assessment of Morison's equation, which describes the forces acting on a pile due to the action of progressive waves. The third term is cubic damping. The total damping may be replaced by an equivalent viscous term in the form

$$B(\dot{\phi}) = c_{\text{eq}}\dot{\phi}, \tag{3.22}$$

where c_{eq} is the equivalent damping coefficient. This coefficient can be expressed in terms of the nonlinear coefficients as

$$c_{\mathrm{eq}} = c_1 + \frac{8}{3\pi} c_2 (\omega \phi_0) + \frac{3}{4} c_3 (\omega \phi_0)^2, \tag{3.23}$$

where ω is the wave frequency and ϕ_0 is the amplitude of the ship roll angle.

Dalzell [37] replaced the nonlinear damping term $\dot{\phi}|\dot{\phi}|$ by an equivalent smooth nonlinear polynomial given by

$$\dot{\phi}|\dot{\phi}| = \sum_{k=1,3,\ldots} \alpha_k \frac{\dot{\phi}^k}{(\dot{\phi}_c)^{k-2}} \cong \frac{5}{16} \dot{\phi}_c \dot{\phi} + \frac{35}{48} \frac{\dot{\phi}^3}{\dot{\phi}_c}, \tag{3.24}$$

where $\dot{\phi}_c$ is the maximum amplitude of roll velocity. The numerical coefficients α_k were estimated by using least-square fitting.

Haddara [81] employed the concept of the random decrement in the damping identification of linear systems. He extended the concept of the random decrement for a ship performing rolling motion in random beam waves. Wave excitation was assumed to be a Gaussian white noise process. The equations were used to identify the parameters of the nonlinear roll damping moment. Wu et al. [82] conducted an experimental investigation to measure the nonlinear roll damping of a ship in regular and irregular waves.

3.3. *Ship Inertia Forces and Moments*

The inertia forces and moments in heave, roll, and pitch motions are mainly due to the ship mass and mass moment of inertia and the corresponding added mass terms. These are well documented in Neves and Rodríguez [11] and are given in the form

$$\begin{aligned} F_{ZI} &= (m + Z_{\ddot{z}})\ddot{z} + Z_{\ddot{\theta}}\ddot{\theta}, \\ M_{xI} &= \left(J_{xx} + K_{\ddot{\phi}}\right)\ddot{\phi}, \\ M_{yI} &= (J_{yy} + K_{\ddot{\theta}})\ddot{\theta} + M_{\ddot{z}}\ddot{z}, \end{aligned} \tag{3.25}$$

where m is the ship mass, J_{xx} and J_{yy} are the ship mass moment of inertia about roll and pitch axes, $Z_{\ddot{z}}$ is the hydrodynamic added mass in heave, $Z_{\ddot{\theta}}$ is the hydrodynamic added inertia in heave due to pitch motion, inertia $K_{\ddot{\phi}}$ and $K_{\ddot{\theta}}$ are the hydrodynamic added polar mass moment of inertia about the ship roll and pitch axes, respectively, and $M_{\ddot{z}}$ is the added inertia in pitch due to heave motion. The added inertia parameters may be evaluated using the potential theory as described by Salvesen et al. [83] and Meyers et al. [84].

3.4. Governing Equations of Motion

Applying Newton's second law, the equations governing heave-roll-pitch motion may be written in a form.

The heave equation of motion is

$$\begin{aligned}
(m + Z_{\ddot{z}})\ddot{z} + Z_{\ddot{\theta}}\ddot{\theta} + C_{\dot{z}}\dot{z} + \rho g\Bigg\{ & zA_0 - A_y\big|_0 \sin\theta + \frac{\partial A}{\partial z}\bigg|_0 z^2 + \frac{\partial A}{\partial \theta}\bigg|_0 z(\theta + \sin\theta) + \frac{\partial A_x}{\partial \phi}\bigg|_0 \phi\sin\phi \\
& - \frac{\partial A_y}{\partial \theta}\bigg|_0 \theta\sin\theta + \frac{1}{6}\frac{\partial^2 A}{\partial z^2}\bigg|_0 z^3 + \frac{\partial^2 A}{\partial z\partial\theta}\bigg|_0 z^2(\theta + \sin\theta) \\
& + \frac{1}{2}\frac{\partial^2 A}{\partial \phi^2}\bigg|_0 z\phi\left(\frac{\phi}{2} + \sin\phi\right) + \frac{\partial^2 A_x}{\partial\phi\partial\theta}\bigg|_0 \phi\left(\theta\sin\phi + \frac{\phi}{2}\sin\theta\right) \\
& + \frac{\partial^2 A}{\partial \theta^2}\bigg|_0 z\theta\left(\frac{\theta}{2} + \sin\theta\right) - \frac{1}{2}\frac{\partial^2 A_y}{\partial\theta^2}\bigg|_0 \theta^2\sin\theta\Bigg\} \\
+ 2\rho g z\int_L \frac{\partial y}{\partial z}\eta(t)dx - 2\rho g\theta\int_L x\frac{\partial y}{\partial z}\eta(t)dx & - \rho g\phi^2\int_L \left[2y\left(\frac{\partial y}{\partial z}\right)^2 + y\right]\eta(t)dx = Z(t).
\end{aligned} \tag{3.26}$$

The roll moment equation of motion taking into account the beam sea hydrodynamic wave excitation moment, $\Phi(t)$, is

$$\begin{aligned}
\left(J_{xx} + K_{\ddot{\phi}}\right)\ddot{\phi} + \sum_{k=1}^{K} c_k\dot{\phi}\left|\dot{\phi}\right|^{k-1} + \rho g\Bigg\{ & v_0(\overline{z}_{BO} - \overline{z}_G)\sin\phi + I_{xx}\big|_0 z\sin\phi + \frac{\partial I_{xx}}{\partial z}\bigg|_0 z(\phi + \sin\phi) \\
& + \frac{\partial I_{xx}}{\partial \theta}\bigg|_0 (\theta\sin\phi + \phi\sin\theta) + \frac{\partial^2 I_{xx}}{\partial z^2}\bigg|_0 z^2\left(\phi + \frac{1}{2}\sin\phi\right) \\
& + \frac{1}{2}\frac{\partial^2 I_{xx}}{\partial\theta^2}\bigg|_0 \theta(\theta\sin\phi + \phi\sin\theta) \\
& + \frac{\partial^2 I_{xx}}{\partial z\partial\theta}\bigg|_0 z(\phi\theta + \theta\sin\phi + \phi\sin\phi) \\
& + \frac{1}{2}\frac{\partial^2 I_{xx}}{\partial\phi^2}\bigg|_0 \phi^2\sin\phi\Bigg\} \\
+ 2\rho g\phi\int_L y^2\frac{\partial y}{\partial z}\eta(t)dx - 2\rho g z\phi\int_L & \left[2y\left(\frac{\partial y}{\partial z}\right)^2 + y\right]\eta(t)dx + \rho g\phi\int_L\left[2y\left(\frac{\partial y}{\partial z}\right)^2 + y\right]\eta^2(t)dx \\
+ 2\rho g\phi\theta\int_L \left[2xy\left(\frac{\partial y}{\partial z}\right)^2 + xy\right]\eta(t)dx = \Phi(t). &
\end{aligned} \tag{3.27}$$

The pitch moment equation of motion taking into account the beam sea hydrodynamic wave excitation moment, $\Theta(t)$, is

$$
\begin{aligned}
(J_{yy} + K_{\ddot{\theta}})\ddot{\theta} + M_{\ddot{z}}\ddot{z} + C_{\dot{\theta}}\dot{\theta} + \rho g \Bigg\{ & v_0(\overline{z}_{BO} - \overline{z}_G)\sin\theta - A_y\big|_0 z + I_{yy}\big|_0 \sin\theta - \frac{\partial A_y}{\partial z}\bigg|_0 z^2 \\
& + \frac{\partial I_{yy}}{\partial z}\bigg|_0 z(\theta + \sin\theta) - \frac{\partial I_{xy}}{\partial \phi}\bigg|_0 \phi\sin\phi + \frac{\partial I_{yy}}{\partial \theta}\bigg|_0 \theta\sin\theta \\
& - \frac{1}{6}\frac{\partial^2 A_y}{\partial z^2}\bigg|_0 z^3 + \frac{\partial^2 I_{yy}}{\partial z^2}\bigg|_0 z^2\left(\theta + \frac{1}{2}\sin\theta\right) \\
& - \frac{\partial^2 I_{xy}}{\partial z \partial \phi}\bigg|_0 z\phi\left(\frac{\phi}{2} + \sin\phi\right) + \frac{\partial^2 I_{yy}}{\partial z \partial \theta}\bigg|_0 z\theta\left(\frac{\theta}{2} + \sin\theta\right) \\
& - \frac{\partial^2 I_{xy}}{\partial \phi \partial \theta}\bigg|_0 \phi\theta\sin\phi + \frac{1}{2}\frac{\partial^2 I_{yy}}{\partial \phi^2}\bigg|_0 \phi^2\sin\theta + \frac{1}{2}\frac{\partial^2 I_{yy}}{\partial \theta^2}\bigg|_0 \theta^2\sin\theta \Bigg\} \\
- 2\rho g z \int_L x \frac{\partial y}{\partial z}\eta(t)dx + 2\rho g \theta \int_L x^2 \frac{\partial y}{\partial z}\eta(t)dx & + \rho g \phi^3 \int_L \left[2xy\left(\frac{\partial y}{\partial z}\right)^2 + xy \right]\eta(t)dx = \Theta(t),
\end{aligned}
\tag{3.28}
$$

where $C_{\dot{z}}$ and $C_{\dot{\theta}}$ are linear damping coefficients associated with heave and pitch motions, respectively. $Z(t)$, $\Phi(t)$, and $\Theta(t)$ are the external excitations due to sea waves. One can extract from the above three equations the coupled roll-pitch equations of motion or the purely roll equation of motion. Nayfeh et al. [85] described two different mechanisms that cause roll instabilities in ships. An approximate solution based on the method of multiple scales was presented together with different simulations using the Large Amplitude Motions Program (LAMP) code to determine linear parameters of the heave, pitch, and roll response. A methodology for nonlinear system identification that combines the method of multiple scales and higher-order statistics was also proposed.

3.4.1. Coupled Roll-Pitch Equations of Motion

Considering the coupled roll-pitch equations of motion, (3.27) and (3.28) take the form

$$
\begin{aligned}
\left(J_{xx} + K_{\ddot{\phi}}\right)\ddot{\phi} + \sum_{k=1}^{K} c_k \dot{\phi}|\dot{\phi}|^{k-1} + \rho g \Bigg\{ & v_0(\overline{z}_{BO} - \overline{z}_G)\sin\phi + \frac{\partial I_{xx}}{\partial \theta}\bigg|_0 (\theta\sin\phi + \phi\sin\theta) \\
& + \frac{1}{2}\frac{\partial^2 I_{xx}}{\partial \theta^2}\bigg|_0 \theta(\theta\sin\phi + \phi\sin\theta) + \frac{1}{2}\frac{\partial^2 I_{xx}}{\partial \phi^2}\bigg|_0 \phi^2\sin\phi \Bigg\}
\end{aligned}
$$

$$+ 2\rho g\phi\int_L y^2\frac{\partial y}{\partial z}\eta(t)dx + \rho g\phi\int_L\left[2y\left(\frac{\partial y}{\partial z}\right)^2 + y\right]\eta^2(t)dx$$
$$+ 2\rho g\phi\theta\int_L\left[2xy\left(\frac{\partial y}{\partial z}\right)^2 + xy\right]\eta(t)dx = \Phi(t),$$
$$(J_{yy} + K_{\ddot{\theta}})\ddot{\theta} + C_{\dot{\theta}}\dot{\theta} + \rho g\left\{v_0(\overline{z}_{BO} - \overline{z}_G)\sin\theta + I_{yy}\big|_0\sin\theta - \left.\frac{\partial I_{xy}}{\partial\phi}\right|_0\phi\sin\phi + \left.\frac{\partial I_{yy}}{\partial\theta}\right|_0\theta\sin\theta\right.$$
$$\left. - \left.\frac{\partial^2 I_{xy}}{\partial\phi\partial\theta}\right|_0\phi\theta\sin\phi + \frac{1}{2}\left.\frac{\partial^2 I_{yy}}{\partial\phi^2}\right|_0\phi^2\sin\theta + \frac{1}{2}\left.\frac{\partial^2 I_{yy}}{\partial\theta^2}\right|_0\theta^2\sin\theta\right\}$$
$$+ 2\rho g\theta\int_L x^2\frac{\partial y}{\partial z}\eta(t)dx + \rho g\phi^3\int_L\left[2xy\left(\frac{\partial y}{\partial z}\right)^2 + xy\right]\eta(t)dx = \Theta(t). \tag{3.29}$$

Note that the nonlinear coupling terms may result in nonlinear internal resonances among pitch and roll motions (see, e.g., [2, 3]).

3.4.2. Roll Equation of Motion

The prediction of ship stability during the early stages of design is very important from the point of a vessel's safety. Of the six motions of a ship, the critical motion leading to capsize is the rolling motion. Thus for studying roll stability in beam seas one should consider the nonlinear roll equation

$$\begin{aligned}&\left(J_{xx} + K_{\ddot{\phi}}\right)\ddot{\phi} + \sum_{k=1}^{K} c_k\dot{\phi}|\dot{\phi}|^{k-1} + \rho g\left[v_0(\overline{z}_{BO} - \overline{z}_G) + \frac{1}{2}\left.\frac{\partial^2 I_{xx}}{\partial\phi^2}\right|_0\phi^2\right]\sin\phi\\&\quad + 2\rho g\phi\int_L y^2\frac{\partial y}{\partial z}\eta(t)dx + \rho g\phi\int_L\left[2y\left(\frac{\partial y}{\partial z}\right)^2 + y\right]\eta^2(t)dx = \Phi(t).\end{aligned} \tag{3.30}$$

In formulating the roll equation in beam seas one should realize that the hydrodynamic roll moments on the ship are dependent on the relative motion of ship and wave, rather than upon the absolute roll motion. In a beam sea the relative roll is defined as $(\phi - \alpha)$, where α is the local wave slope in a long-crested regular beam sea. In this case, the nonlinear equation of roll motion may be written in the form [86]

$$J_{xx}\ddot{\phi} = -\delta J_{xx}(\ddot{\phi} - \ddot{\alpha}) - H(\dot{\phi} - \dot{\alpha}) - E(\phi - \alpha) + B, \tag{3.31}$$

where δJ_{xx} is the roll added inertia and B is the bias moment created by several sources such as a steady beam wind, a shift of cargo, water, or ice on deck. Setting $\phi_r = \phi - \alpha$, (3.31) takes the form (see, e.g., [87])

$$(J_{xx} + \delta J_{xx})\ddot{\phi}_r + H\dot{\phi}_r + E\phi_r = B - J_{xx}\ddot{\alpha}. \tag{3.32}$$

Wright and Marshfield [86] solved (3.32) for small nonlinear restoring moment and small linear and cubic damping near the resonance frequency using three different approximate techniques: perturbation method, averaging method, and harmonic balance. Lin and Salvesen [88] presented an assessment of the Large Amplitude Motion Program (LAMP) for evaluating ship performance in extreme seas. The study included a time domain simulation of a ship capsizing in beam seas. It was shown that capsizing can happen due to dynamic effects even for ships that satisfy the minimum righting arm requirement. Surendran and Reddy [89, 90] evaluated the performance of a ship in beam seas using strip theory. The critical condition in the rolling motion of a ship is when it is subjected to synchronous beam waves (i.e., the encounter frequency coincides with the wave frequency). They considered various representations of damping and restoring terms to identify the effect of wave amplitude, wave frequency, and metacentric height (was represented by a quintic polynomial).

Contento et al. [91] reported some results of experimental tests on nonlinear rolling in a regular beam sea of a Ro-Ro ship model by varying both the wave steepness and the wave frequency. They adopted a parameter estimation technique based on the least squares fitting of the stationary numerical solution of the nonlinear rolling motion differential equation. It was possible to extract information on the damping model and on the linear and nonlinear damping coefficients. These exhibit a quite strong dependence on frequency that reduces the efficiency of constant coefficients rolling equation to simulate large amplitude nonlinear rolling. The results indicate that a good quality prediction model of nonlinear rolling cannot be based on constant coefficients time-domain simulations. The analysis indicates also a marked dependence of the effective wave slope coefficient on wave amplitude. The effect of the excitation modeling on the fitting capability of the nonlinear roll motion equation to experimental data was studied by Francescutto et al. [92]. Several frequency dependent and constant effective wave slope coefficients were derived for five different scale models corresponding to different ship typologies by a parameter identification technique. Later, Francescutto and Contento [93] studied the steady rolling response in a regular beam sea of a 1 : 50 scale model of a destroyer in the bare hull condition. In view of the softening characteristics of the restoring moment, bifurcations with jump in amplitude and phase at two different wave frequencies were observed experimentally. Exact numerical solutions were used to obtain reliable values of the coefficients of the mathematical model to be used for the roll motion simulation.

Mahfouz [94] presented a robust method for the identification of linear and nonlinear damping and restoring parameters in the equation describing the rolling motion of a ship using only its measured response at sea. The parameters were identified using a combination of the random decrement technique, auto- and cross-correlation functions, a linear regression algorithm, and a neural-network technique. The proposed method would be particularly useful in identifying the nonlinear damping and restoring parameters for a ship rolling under the action of unknown excitations effected by a realistic sea.

3.5. Memory Effect

Note that the previous formulation did not account for the hydrodynamic memory effect. The hydrodynamic load due to the ship motion is a function of its frequency of oscillation. When the ship oscillates, waves will be generated on the free surface. As time increases, these waves will propagate outward from the body, but they continue to affect the fluid pressure and hence the body force for all subsequent times [14]. In the time domain, this force or moment can be represented by a convolution integral of the impulse response function as outlined by Cummins [95], that is,

$$F_{ij} = -\alpha_{ij}(\infty)\dot{V}_j - \int_{-\infty}^{t} K_{ij}(t-\tau)V_j(\tau)d\tau, \quad i,j = 1,2,3, \tag{3.33}$$

where $i, j = 1, 2, 3$ indicate surge, sway, and yaw, respectively. $V_j(\tau)$ is the ship velocity along the axis j, α_{ij} is the ship added mass, and $K_{ij}(t-\tau)$ is the retardation function and can be expressed in terms of the velocity potential function φ as

$$K_{ij}(t-\tau) = \rho \iint_S \frac{\partial\varphi_j(t-\tau)}{\partial\tau} s_i \, d\sigma, \tag{3.34}$$

where s_i is the ith component of the normal vector of the surface element $d\sigma$. Chung and Bernitsas [96] evaluated these forces in details.

A component of this force initiated at a certain moment continues to attribute its influence on the system for a period of time. This is referred to as the hydrodynamic memory effect [96]. It was indicated that calculating this effect in the time domain is very time consuming. Tick [97] represented the convolution integral by a set of recursive differential equations with constant coefficients. These coefficients are determined by curve fitting the added mass and damping in the frequency domain. This method was used for estimating the memory effect on ship maneuvering by McCreight [98], a single-point mooring tanker by Jiang et al. [99] and Sharma et al. [100], and other motions by Schmiechen [101, 102].

4. Ship Roll Dynamic Stability

Traditional ship stability analysis compares the vessel righting arm curve to a standard or to a steady wind heeling moment (see, e.g., [103, 104]). Modern analysis methods of ship stability are based on analyzing the vessel's roll motion response either by simulation or using modern methods of dynamical systems. One method of analyzing nonlinear dynamical systems used by numerous researchers is the analysis of the so-called safe basin [105]. This method consists of numerically integrating a grid of initial conditions in order to determine which initial conditions will lead to bounded motions (safe basin) and which will yield unbounded motions (i.e., capsizing). There are two cases that are generally analyzed and these are the unbiased and biased cases. The unbiased case involves the symmetric ship while the biased case is for an asymmetric ship. The asymmetry may be due to a steady wind moment, cargo shifting, or an asymmetric ice accretion.

The upper bound of the wave excitation amplitude, beyond which the ship becomes dynamically unstable, is governed by both the damping factor and the excitation frequency. This is reflected by the reduction of the safe basin area with the excitation amplitude. The stability fraction known in the literature as the normalized safe basin area or Safe Basin Integrity Factor [106–109]. It is usually obtained by estimating the ratio of the area of the stable region in the phase plane (area of the safe basin) to the total area encompassed by the homoclinic orbit, which is defined to be the safe basin in the absence of excitation. The stability fraction is strongly dependent on the excitation amplitude. For excitation amplitudes less than a critical value, governed by the excitation frequency, there is no erosion at all for the safe basin. Above this critical value of the excitation amplitude, the area of the safe basin shrinks and the stability fraction drops. As the excitation frequency changes so does the critical excitation amplitude.

Froude [110] observed that ships have undesirable roll characteristics when the frequency of a small free oscillation in pitch is twice the frequency of a small free oscillation in roll. It was Paulling and Rosenberg [8] who formulated the analytical modeling of the coupled roll-pitch motion. This coupled motion is described by a set of nonlinear equations. If the nonlinear effect of roll is neglected, the pitch equation of motion is reduced to a linear differential equation, which is free from roll motion terms. When pitch equation is solved, its response appears as a coefficient to the restoring moment in the roll equation of motion in the well-known Mathieu equation. Nayfeh et al. [1] analyzed the nonlinear coupling of the pitch and roll modes in regular seas when their frequencies are in the ratio of two to one. When the frequency of encounter (excitation frequency) is near the pitch frequency, the pitch mode is excited if the encountered wave amplitude (excitation amplitude) is small. As the excitation amplitude increases, the amplitude of the pitch mode increases until it reaches a critical small value. As the excitation amplitude increases further, the pitch amplitude reaches a saturated value, and all of the extra energy is transferred to the roll mode.

4.1. Stochastic Roll Stability

Sea waves are not sinusoidal and may cause severe or dangerous ship motions. The probabilistic theory of ship dynamics was documented by Price and Bishop [111] and Lloyd [112]. If the nonlinear effect of roll is neglected, the pitch equation of motion is reduced to a linear differential equation, which is free from roll motion terms. When the pitch equation is solved, its response appears as a coefficient to the restoring moment of the roll motion, and the roll equation of motion is reduced to the Mathieu equation

$$\ddot{\phi} + 2\bar{\zeta}\omega_n\dot{\phi} + \omega_n^2[1 + \varepsilon\Theta(t)]\phi = \varepsilon M(t), \tag{4.1}$$

where ϕ is the roll angle, $\Theta(t)$ represents the pitch angle which is assumed to be a random stationary process, $M(t)$ represents the wave random excitation, $\bar{\zeta}$ is a linear damping factor, ω_n is the natural frequency of the ship roll oscillation, and ε is a small parameter.

The analysis of the ship roll stability in an irregular sea was addressed by some investigators (see, e.g., [113–115]). The stochastic stability and response of the ship roll motion in random seas have been predicted analytically using the stochastic averaging method [46, 47, 49, 116]. Roberts [47] analyzed (4.1) for the stochastic stability and statistical response moments using the stochastic averaging method. The sample stability condition of the roll angle amplitude was obtained in the form

$$\bar{\zeta} > \frac{\varepsilon^2\omega_n S_\Theta(2\omega_n)}{8}, \tag{4.2}$$

where $S_\Theta(2\omega_n)$ is the power spectral density of the random pitch process $\Theta(t)$ at frequency $2\omega_n$. Condition (4.2) reveals that the onset of instability is not affected by the forcing excitation $M(t)$. If this excitation is removed, the probability density function of the response degenerates into a delta function. The stability conditions of the first moment of the response amplitude is

$$\bar{\zeta} > \frac{3}{16}\varepsilon^2 \omega_n S_{\Theta}(2\omega_n). \tag{4.3}$$

The stability condition of the second moment is

$$\bar{\zeta} > \frac{1}{4}\varepsilon^2 \omega_n S_{\Theta}(2\omega_n). \tag{4.4}$$

These stability conditions apply only for the case of random sea waves in the absence of ice effects. Haddara [117] obtained the autocorrelation function of the roll motion in irregular seas.

4.2. Probabilistic Roll Dynamics

In addition to the modes of stochastic stability outlined in the previous subsection, it is important to examine the ship probabilistic description in random seas. It is of great importance to estimate the probability of capsizing. Equally important is to identify ship's parameters in roll motion. Different probability approaches have been found very effective in studying these issues. For example, the path integral technique was applied to the roll nonlinear motion of a ship in irregular waves by Kwon et al. [118]. The exciting moment due to irregular waves was modeled as a nonwhite noise. Both damping and nonlinear restoring functions were included with the equivalent white-noise intensity. Lin and Yim [119] developed a stochastic analysis to examine the properties of chaotic roll motion and capsize of ships subjected to a periodic excitation with a random noise disturbance. They used a generalized Melnikov method to provide an upper bound on the domain of the potential chaotic roll motion. The associated Fokker-Planck equation governing the evolution of the probability density function of the roll motion was numerically solved by the path integral solution procedure to obtain joint probability density functions in state space. A chaotic response was found to take place near the homoclinic and heteroclinic orbits. The heteroclinic model emulates symmetric vessel capsize and the homoclinic model represents a vessel with an initial bias caused by water on deck. It was found that the presence of noise enlarges the boundary of the chaotic domains and bridges coexisting attracting basins in the local regimes. The probability of capsize was considered as an extreme excursion problem with the time-averaged probability density function as an invariant measure. In the presence of noise, the numerical results revealed that all roll motion trajectories that visit the regime near the heteroclinic orbit will eventually lead to capsize.

Another version of the path integration approach based on the Gauss-Legendre quadrature integration rule was proposed by Gu [120]. It was applied for estimating the probability density of the nonlinear roll motion of ships in stochastic beam seas. The ship roll motion was described by a nonlinear random differential equation that includes a nonlinear damping moment and restoring moment. The results include the time evolution of the ship response probability density as well as the tail region, which is very important for the system reliability analysis. Gu [121] derived an approximate stationary probability density function and stationary mean out-crossing rate of the response of nonlinear roll-motion subjected to additive stochastic white noise excitations.

Yim et al. [122, 123] developed an analytical approach for the identification of ship parameters and calibration of their prediction capability using experimental results. They examined a three-degree-of-freedom fully coupled roll-heave-sway model, which features realistic and practical high-degree polynomial approximations of rigid body motion relations, hydrostatic and hydrodynamic forces and moments. System parameters of the model were identified using physical model test results from several regular wave cases. The predictive capability of the model is then calibrated using results from a random wave test case. Yim et al. [123] presented a computationally quasi-two-degree-of-freedom stochastic model describing the coupled roll-heave motions and a stability analysis of barges in random seas. Stochastic differential equations governing the evolution of probability densities of roll-heave and roll responses were derived using the Fokker-Planck formulation. Numerical results of roll responses using direct simulation in the time domain and the path integral solution technique in the probability domain were compared to determine the effects of neglecting the influence of heave on roll motion.

The case of small ships with water on deck subjected to random beam waves described by a periodic force and white noise perturbation was considered by Liu and Yougang [124] using the path integral solution. The random Melnikov mean square criterion was used to determine the parameter domain for the ship's stochastic chaotic motion. The evolution of the probability density function of the roll response was calculated by solving the stochastic differential equations using the path integral method. It was found that in the probability density function of the system has two peaks for which the response of the system was found to jump from one peak to another for large amplitudes of periodic excitation. Mamontov and Naess [125] developed a combined analytical-numerical approach referred to as the successive-transition method, which is essentially a version of the path-integration solution and is based on an analytical approximation for the transition probability density. The method was applied to a one-dimensional nonlinear Ito's equation describing the velocity of a ship maneuvering along a straight line under the action of the stochastic drag due to wind or sea waves. It was also used for the problem of ship roll motion up to its possible capsizing. It was indicated that the advantage of the proposed successive transition is that it provides an account for the damping matrix in the approximation.

Haddara and Zhang [126] developed an expression for the joint conditional probability density function for the ship roll angle and roll velocity in beam seas. The joint probability density function was expressed as a double series in the nondimensional roll angle and roll velocity. Jiang et al. [127] examined ships capsizing in random beam seas using the Melnikov function and the concept of phase-flux rates. Damping and wave excitation moments were treated as perturbations since they are relatively small compared with inertial effects and hydrostatic righting moments. Safe and unsafe areas were defined in the phase plane of the unperturbed system model to distinguish the qualitatively different ship motions of capsize and noncapsize. They derived expressions for the phase space flux rate. The correlation of phase space flux and capsize was investigated through extensive simulations. It was shown that these analytical tools provide reliable predictive information regarding the likelihood of a vessel capsize in a given sea state. Gu [128] and Tang et al. [129] employed the Melnikov function and phase space flux to examine the nonlinear roll motion of a fishing ship in random beam seas. They showed that the phase space flux is monotonically increasing as the significant wave height increases, while the safe basin is decreasing rapidly.

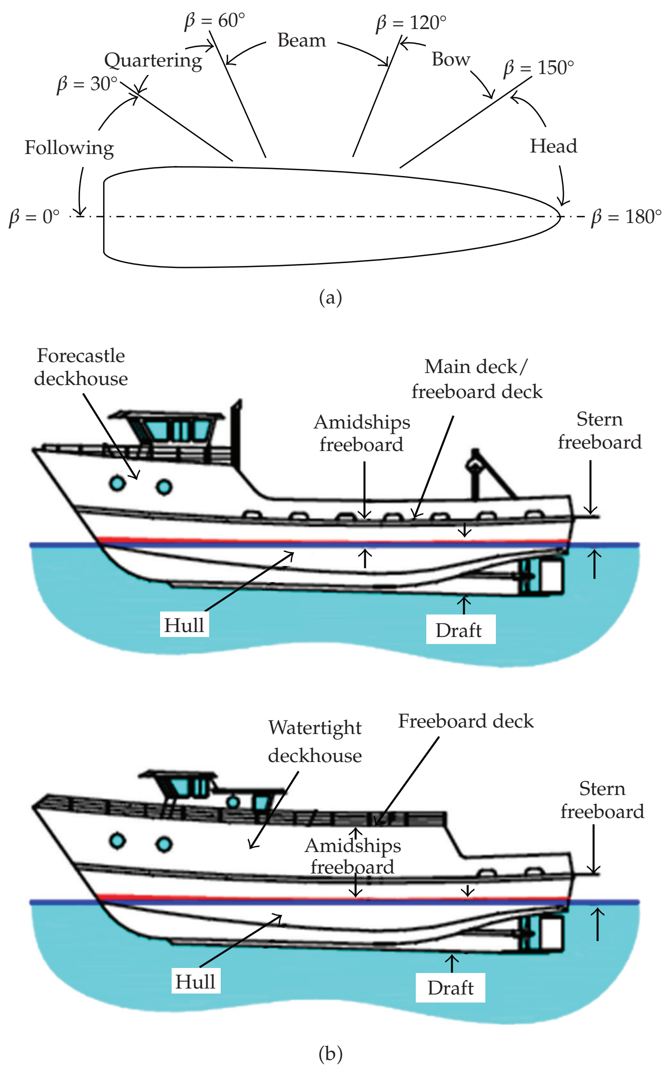

Figure 6: (a) Definition of incident wave directions. (b) Key parts of a ship structure.

Liu et al. [130] considered some methods for constructing safe basins and predicting the survival probability of ships in random beam waves. The nonlinear random roll differential equation was numerically solved in the time domain by considering the instantaneous state of ships and the narrowband wave energy spectrum. The safe basins were constructed for safe navigation, and the survival probability of ships was also estimated. In another work, Liu et al. [131] considered the random differential equation of roll motion in beam seas and the random Melnikov mean-square criterion was used to determine the threshold intensity for the onset of chaos. It was found that ships undergo stochastic chaotic motion when the real intensity of the white noise exceeds the threshold intensity. The stable probability density function of the roll response was found to possess two peaks and the random jump happened in the response of the system for high intensity of the white noise excitation.

Reliability of ship operations under Gaussian or nonGaussian random sea waves deals with the probability that the ship will not capsize. One may estimate the ship reliability in terms of the probabilistic characteristics of the time at which the roll motion first exits from the safe domain. When capsizing is defined by the first exit of response from a safe domain of operation, the reliability is referred to as a first-passage problem. For ships whose response is described by a Markov process, the mean value of the exit time is usually governed by a partial differential equation known as the Pontryagin-Vitt (PV) equation [132].

The first-passage problem of nonlinear roll oscillations in random seas has been considered by Roberts [46, 47], Cai and Lin [133], Cai et al. [134], and Moshchuk et al. [51, 135]. Roberts [46, 47] developed an approximate theory based on a combination of averaging techniques and the theory of Markov processes. His analysis resulted in a simple expression for the distribution of the ship roll angle. Cai et al. [134] adopted the same modeling and introduced a parametric excitation term. They used the modified version of quasiconservative averaging. Moshchuk et al. [51] determined the mean exit time of the perturbed ship motion by solving Pontryagin's partial differential equation using the method of asymptotic expansion. It was found that the mean exit time is extremely large for any excitation intensity less than a critical value above which it experiences exponential decay.

5. Closing Remarks

The nonlinear dynamic modeling of ship motions in roll, pitch, and heave has been formulated based on physical ground. The formulation has been adopted from the work of Neves and Rodríguez [11, 12]. One can use the coupled nonlinear equations motion to examine only the ship motion in roll oscillations under regular and random sea waves. Other issues related to this modeling deal with the effect of roll damping and hydrodynamic memory effect arising from the ship motion. An overview of the roll dynamic stability under random sea waves has been presented in terms of the sample stability condition and response statistical moments. Equally important are the probability of capsizing and the identification of parameters in roll motion. Different probability approaches have been found very effective in studying these issues such as the path integral technique and the generalized Melnikov method, which provides an upper bound on the domain of the potential chaotic roll motion. This paper has not addressed the interaction of roll dynamics with floating ice and the reader can consult the review article by Ibrahim et al. [136].

Appendix

Figure 6 clarifies some of the terminology defined in this Appendix.
Aft: toward the stern of the boat.
Beam: the width of a vessel also a structural component. Both Uses come from the Anglo-Saxon word beam, meaning, "tree".
Beam Sea: sea coming on the side of the ship.
Bilge: the lower point of inner hull of a ship.
Bow: the forward part of a boat. The word may come from the Old Icelandic bogr, meaning "shoulder."
Broach: the action of turning a vessel broadside to the waves.
Broadside: presenting the side of the ship.

Buoyancy: the upward push of water pressure, equal to the weight of the volume of water the ship displaces (W).
Capsize: to turn over.
Center of Buoyancy (B): the geometric center of the submerged hull, acting vertically upward.
Center of Flotation (F): the geometric center of the waterline plane, about which the ship trims fore and aft.
Center of Gravity (G): the center of all mass of the ship, acting vertically downward.
Displacement Volume (v): the volume of the underwater hull at any given waterline.
Displacement (W): the weight of water of the displaced volume of the ship, which equals the weight of the ship and cargo.
Draft: the depth of water a boat draws.
Fathom: six feet.
Following Sea: sea coming on the stern.
Forecastle: pronounced "fo'c's'l" and usually now spelled that way. Now the foredeck of a vessel, the term originally referred to a raised and fortified platform at the ship's bow. Used by archers in combat at sea as early as the 13th century.
Freeboard: that part of a ships sides above water, from the Anglo-Saxon framebord, meaning "the frame's side."
Head: (1) the uppermost or forward-most part of a ship (or of some specific part of a ship, such as the masthead, beakhead, stemhead, or whatever. (2) The bathroom. In the age of sail, the crew was quartered forward in the forecastle, and their latrine was located on the beakhead, over hanging the water (for obvious reasons).
Heading: the direction in which a vessel's bow points at any given time.
Headway: the forward motion of a boat. Opposite of sternway.
Heel: constant roll angle—such as caused by a side wind or turning of the vessel.
Hull: the main body of a vessel.
Keel: the centerline of a boat running fore and aft; the backbone of a vessel.
Knot: a measure of speed equal to one nautical mile (6076 feet) per hour.
Lee: the side sheltered from the wind.
Leeward: the direction away from the wind. Opposite of windward.
Leeway: the sideways movement of the boat caused by either wind or current.
List, Heel, and Roll: it is both a noun and a verb referring to ships upping to one side or the other due to poor trim, shifting cargo, or sinking. The word comes from the Anglo-Saxon lystan, meaning "to lean." Angular transverse inclinations. List describes a static inclination such as list due to side damage. Heel describes a temporary inclination generally involving motion, such as wind or turning, while roll indicates periodic inclination from side to side such as wave action.
Metacenter (M): when the ship is inclined at small angles, the metacenter is the intersection of the buoyant force with the ship centerline. If the metacenter is above the center of gravity then the ship is stable.
Midship: approximately in the location equally distant from the bow and stern.
Nautical Mile: one minute of latitude; approximately 6076 feet: about 1/8 longer than the statute mile of 5280 feet.
Naval Architecture: ship design: especially hull design, overall layout with attention to stability, sea keeping, and strength.
Port: the left side of a boat looking forward.
Quarter: the sides of a boat aft of amidships.
Quarter Sea: sea coming on a boat's quarter.

Reserve Buoyancy: the watertight volume between the waterline and the uppermost continuous watertight deck.
Righting Arm (also *Restoring Lever*): it is the horizontal distance between the vertical line passing through the buoyant center and the vertical line passing through the ship center of gravity.
Starboard: the right side of a boat when looking forward.
Stern: the rear of any vessel. The word came from the Norse Stjorn (pronounced “Styorn”), meaning “steering.” It is the after part of the boat.
Thwartships: it means across the ship.
Trim: longitudinal tilt. Stern draft, bow draft.
Wake: moving waves, track, or path that a boat leaves behind it, when moving across the waters.
Waterline: a line painted on a hull which shows the point to which a boat sinks when it is properly trimmed.
Way: movement of a vessel through the water such as headway, sternway, or leeway.
Windward: toward the direction from which the wind is coming.
Yaw: to swing or steer off course, as when running with a quartering sea.

Acknowledgment

This work is supported by a grant from ONR under Award no: N00014-05-1-0040. Dr. Kelly B. Cooper is the Program Director.

References

[1] A. H. Nayfeh, D. T. Mook, and L. R. Marshall, “Nonlinear coupling of pitch and roll modes in ship motions,” *Journal of Hydronautics*, vol. 7, no. 4, pp. 145–152, 1973.
[2] A. H. Nayfeh, D. T. Mook, and L. R. Marshall, “Perturbation-energy approach for the development of the nonlinear equations of ship motion,” *Journal of Hydronautics*, vol. 8, no. 4, pp. 130–136, 1974.
[3] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, Pure and Applied Mathematics, John Wiley & Sons, New York, NY, USA, 1979.
[4] B. M. Suleiman, *Identification of finite-degree-of-freedom models for ship motions*, Ph.D. thesis, Virginia Polytechnic Institute and State University, Blacksburg, Va, USA, 2000.
[5] T. I. Fossen, *Guidance and Control of Ocean Vehicles*, John Wiley & Sons, New York, NY, USA, 1994.
[6] E. M. Lewandowski, *The Dynamics of Marine Craft Maneuvering and Seakeeping*, vol. 2 of *Advanced Series on Ocean Engineering*, World Scientific, Singapore, 2004.
[7] T. Perez, *Ship Motion Control; Course Keeping and Roll Stabilization Using Rudder and Fins*, Advances in Industrial Control, Springer, Berlin, Germany, 2005.
[8] J. R. Paulling and R. M. Rosenberg, “On unstable ship motions resulting from nonlinear coupling,” *Journal of Ship Research*, vol. 3, no. 1, pp. 36–46, 1959.
[9] W. D. Kinney, “On the unstable rolling motions of ships resulting from nonlinear coupling with pitch including the effect of damping in roll,” Institute of Engineering Research, University of California, Berkeley, Calif, USA, 1961.
[10] M. R. Haddara, “On the stability of ship motion in regular oblique waves,” *International Shipbuilding Progress*, vol. 18, no. 207, pp. 416–434, 1971.
[11] M. A. S. Neves and C. A. Rodríguez, “On unstable ship motions resulting from strong non-linear coupling,” *Ocean Engineering*, vol. 33, no. 14-15, pp. 1853–1883, 2006.
[12] M. A. S. Neves and C. A. Rodríguez, “Influence of non-linearities on the limits of stability of ships rolling in head seas,” *Ocean Engineering*, vol. 34, no. 11-12, pp. 1618–1630, 2007.
[13] J. H. Vugts, “The hydrodynamic forces and ship motions on oblique waves,” Tech. Rep. 150 S, Netherlands Ship Research Center, 1971.

[14] J. N. Newman, *Marine Hydrodynamics*, The MIT Press, Cambridge, Mass, USA, 1977.
[15] L. V. Edward, *Principles of Naval Architecture*, The Society of Naval Architects and Marine Engineers, Jersey City, NJ, USA, 1989.
[16] H. Chu, *Numerical modeling of ship motions in shallow water waves*, Ph.D. thesis, Florida Institute of Technology, Melbourne, Fla, USA, 1998.
[17] Y. Lee, *Non linear ship motion models to predict capsize in regular beam seas*, Ph.D. thesis, University of Michigan, Ann Arbor, Mich, USS, 2001.
[18] M. S. Khalid, *Simulation of Euler equations of motion and blended—nonlinear hydrodynamics for multi-hulled vessels*, Ph.D. thesis, University of Michigan, Ann Arbor, Mich, USA, 2007.
[19] M. A. Abkowitz, "Lectures on ship hydrodynamics—steering and maneuverability," Tech. Rep. Hy-5, Hydro- and Aerodynamics Laboratory, Lyngby, Denmark, 1964.
[20] M. A. Abkowitz, "System identification techniques for ship maneuvering trials," in *Proceedings of the Symposium on Control Theory and Navy Applications*, pp. 337–393, Monterey, Calif, USA, 1975.
[21] M. A. Abkowitz, "Measurements of hydrodynamic characteristics from ship maneuvering trials by system identification," *Transactions of the Society of Architects and Marine Engineering*, vol. 88, pp. 283–318, 1981.
[22] W. Y. Hwang, *Application of system identification to ship maneuvering*, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, Mass, USA, 1980.
[23] C. G. Källström and K. J. Åström, "Experiences of system identification applied to ship steering," *Automatica*, vol. 17, no. 1, pp. 187–198, 1981.
[24] K.-H. Son and K. Nomoto, "On the coupled motion of steering and rolling of a high-speed container ship," *Naval Architecture and Ocean Engineering*, vol. 20, pp. 73–83, 1982.
[25] A. Ross, *Nonlinear Manoeuvring Models for Ships: a Lagrangian Approach*, Ph.D. thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2008.
[26] G. Kirchhoff, *Advanced Engineering Mathematics*, John Wiley & Sons, New York, NY, USA, 1869.
[27] L. M. Milne-Thomson, *Theoretical Hydrodynamics*, MacMillan, London, UK, 1968.
[28] T. I. Fossen, *Marine Control Systems: Guidance, Navigation and Control of Ships*, Rigs and Underwater Vehicles, Marine Cyberentics AS, Trondheim, Norway, 2002.
[29] Z. Rong, *Studies of weak and strong nonlinear sea loads on floating marine structures*, M.S. thesis, Norges Tekniske Hoegskol, Trodheim, Norway, April 1994.
[30] L. Alford, A. Banik, V. Belenky, et al., "Discussion on analytical approaches to the study of vessel dynamics-outcomes of the two-part mini symposium at the 2007 SIAM conference on applications of dynamical systems," *Marine Technology*, vol. 45, no. 4, pp. 211–220, 2008.
[31] J. O. de Kat and J. R. Paulling, "The simulation of ship motions and capsizing in severe seas," *Transactions of the Society of Architects and Marine Engineering*, vol. 97, pp. 139–168, 1989.
[32] J. A. Witz, C. B. Ablett, and J. H. Harrison, "Roll response of semisubmersibles with nonlinear restoring moment characteristics," *Applied Ocean Research*, vol. 11, pp. 153–166, 1989.
[33] J.-P. F. Denise, "On the roll motion of barges," *Transactions of the Royal Institution of Naval Architects*, vol. 125, pp. 255–268, 1983.
[34] D. W. Bass and M. R. Haddara, "On the modeling of the nonlinear damping moment for the rolling motion of a ship," in *Proceedings of the International Association of Scientific Technology (IASTED) Symposium: Identification, Modeling and Simulation*, pp. 346–349, Paris, France, 1987.
[35] D. W. Bass and M. R. Haddara, "Nonlinear models of ship roll damping," *International Shipbuilding Progress*, vol. 35, no. 401, pp. 5–24, 1988.
[36] M. Taylan, "The effect of nonlinear damping and restoring in ship rolling," *Ocean Engineering*, vol. 27, no. 9, pp. 921–932, 2000.
[37] J. F. Dalzell, "A note on the form of ship roll damping," *Journal of Ship Research*, vol. 22, no. 3, pp. 178–185, 1978.
[38] A. Cardo, A. Francescutto, and R. Nabergoj, "On damping models in free and forced rolling motion," *Ocean Engineering*, vol. 9, no. 2, pp. 171–179, 1982.
[39] J. B. Mathisen and W. G. Price, "Estimation of ship roll damping coefficients," *Transactions Royal Institution of Naval Architects*, vol. 127, pp. 295–307, 1984.
[40] A. F. El-Bassiouny, "Nonlinear analysis for a ship with a general roll-damping model," *Physica Scripta*, vol. 75, no. 5, pp. 691–701, 2007.
[41] M. Taylan, "Effect of forward speed on ship rolling and stability," *Mathematical and Computational Applications*, vol. 9, no. 2, pp. 133–145, 2004.
[42] A. H. Nayfeh and B. Balachandran, *Applied Nonlinear Dynamics*, Wiley Series in Nonlinear Science, John Wiley & Sons, New York, NY, USA, 1995.

[43] L. Arnold, I. Chueshov, and G. Ochs, "Stability and capsizing of ships in random sea-a survey," *Nonlinear Dynamics*, vol. 36, no. 2–4, pp. 135–179, 2004.

[44] S. Surendran, S. K. Lee, J. V. R. Reddy, and G. Lee, "Non-linear roll dynamics of a Ro-Ro ship in waves," *Ocean Engineering*, vol. 32, no. 14-15, pp. 1818–1828, 2005.

[45] G. Bulian, "Nonlinear parametric rolling in regular waves–a general procedure for the analytical approximation of the GZ curve and its use in time domain simulations," *Ocean Engineering*, vol. 32, no. 3-4, pp. 309–330, 2005.

[46] J. B. Roberts, "A stochastic theory for nonlinear ship rolling in irregular seas," *Journal of Ship Research*, vol. 26, no. 4, pp. 229–245, 1982.

[47] J. B. Roberts, "Effect of parametric excitation on ship rolling motion in random waves," *Journal of Ship Research*, vol. 26, no. 4, pp. 246–253, 1982.

[48] J. Falzarano and F. Zhang, "Multiple degree of freedom global analysis of transient ship rolling motion," in *Proceedings of the ASME Winter Annual Meeting on Nonlinear Dynamics of Marine Vehicles*, J. M. Falzarano and F. Papoulias, Eds., vol. 51, pp. 57–72, New York, NY, USA, 1993.

[49] X. Huang, X. Gu, and W. Bao, "The probability distribution of rolling amplitude of a ship in high waves," in *Proceedings of the 5th International Conference on Stability of Ships and Ocean Vehicles*, Melbourne, Fla, USA, 1994.

[50] I. Senjanović, G. Ciprić, and J. Parunov, "Survival analysis of fishing vessels rolling in rough seas," *Philosophical Transactions of the Royal Society A*, vol. 358, no. 1771, pp. 1943–1965, 2000.

[51] N. K. Moshchuk, R. A. Ibrahim, R. Z. Khasminskii, and P. L. Chow, "Asymptotic expansion of ship capsizing in random sea waves-I. First-order approximation," *International Journal of Non-Linear Mechanics*, vol. 30, no. 5, pp. 727–740, 1995.

[52] G. H. Bryan, "The action of bilge keels," *Transactions of the Institute of Naval Architects*, vol. 42, pp. 198–239, 1900.

[53] T. Hishida, "Study on the wave making resistance for rolling ships—parts 1 and 2," *Journal of Society of Naval Architecture of Japan*, vol. 85, pp. 45–61, 1952.

[54] T. Hishida, "Study on the wave making resistance for rolling ships—parts 3 and 4," *Journal of Society of Naval Architecture of Japan*, vol. 86, pp. 277–283, 1954.

[55] T. Hishida, "Study on the wave making resistance for rolling ships—parts 5 and 6," *Journal of Society of Naval Architecture of Japan*, vol. 87, pp. 67–68, 1955.

[56] H. Kato, "On the frictional resistance of the roll of ships," *The Science of Naval Architects of Japan*, vol. 102, pp. 115–123, 1958.

[57] M. Martin, "Roll damping due to bilge keels," Tech. Rep., Iowa University, Institute of Hydraulics Research, Iowa City, Iowa, USA, 1958.

[58] M. Tanaka, "A study on the bilge keel—part 1," *Journal of Society of Naval Architecture of Japan*, vol. 101, pp. 99–105, 1957.

[59] M. Tanaka, "A study on the bilge keel—part 2," *Journal of Society of Naval Architecture of Japan*, vol. 103, p. 343, 1958.

[60] M. Tanaka, "A study on the bilge keel—part 3," *Journal of Society of Naval Architecture of Japan*, vol. 105, pp. 27–32, 1959.

[61] M. Tanaka, "A study on the bilge keel—part 4," *Journal of Society of Naval Architecture of Japan*, vol. 104, pp. 205–212, 1961.

[62] H. Kato, "Effect of bilge keels on the rolling of ships," *Journal of Society of Naval Architecture of Japan*, vol. 117, pp. 93–114, 1965.

[63] C. G. Moody, "The effect of bilge keels on the roll damping characteristics of the large bulk-oil carrier SS World Glory," Tech. Rep., David Taylor Model Basin, West Bethesda, Md, USA, 1961.

[64] L. E. Motter, "Ship motion and bilge keel studies for the CVA," Tech. Rep. OPNAVINST S5513.3A-17, David Taylor Model Basin, West Bethesda, Md, USA, 1967.

[65] H. Jones, "Bilge keel size investigation for an attack aircraft carrier (CVA-41)," Tech. Rep. DTNSRDC/SPD-0861-01, David Taylor Model Basin, West Bethesda, Md, USA, 1978.

[66] H. Jones, "Bilge keel size investigation for an attack aircraft carrier and vstol (vertical short takeoff and landing," Tech. Rep. DTNSRDC/SPD-0861-02, David Taylor Model Basin, West Bethesda, Md, USA, 1979.

[67] R. E. D. Bishop and W. G. Price, "A note on structural damping of ship hulls," *Journal of Sound and Vibration*, vol. 56, no. 4, pp. 495–499, 1978.

[68] Y. Himeno, "Prediction of ship roll damping: state-of-the-art," Tech. Rep. 239, Department of Naval Architecture and Marine Engineering, The University of Michigan, Ann Arbor, Mich, USA, September 1981.

[69] R.-Q. Lin, W. Kuang, and A. M. Reed, "Numerical modeling of nonlinear interactions between ships and surface gravity waves—part I: ship waves in calm water," *Journal of Ship Research*, vol. 49, no. 1, pp. 1–11, 2005.

[70] R.-Q. Lin and W. Kuang, "Nonlinear ship-wave interaction model—part 2: ship boundary condition," *Journal of Ship Research*, vol. 50, no. 2, pp. 181–186, 2006.

[71] R.-Q. Lin and W. Kuang, "Modeling nonlinear roll damping with a self-consistent, strongly nonlinear ship motion model," *Journal of Marine Science and Technology*, vol. 13, no. 2, pp. 127–137, 2008.

[72] N. Salvesen, "Five years of numerical naval ship hydrodynamics at DTNSRDC," Tech. Rep. DTNSRDC-79/082, David W. Taylor Naval Ship Research and Development Center, Bethesda, Md, USA, 1979.

[73] W. M. Lin, M. J. Meinhold, and N. Salvesen, "Nonlinear time-domain simulation of fishing vessel capsizing in quartering seas," Tech. Rep. SAIC-CG-D-17-98, Science Applications International, Annapolis, Md, USA, April 1998.

[74] S. Chakrabarti, "Empirical calculation of roll damping for ships and barges," *Ocean Engineering*, vol. 28, no. 7, pp. 915–932, 2001.

[75] Y. Ikeda, "Roll damping of ships," in *Proceedings of the Ship Motions, Wave Loads and Propulsive Performance in a Seaway, 1st Marine Dynamics Symposium*, pp. 241–250, The Society of Naval Architecture in Japan, 1984.

[76] Y. Ikeda, Y. Himeno, and N. Tanaka, "Components of roll damping of ships at forward speed," *Journal of Society of Naval Architecture of Japan*, vol. 143, pp. 121–133, 1978.

[77] Y. Ikeda, Y. Himeno, and N. Tanaka, "A prediction method for ships roll damping," Tech. Rep. 00405, Department of Naval Architecture, University of Osaka Prefecture, Osaka, Japan, 1978.

[78] Y. Ikeda, T. Fujiwara, and T. Katayama, "Roll damping of a sharp-cornered barge and roll control by a new-type stabilizer," in *Proceedings of the 3rd International Offshore and Polar Engineering Conference*, pp. 634–639, Singapore, June 1993.

[79] J. P. Morison, M. P. O'Brien, J. W. Johnson, and S. A. Schaaf, "The force exerted by surface waves on a pile," *Petroleum Transactions*, vol. 189, pp. 149–154, 1950.

[80] T. Sarpkaya and M. Isaacson, *Mechanics of Wave Forces on Offshore Structures*, Van Nostrand Reinhold, New York, NY, USA, 1981.

[81] M. R. Haddara, "On the random decrement for nonlinear rolling motion," in *Proceedings of the 11th International Offshore Mechanics and Arctic Engineering Symposium*, vol. 2 of *Safety and Reliability*, pp. 321–324, Calgary, Canada, 1992.

[82] X. Wu, L. Tao, and Y. Li, "Nonlinear roll damping of ship motions in waves," *Journal of Offshore Mechanics and Arctic Engineering*, vol. 127, no. 3, pp. 205–211, 2005.

[83] N. Salvesen, O. E. Tuck, and O. Faltinsen, "Ship motions and sea loads," *Transactions Society of Architects and Marine Engineering*, vol. 78, pp. 250–287, 1970.

[84] W. G. Meyers, D. J. Sheridan, and N. Salvesen, "Manual: NSRDC ship-motion and sea-load computer program," Tech. Rep. 3376, Naval Ship Research and Development Center, Carderock, Md, USA, 1975.

[85] A. H. Nayfeh, S. A. Ragab, and A. A. Al-Maaitah, "Effect of bulges on the stability of boundary layers," *Physics of Fluids*, vol. 31, no. 4, pp. 796–806, 1988.

[86] J. H. G. Wright and W. B. Marshfield, "Ship roll response and capsize beviour in beam seas," *Transactions of the Institute of Naval Architects*, no. 3, pp. 129–148, 1979.

[87] C. Kuo and Y. Odabasi, "Theoretical studies on intact stability of ships (Phase 2)," Department of Shipbuilding and Naval Architecture, University of Strathclyde, October 1974.

[88] W. M. Lin and N. Salvesen, "Nonlinear time-domain simulation of ship capsizing in beam seas," Final Report, Science Applications International, Annapolis, Md, USA, November 1997.

[89] S. Surendran and J. V. R. Reddy, "Roll dynamics of a Ro-Ro ship," *International Shipbuilding Progress*, vol. 49, no. 4, pp. 301–320, 2002.

[90] S. Surendran and J. V. R. Reddy, "Numerical simulation of ship stability for dynamic environment," *Ocean Engineering*, vol. 30, no. 10, pp. 1305–1317, 2003.

[91] G. Contento, A. Francescutto, and M. Piciullo, "On the effectiveness of constant coefficients roll motion equation," *Ocean Engineering*, vol. 23, no. 7, pp. 597–618, 1996.

[92] A. Francescutto, G. Contento, M. Biot, L. Schiffrer, and G. Caprino, "Effect of excitation modeling in the parameter estimation of nonlinear rolling," in *Proceedings of the 8th International Offshore and Polar Engineering Conference*, vol. 3, pp. 490–498, Monteral, Canada, May 1998.

[93] A. Francescutto and G. Contento, "Bifurcations in ship rolling: experimental results and parameter identification technique," *Ocean Engineering*, vol. 26, no. 11, pp. 1095–1123, 1999.

[94] A. B. Mahfouz, "Identification of the nonlinear ship rolling motion equation using the measured response at sea," *Ocean Engineering*, vol. 31, no. 17-18, pp. 2139–2156, 2004.
[95] W. E. Cummins, "The impulse response function and ship motions," *Schiffstechnik*, vol. 9, pp. 101–109, 1962.
[96] J.-S. Chung and M. M. Bernitsas, "Hydrodynamic memory effect on stability, bifurcation, and chaos of two-point mooring systems," *Journal of Ship Research*, vol. 41, no. 1, pp. 26–44, 1997.
[97] L. J. Tick, "Differential equations with frequency-dependent coefficients," *Journal of Ship Research*, vol. 3, no. 2, pp. 45–46, 1959.
[98] W. R. McCreight, "Ship maneuvering in waves," in *Proceedings of the 16th Symposium on Naval Hydrodynamics: Ship Wakes; Large Amplitude Waves; Real Fluid Effects in Ship Hydrodynamics; Fluid-Structure Interaction; Frontier Problems in Hydrodynamics*, pp. 456–469, National Research Council, US Naval Studies Board, University of California, 1986, US Office of Naval Research, USA.
[99] T. Jiang, T. E. Schellin, and S. D. Sharma, "Maneuvering simulation of a tanker moored in a steady current including hydrodynamic memory effect and stability analysis," in *Proceedings of the Royal Institution of Naval Architects (RINA) International Conference on Ship Maneuvering*, London, UK, 1987.
[100] S. D. Sharma, T. Jiang, and T. E. Schellin, "Dynamic instability and chaotic motions of a single-point-moored tanker," in *Proceedings of the 17th Office of Naval Research (ONR) Symposium on Naval Hydrodynamics*, The Hague, The Netherlands, 1988.
[101] M. Schmiechen, "Zur kollisiondynamik von Schiffen," *Jahrbuch der Schiffbautechnischen Gesellschaft*, vol. 68, 1974.
[102] M. Schmiechen, "Equation for non-quasi-steady ship motion," in *Proceedings of the 14th International Towing Tank Conference*, Ottawa, Canada, 1975.
[103] A. Francescutto, A. Serra, and S. Scarpa, "A critical analysis of weather criterion for intact stability of large passenger vessels," in *Proceedings of the International Conference on Offshore Mechanics and Arctic Engineering (OMAE '01)*, vol. 1, pp. 829–836, 2001.
[104] K. Spyrou, "A basis for developing a rational alternative to the weather criterion: problems and capabilities," in *Proceedings of the 6th International Ship Stability Workshop*, October 2002.
[105] J. M. T. Thompson, "Designing against capsize in beam seas: recent advances and new insights," *Applied Mechanics Reviews*, vol. 50, no. 5, pp. 307–324, 1997.
[106] A. N. Lansbury, J. M. T. Thompson, and H. B. Stewart, "Basin erosion in the twin-well Duffing oscillator: two distinct bifurcations scenarios," *International Journal of Bifurcation and Chaos*, vol. 2, pp. 505–532, 1992.
[107] J. M. T. Thompson, "Loss of engineering integrity due to the erosion of absolute and transient basin boundaries," in *Proceedings of IUTAM Symposium 'Nonlinear Dynamics in Engineering Systems'*, W. Schiehlen, Ed., Springer, Berlin, Germany, August 1989.
[108] J. M. T. Thompson, "Chaotic phenomena triggering the escape from a potential well," *Proceedings Royal Society London A*, vol. 421, pp. 195–225, 1989.
[109] J. M. T. Thompson and F. A. McRobie, "Intermediate bifurcations and global dynamics of driven oscillators," in *Proceedings of the 1st European Nonlinear Oscillations Conference*, E. Kreuzer and G Schmidt, Eds., vol. 72 of *Mathematical Research*, pp. 107–128, Academie, Hamburg, Germany, 1993.
[110] W. Froude, "Remarks on Mr. Scott Russell's paper on rolling," *Transactions of the Institute of Naval Research*, vol. 4, pp. 232–275, 1863.
[111] W. G. Price and R. E. D. Bishop, *Probabilistic Theory of Ship Dynamics*, Chapman & Hall, London, UK, 1974.
[112] A. R. J. M. Lloyd, *Seakeeping—Ship Behavior in Rough Weather*, Ellis Howood, Chichester, UK, 1989.
[113] W. G. Price, "A stability analysis of the roll motion of a ship in an irregular seaway," *International Shipbuilding Progress*, vol. 22, no. 247, pp. 103–112, 1975.
[114] M. R. Haddara, "A study of stability of the mean and variance of rolling motion in random waves," in *Proceedings of the International Conference on Stability of Ships and Ocean Vehicles*, University of Strathclyde, Glasgow, UK, 1975.
[115] P. K. Muhuri, "A study of the stability of the rolling motion of a ship in an irregular seaway," *International Shipbuilding Progress*, vol. 27, no. 310, pp. 139–142, 1980.
[116] J. B. Roberts and N. M. C. Dacunha, "Roll motion of a ship in random beam waves: comparison between theory and experiment," *Journal of Ship Research*, vol. 29, no. 2, pp. 112–126, 1985.
[117] M. R. Haddara, "Note on the power spectrum of nonlinear rolling motion," *International Shipbuilding Progress*, vol. 30, no. 342, pp. 41–44, 1983.

[118] S. Kwon, D. Kim, and J. Chung, "Application of path integral solution to ship rolling motion," in *Proceedings of the 3rd International Offshore and Polar Engineering Conference (ISOPE '93)*, pp. 657–660, Singapore, 1993.

[119] H. Lin and S. C. S. Yim, "Chaotic roll motion and capsize of ships under periodic excitation with random noise," *Applied Ocean Research*, vol. 17, no. 3, pp. 185–204, 1995.

[120] J.-Y. Gu, "Calculation of ship rolling probability using a new path integration method," *Journal of Ship Mechanics*, vol. 10, no. 6, pp. 43–52, 2006.

[121] J.-Y. Gu, "Probability analysis of ship nonlinear roll-motion excited by white noise," *Journal of Ship Mechanics*, vol. 10, no. 3, pp. 17–25, 2006.

[122] S. C. Yim, T. Nakhata, W. A. Bartel, and E. T. Huang, "Coupled nonlinear barge motions—part I: deterministic models development, identification and calibration," in *Proceedings of the 23rd ASME International Conference on Offshore Mechanics and Arctic Engineering (OMAE '04)*, vol. 1, pp. 281–291, Vancouver, Canada, 2004.

[123] S. C. Yim, T. Nakhata, and E. T. Huang, "Coupled nonlinear barge motions—part II: deterministic models stochastic models and stability analysis," in *Proceedings of the 23rd ASME International Conference on Offshore Mechanics and Arctic Engineering (OMAE '04)*, vol. 1A, pp. 293–302, Vancouver, Canada, 2004.

[124] L. Liu and T. Yougang, "Stability of ships with water on deck in random beam waves," *Journal of Vibration and Control*, vol. 13, no. 3, pp. 269–280, 2007.

[125] E. Mamontov and A. Naess, "An analytical-numerical method for fast evaluation of probability densities for transient solutions of nonlinear Ito's stochastic differential equations," *International Journal of Engineering Science*, vol. 47, no. 1, pp. 116–130, 2009.

[126] M.R. Haddara and Y. Zhang, "On the joint probability density function of non-linear rolling motion," *Journal of Sound and Vibration*, vol. 169, no. 4, pp. 562–569, 1994.

[127] C. Jiang, A. W. Troesch, and S. W. Shaw, "Highly nonlinear rolling motion of biased ships in random beam seas," *Journal of Ship Research*, vol. 40, no. 2, pp. 125–135, 1996.

[128] J.-Y. Gu, "Nonlinear rolling motion of ship in random beam seas," *Journal of Marine Science and Technology*, vol. 12, no. 4, pp. 273–279, 2004.

[129] Y.-G. Tang, J.-Y. Gu, H.-Y. Zheng, and H.-X. Li, "Study on the ship capsize in random beam seas using Melnikov method," *Journal of Ship Mechanics*, vol. 8, no. 5, pp. 27–34, 2004 (Chinese).

[130] L.-Q. Liu, Y.-G. Tang, H.-Y. Zheng, and J.-Y. Gu, "Analysis method of capsizing probability in time domain for ships in random beam waves," *Journal of Tianjin University Science and Technology*, vol. 39, no. 2, pp. 165–169, 2006 (Chinese).

[131] L.-Q. Liu, Y.-G. Tang, and H.-X. Li, "Stochastic chaotic motion of ships in beam seas," *Journal of Marine Science and Technology*, vol. 15, no. 2, pp. 123–128, 2007.

[132] L. S. Pontryagin, A. A. Andonov, and A. A. Vitt, "On statistical consideration of dynamical systems," *Journal of Experimental and Theoretical Physics*, vol. 3, no. 3, pp. 165–172, 1933.

[133] G. Q. Cai and Y. K. Lin, "On statistics of first-passage failure," *Journal of Applied Mechanics*, vol. 61, pp. 93–99, 1994.

[134] G. Q. Cai, J. S. Yu, and Y. K. Lin, "Ship rolling in random sea," in *Stochastic Dynamics and Reliability of Nonlinear Ocean Systems*, R. A. Ibrahim and Y. K. Lin, Eds., vol. 77, pp. 81–88, ASME, New York, NY, USA, 1994.

[135] N. K. Moshchuk, R. Z. Khasminskii, R. A. Ibrahim, and P. L. Chow, "Asymptotic expansion of ship capsizing in random sea waves-II. Second-order approximation," *International Journal of Non-Linear Mechanics*, vol. 30, no. 5, pp. 741–757, 1995.

[136] R. A. Ibrahim, N. G. Chalhoub, and J. Falzarano, "Interaction of ships and ocean structures with ice loads and stochastic ocean waves," *Applied Mechanics Reviews*, vol. 60, no. 5, pp. 246–290, 2007.