Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 972873, 23 pages
doi:10.1155/2010/972873

Research Article

A Model of Gear Transmission: Fractional Order System Dynamics

Katica (Stevanović) Hedrih[1,2] and Vera Nikolić-Stanojević[3,4]

[1] *Mathematical Institute SANU, Belgrade, Serbia*
[2] *Faculty of Mechanical Engineering, University of Niš, Yu-18 000, ul. Vojvode Tankosića 3/22, Niš, Serbia*
[3] *State University of Novi Pazar, Novi Pazar, Serbia*
[4] *Faculty of Mechanical Engineering, University of Kragujevac, Kragujevac, Serbia*

Correspondence should be addressed to Katica (Stevanović) Hedrih, khedrih@eunet.rs

Received 24 January 2010; Revised 2 March 2010; Accepted 6 May 2010

Academic Editor: Alexander P. Seyranian

A theoretical model of multistep gear transmission dynamics is presented. This model is based on the assumption that the connection between the teeth of the gears is with properties within the range from ideal clasic to viscoelastic so that a new model of connection between the teeth was expressed by means of derivative of fractional order. For this model a two-step gear transmision with three degrees of freedom of motion has been used. The obtained solutions are in the analytic form of the expansion according to time. As boundary cases this model gives results for the case of ideally elastic connection of the gear teeth and for the case of viscoelastic connection of the gear teeth, as well. Eigen fractional modes are obtained and a vizualization is done.

1. Introduction

Gear transmissions have a long history dating back since the time of the first engineering systems. Their practical usage in the present day modern engineering systems is enormous. In accordance with contemporary development of mechanical engineering technics ever growing requirements have been imposed concerning characteristics and working specifications. The machines which utilize high-power duty gear transmissions (excavating machines, crushing mashines, rolling machines, ships, etc.) operate under nonstationary conditions so that the loads of the elements of these gear transmissions are variable. For example, abrupt accelerations and abrupt decelerations of machine parts, that is, masses of the gear transmissions cause inertial forces which, in addition to the conditions of operation, influence the magnitude of actual leads of the elements of gear transmissions. All this, together with the changes of the torque of drive and operating machine, the forces induced by dynamic

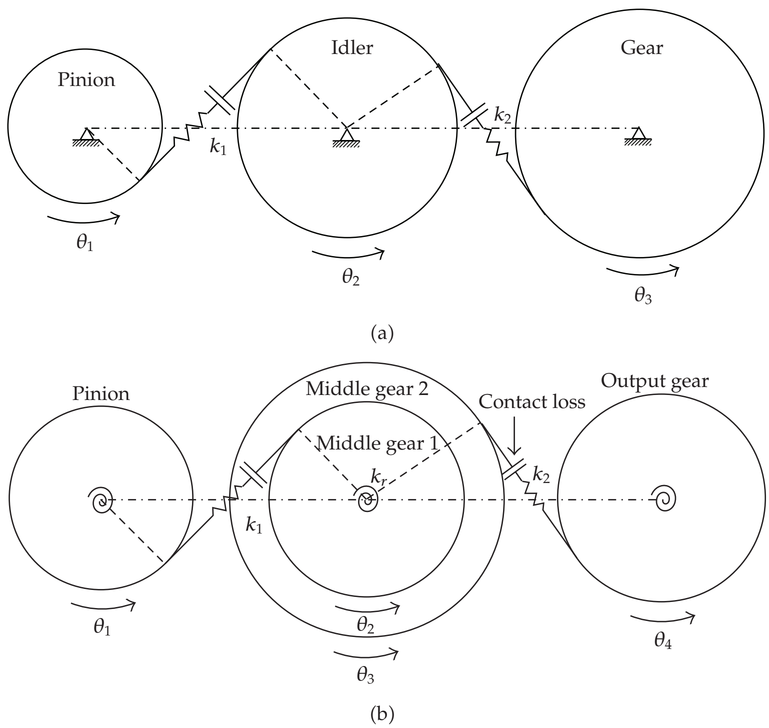


Figure 1: Two models of the gear power transmission with visco-elastic fractional order tooth coupling.

behaviours of the complete system, and so forth, lead to the simulation where the stresses in the gears are higher than critical stresses; after certain time this may result in breakage of the teeth.

1.1. Introduction into Nonlinear Dynamics of the Rotors

Dynamics of coupled rotors (see Figure 1) and of gyrorotors are very old engineering problems with many different research results and discoveries of new nonlinear phenomena, and of stationary and no stationary vibrations regimes with different kinetic parameters of the dynamical system (see [1–14]). However, even nowadays many researchers pay attention to this problem again.

Chaotic clock models, as well as original ideas on a paradigm for noise in machines were presented by Moon (see [15]): "All machines exhibit a greater or lesser amount of noise. The question arises as to whether a certain level of noise is natural or inevitable in a complex assembly of mechanical or electromechanical devices?" In the cited paper, the nature of noise or chaos in a specific class of complex multibody machines, namely the clock was examined. For examining natural clocks of reductors (power transmission), as well as source of nonlinear vibrations and noise in its dynamics, it is necessary to investigate properties of nonlinear dynamics, and phase portraits, as well as structures of homoclinic orbits, layering and sensitivity of this layering of homoclinic orbits and bifurcation of homoclinic points, as it is presented in [6, 9–11].

Following up the idea of Mossera that the distance between trajectories be measured maintaining different time scales or "clock" with which time is measured along each motion, Leela (see [16]) defines the new concepts of orbital stability in terms of given topology of the function space. Leela's paper pointed out the different kind of clock. Perfect clock corresponds to stable system dynamics, entire clock space corresponds to the chaotic topology and chaotic-like dynamics of the system.

By using examples of the rotor system which rotates about two axes with section or without section, we applied the vector method of the kinetic parameters analysis of the rotors with many axes which is done in [12, 13, 17–28]. In the previous listed papers, the expressions for the corresponding linear momentum and angular momentum, as well as their derivatives in time for the rotors with coupled rotation are used in the vector form. By these expressions, the vector equations of the gyrorotor system dynamics are derived, as well as the expression for the kinetic pressures on the gyro rotor system bearings. The mass moment vectors introduced and defined by first author (see [6, 7, 17, 18, 29–36]), are used to present a vector method for the analysis of kinetic parameter of coupled rigid rotors dynamics with deviational mass properties of rotor, as well as of the dynamics of rotor with changeable mass distribution (see [32, 32]).

By using vector equations (see [21, 22, 24, 25]), two scalar differential equations of the heavy rotor system nonlinear dynamic for the case that disc is skewly eccentrically positioned on the own polhode shaft axis (gyrodisk-rotor) is studied. For the case when one rotation about axis is controlled by constant angular velocity, the nonlinear dynamics of the rotation about other axis is studied. Non-linear gyrodisc-rotor system dynamics is presented by phase portrait in the phase plane, with trigger of the coupled singularities, as well as with homoclinic orbits and homoclinic points of the no stable type saddle. For the case of gyrodisc-rotor system dynamics under the action of the perturbed couple the sensitive dependence in the vicinity of the equilibrium no stable position which corresponds to homoclinic point of the type no stable saddle, the possibility of the chaotic character behavior is pointed out.

Expressions of the kinetic pressures of shaft bearing are determined.

The analogy between motions of heavy material point: on the circle in vertical plane which rotates around vertical axis in the plane (see [37, 38]) and corresponding motions case of the heavy rotor around two axes with cross section, as well as of the gyrodisc-rotor which rotates around two axes is pointed out (see [5, 21, 24, 25, 39, 40]).

Dynamics of disc on the one, or more, shaft is a classical engineering problem. This problem attracts attention of many researchers and permanently takes place in world scientific and engineering professional literature (see [3, 41, 42]). Some of these problems are classical and can be found in university text books of mechanics (see [42]). As we can see, these problems are in the nonlinear dynamics described by nonlinear differential equations without analytical solutions. In present time these problems were conditionally and approximately solved by approximate solutions or by linearizations (first by Simes, Stodola, Rubanik and others [38, 43]). Problem of dynamics of the eccentric, skewly positioned disc on one-shaft rotation is classical problem with gyroscopic effect (see classical text books [17, 38, 43, 44]) which takes place in all text books of Dynamics and Theory of Oscillations with applications in engineering, but their presentations are finished only by nonlinear differential equations without their solutions and expression for kinetic pressures. Nowadays, numerous new published papers containing different approximations of the solutions of different classes of the mathematical descriptions of rotor dynamics are not enough to take are into account all real influential factors to describe real system dynamics. This is inspiration for new research in this area.

By using knowledge of nonlinear mechanics (see [37, 45]), as well as by using introduced mass moments vectors and vector rotators in the series of the published papers [8, 19, 23–25, 34, 36, 46–49] phase portrait of gyrorotor dynamics with analysis of static and dynamical equilibrium positions depending on system kinetic parameters are presented in new light and new approach.

Using new knowledge in the nonlinear mechanics, theory of chaos and dynamical systems published in [19, 50, 51], the sensitive dependence of the initial conditions and of the forced motion—oscillation/rotation/stochasticlike-chaoticlike motion of the heavy rotor with vibrating axis as well as gyrorotor in the "vicinity" of the homoclinic point and orbit are analyzed. We followed the ideas of Holmes from [52] on the example pendulum excited by one frequency force, and which showed us that Poincare maps contain the Smale horseshoe map as well as global analysis processes of the dynamical systems which posses on the homoclinic orbit is suitable for applying to study of the rotor dynamic. By using ideas of Holmes from [52], it is easy to prove that forced dynamic of the heavy gyrorotor has in the vicinity of homoclinic point sensitive dependence of initial conditions.

In the paper [6] the motion of a heavy body around a stationary axis in the field with turbulent damping [53] is investigated and kinetic pressures on bearings are expressed by mass moment vectors for the pole in the stationary bearing and for the axis of the body rotation. The motion equations of a variable mass object rotating around a fixed axis are expressed by mass moment vector for the pole and the axis and presented in [20].

A trigger of coupled singularities, on an example of coupled rotors with deviational material particles are presented in [54]. Non-linear phenomena in rotor dynamics were investigated in the series of [6].

From time to time it is useful to pay attention again to classical models of dynamics of mechanical systems and evaluate possibilities for new approaches to these classical results by using other than the methods usually used in the classical literature.

The interest in the study of vector and tensor methods with applications in the *Dynamics* especially in *Kinetics* of rigid and solid body rotational motions and deformation displacements as a new qualitative approach to the optimization of the time for study process grew exponentially over the last few years because theoretical challenges involved in the study of technical sciences need such optimization of university systems study. Short time for fundamental knowledge transfer during one term (semester) courses with high level of apparent study results requires the optimization of the time for introducing new basic high level scientific ideas (logic and philosophical) which are easy to understand to most of students in the study process and for engineering applications this is very important.

Also, we can conclude that the impact of different possibilities to establish the phenomenological analogy of different model dynamics expressed by vectors connected to the pole and the axis and the influence of such possibilities to applications allows professors, researchers and scientists to obtain larger views within their specialization fields.

This is the reason to introduce mass moment vectors to presentation of the kinetic parameters of the rotor dynamics and multistep gear transmission. On the basis of this approach we built the first model presented in this paper.

In industry there is an increased need for detailed investigation of the toothed coupling through models that involute the coupling of more than two teeth and for more than two, the systems which give high revolution numbers and others. Relatively new models (see [1–4, 14, 15, 26–28, 41, 55, 56]) have been established to study numerous problems in the gear transmission dynamics.

1.2. Introduction into Fractional Order Dynamics of the Rotors

In use, gear transmissions are very often exposed to action of forces that change with time (dynamic load). There are also internal dynamic forces present. The internal dynamic forces in gear teeth meshing, are the consequence of elastic deformation of the teeth and defects in manufacture such as pitch differences of meshed gears and deviation of shape of tooth profile. Deformation of teeth results in the so-called collision of teeth which is intensified at greater difference in the pitch of meshed gears. Occurrence of internal dynamic forces results in vibration of gears so that the meshed gears behave as an oscillatory system. This model consists of reduced masses of the gear with elastic and damping connections (see [2, 4, 55]). By applying the basic principles of mechanics and taking into consideration initial and boundary conditions, the system of equation is established which describes physicality of the gear meshing process. On the other hand, extremely cyclic loads (dynamic forces) can result in breakage of teeth, thus causing failure of the mechanism or system.

Primary dependences between geometrical and physical quantities in the mechanics of continuum (and with gear transmissions as well) include mainly establishing the constitutive relation between the stress state and deformation state of the tooth's material in the two teeth in contact for each particular case.

Thus, solving this task, it is necessary to reduce numerous kinetic parameters to minimal numbers and obtain a simple abstract model describing main properties for investigation of corresponding dynamical influences. Analytic methods include determination of mathematical functions which detemine the solution in closed form. They are based on the constitutive laws and relations of the stress-strain states in gear's materials, and they can give solutions for a very small number of boundary tasks. But, always each aproach needs certain assumotions-approximations concerning description of real contours, properties of teeth is contacts and initial conditions. For this reason numerous researchers resort to application of numeric method in solving differential equation of the gear transmission motion. The basic characteristic of the numeric methods is that the fundamental equations of the Elasticity theory, including the boundary conditions, are solved by approximative numeric methods. The solutions obtained are approximate.

Based on previous analysis at starting this part, we take into account that contact between two teeth is possible to be constructed by standard light element with constitutive stress—strain state relations which can be expressed by fractional order derivatives.

For that Reason, Let us make a short survey of the present results published in the literatute.

The monographs [57, 58] contain a basic mathematical description of fractional calculus and some solutions of the fractional order differential equations necessary for applications of the corresponding mathematical description of a model of gear transmission based on the teeths coupling by standard light fractional order element.

In series of the papers (see [59–62]) and in the monograph [63] analytical mechanics of discrete hereditary systems is constructed and based on the standard light hereditary elements in the form of neglected mass and with viscoelastic properties with corresponding constitutive relations between forces and element deformations. Special case are constitutive relations expressed by fractional order derivatices.

In [61] discrete continuum method was presented by use of the system of the material particles coupled viscoelastically or creeping mass less standard light elements with different stress-strain constitutive relations expressed by corresponding mathematical relations.

Standard light element with constitutive stress-strain relation expressed by members with fractional order derivatives are also used.

In the series of [45, 64–69] a series of the mixed discrete-continuum or continuum mechanical systems with fractional order creep properties are mathematically described by members contained in fractional order derivatives and analytically solved. These examples with mathematical descriptions and solutions are basic for new model of the gear transmission with fractional order properties.

2. Model of the Gear Transmission of the Fractional Order Tooth Coupling

2.1. Description of the Gear Transmission Model of the Fractional Order Tooth Coupling

Let us consider a model who is based on the three-step coupled rigid rotors but couplings between gear teeth are realized by standard light elements fractional order constitutive stress-strain relations, Figure 1(a). The second model of gear transmissions dynamics consists of three rigid disks coupled by two standard light fractional order elements, as it is presented in Figure 1(b). (see Appendix B).

2.2. Standard Light Fractional Order Element

Basic elements of multistep gear transmission system are

(i) gears in the form of disks with mass axial inertia moments $\mathbf{J_k}$, $\mathbf{k} = 1, 2, 3$,

(ii) *standard light coupling elements* of negligible mass in the form of axially stressed rod without bending, and which has the ability to resist deformation under static and dynamic conditions; Constitutive stress-strain relation between restitution force $\mathbf{P}$ and element elongation x can be written in the general form $f_{\text{psr}}(\mathbf{P}, \dot{\mathbf{P}}, x, \dot{x}, x_t^{\alpha}, \mathfrak{D}, \mathfrak{D}_t^{\alpha}, \mathcal{J}, n, c, \tilde{c}, \mu, \alpha, c_{\alpha}, T, \mathcal{U}, \ldots) = 0$, where $\mathfrak{D}$, $\mathfrak{D}_t^{\alpha}$ and $\mathcal{J}$ are differential, fractional order and integral operators (for detail see monographs [45, 58–67, 70, 71]) which find their justification in experimental verifications of material behavior, while $n, c, \tilde{c}, \mu, c_{\alpha}, \alpha, \ldots$ are material constants, which are also determined experimentally.

For each single standard coupling light element of negligible mass, we shall define a particular stress-strain constitutive relation-law of material properties. This means that we will define stress-strain constitutive relation as description relation between forces and deformations of two gears teeth in contact determined and constrained by rotation angles of the gear model in the form of disk and with changes of distances in time, with accuracy up to constants which depend on the accuracy of their determination through experiment.

The accuracy of those constants laws and with them the relation between forces and elongations will depend not only on knowing the nature of object, but also on our having the knowledge necessary for dealing with very complex stress-strain relations in the coupling gears teeth (for details see [2, 4, 55]). In this paper we shall use three types of such light standard constraint elements: *light standard creep constraint element* for which the stress-strain relation for the restitution force in the function of element elongation is given by fractional order derivatives (see [62]) in the form

$$P(t) = -\left\{c_0 x(t) + c_\alpha \mathfrak{D}_\alpha^t [x(t)]\right\}, \tag{2.1}$$

where $\mathfrak{D}_t^\alpha[\cdot]$ is fractional order differential operator of the αth derivative with respect to time t in the following form:

$$\mathfrak{D}_\alpha^t [x(t)] = \frac{d^\alpha x(t)}{dt^\alpha} = x^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau, \tag{2.2}$$

where c, c_α are rigidity coefficients is momentary and prolonged one, and α a rational number between 0 and 1, $0 < \alpha < 1$.

2.3. Governing Equations of the Two-Step Gear Transmission with Fractional Order Tooth Coupling

For defined model of the two-step gear transmission fractional order system vibrations, we use three generalized coordinates—angle of gear disks rotation ϑ_i, $i = 1, 2, 3$, and we take into account that defined system poses three degrees of freedom.

Kinetic energy of the of the two-step gear transmission fractional order system vibrations is in the form

$$\mathbf{E_k} = \frac{1}{2} \sum_{k=1}^{k=2} \mathbf{J}_k \dot{\vartheta}_k^2 + \frac{1}{2} \mathbf{J}_3 \dot{\vartheta}_2^2 + \frac{1}{2} \mathbf{J}_4 \dot{\vartheta}_3^2. \tag{2.3}$$

The first standard light fractional order coupling element is between first gear disk and second and is strained for $x_1 = R_1((R_2/R_1)\vartheta_2 - \vartheta_1)$, and the second standard light fractional order coupling element is between the third gear disk and fourth and is strained for $x_2 = R_3((R_4/R_3)\vartheta_3 - \vartheta_2)$. On the basis of the previous constitutive stress-strain relation of the first and second standard light fractional order coupling elements between geared disks in the two-step gear power transmission are

$$\begin{aligned} P_1 &= -c x_1 - c_\alpha \mathfrak{D}_\alpha^t [x_1] = -c R_1 \left(\frac{R_2}{R_1} \vartheta_2 - \vartheta_1 \right) - c_\alpha \mathfrak{D}_\alpha^t \left[R_1 \left(\frac{R_2}{R_1} \vartheta_2 - \vartheta_1 \right) \right], \\ P_2 &= -c x_2 - c_\alpha \mathfrak{D}_\alpha^t [x_2] = -c R_3 \left(\frac{R_4}{R_3} \vartheta_3 - \vartheta_2 \right) - c_\alpha \mathfrak{D}_\alpha^t \left[R_3 \left(\frac{R_4}{R_3} \vartheta_3 - \vartheta_2 \right) \right]. \end{aligned} \tag{2.4}$$

Governing system of the double gear transmission fractional order differential equations is in the following form:

$$\begin{aligned}
\mathbf{J}_1\ddot{\vartheta}_1 &= -P_1 = cR_1\left(\frac{R_2}{R_1}\vartheta_2 - \vartheta_1\right) + c_\alpha \mathfrak{D}_\alpha^t\left[R_1\left(\frac{R_2}{R_1}\vartheta_2 - \vartheta_1\right)\right],\\
(\mathbf{J}_2 + \mathbf{J}_3)\ddot{\vartheta}_2 &= P_1 - P_2 = -cR_1\left(\frac{R_2}{R_1}\vartheta_2 - \vartheta_1\right) - c_\alpha \mathfrak{D}_\alpha^t\left[R_1\left(\frac{R_2}{R_1}\vartheta_2 - \vartheta_1\right)\right]\\
&\quad + cR_3\left(\frac{R_4}{R_3}\vartheta_3 - \vartheta_2\right) + c_\alpha \mathfrak{D}_\alpha^t\left[R_3\left(\frac{R_4}{R_3}\vartheta_3 - \vartheta_2\right)\right],\\
\mathbf{J}_4\ddot{\vartheta}_3 &= P_2 = -cR_3\left(\frac{R_4}{R_3}\vartheta_3 - \vartheta_2\right) - c_\alpha \mathfrak{D}_\alpha^t\left[R_3\left(\frac{R_4}{R_3}\vartheta_3 - \vartheta_2\right)\right].
\end{aligned} \tag{2.5}$$

After introducing the following notations:

$$\omega_0^2 = \frac{c}{\mathbf{J}_1}R_1, \quad \omega_{0\alpha}^2 = \frac{c_\alpha}{\mathbf{J}_1}R_1, \quad k_{21} = \frac{R_2}{R_1}, \quad k_{31} = \frac{R_3}{R_1}, \quad k_{41} = \frac{R_4}{R_1}, \quad \lambda_{23,1} = \frac{(\mathbf{J}_2 + \mathbf{J}_3)}{\mathbf{J}_1}, \quad \lambda_{4,1} = \frac{\mathbf{J}_4}{\mathbf{J}_1} \tag{2.6}$$

governing system of the l fractional order differential equations is possible to write in the following form:

$$\begin{gathered}
\ddot{\vartheta}_1 - \omega_0^2 k_{21}\vartheta_2 + \omega_0^2\vartheta_1 = \omega_{0\alpha 1}^2 \mathfrak{D}_\alpha^t[(k_{21}\vartheta_2 - \vartheta_1)],\\
\ddot{\vartheta}_2 - \omega_0^2\lambda_{23,1}\vartheta_1 + \omega_0^2\lambda_{23,1}(k_{21} + k_{31})\vartheta_2 - \omega_0^2\lambda_{23,1}k_{41}\vartheta_3 = \omega_{0\alpha}^2\lambda_{23,1}\mathfrak{D}_\alpha^t[\vartheta_1 + (k_{21} + k_{31})\vartheta_2 - k_{41}\vartheta_3],\\
\ddot{\vartheta}_3 + \omega_0^2\lambda_{413}(k_{41}\vartheta_3 - k_{31}\vartheta_2) = -\omega_{0\alpha}^2\lambda_{4,1}\mathfrak{D}_\alpha^t[(k_{41}\vartheta_3 - k_{31}\vartheta_2)].
\end{gathered} \tag{2.7}$$

2.4. Solutions of the Governing System of Differential Equations of Two-Step Gear Transmission Dynamics, with Fractional Order Tooth Coupling

Now, for beginning let us consider corresponding basic systems of the differential equations in linear form:

$$\begin{gathered}
\ddot{\vartheta}_1 - \omega_0^2 k_{21}\vartheta_2 + \omega_0^2\vartheta_1 = 0,\\
\ddot{\vartheta}_2 - \omega_0^2\lambda_{23,1}\vartheta_1 + \omega_0^2\lambda_{23,1}(k_{21} + k_{31})\vartheta_2 - \omega_0^2\lambda_{23,1}k_{41}\vartheta_3 = 0,\\
\ddot{\vartheta}_3 + \omega_0^2\lambda_{413}(k_{41}\vartheta_3 - k_{31}\vartheta_2) = 0,
\end{gathered} \tag{2.8}$$

and with proposed solutions in the following form:

$$\vartheta_k(t) = A_k\cos(\omega t + \alpha), \tag{2.9}$$

and taking the following notation $u = \omega^2/\omega_0^2$, we can write the following systems of algebra of algebra equations with respect to unknown amplitudes A_k in the matrix form

$$\begin{pmatrix} (1-u) & -k_{21} & 0 \\ -\lambda_{23,1} & [\lambda_{23,1}(k_{21}+k_{31})-u] & -\lambda_{23,1}k_{41} \\ 0 & -\lambda_{413}k_{31} & (\lambda_{413}k_{41}-u) \end{pmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \{0\}, \tag{2.10}$$

and corresponding frequency equation in the developed form

$$\begin{aligned} f(u) = {} & (1-u)[\lambda_{23,1}(k_{21}+k_{31})-u](\lambda_{413}k_{41}-u) \\ & - k_{21}\lambda_{23,1}(\lambda_{413}k_{41}-u) - \lambda_{413}k_{31}\lambda_{23,1}k_{41}(1-u) = 0. \end{aligned} \tag{2.11}$$

From the previous frequency equation, we can obtain the following three roots u_s, $s = 1,2,3$ and corresponding eigen circular frequencies: $\omega_s^2 = u_s\omega_0^2$, $s = 1,2,3$, and corresponding cofactors are: $K_{31}^{(s)} = -k_{21}(\lambda_{413}k_{41} - u_s)$; $K_{32}^{(s)} = -\lambda_{23,1}k_{41}(1-u)$; $K_{33}^{(s)} = -k_{21}\lambda_{23,1}k_{41}$. Then, solution of the basic linear differential equations is

$$\vartheta_k(t) = \sum_{s=1}^{s=3} \vartheta_k^{(s)}(t) = \sum_{s=1}^{s=3} A_k^{(s)} \cos(\omega_s t + \alpha_s) = \sum_{s=1}^{s=3} K_{3k}^{(s)} C_s \cos(\omega_s t + \alpha_s) = \sum_{s=1}^{s=3} K_{3k}^{(s)} \xi_s, \tag{2.12}$$

where $\xi_s = C_s \cos(\omega_s t + \alpha_s)$, $s = 1,2,3$ are main coordinates of the linear system.

By using the expression for generalized coordinates ϑ_i, $i = 1,2,3$ by normal coordinates of the linear system, the governing system of the fractional differential equations (2.12) is possible to be transform as in the following form:

$$\ddot{\xi}_s + \omega_s^2 \xi_s = -\omega_{\alpha s}^2 D_\alpha^t[\xi_s], \quad s = 1,2,3, \tag{2.13}$$

where

$$\omega_s^2 = \frac{\sum_{i=1}^{i=3}\sum_{j=1}^{j=3} c_{ij} K_{3i}^{(s)} K_{3j}^{(s)}}{\sum_{i=1}^{i=3}\sum_{j=1}^{j=3} a_{ij} K_{3i}^{(s)} K_{3j}^{(s)}}, \quad s = 1,2,3, \qquad \omega_{\alpha s}^2 = \frac{\sum_{i=1}^{i=3}\sum_{j=1}^{j=3} c_{\alpha ij} K_{3i}^{(s)} K_{3j}^{(s)}}{\sum_{i=1}^{i=3}\sum_{j=1}^{j=3} a_{ij} K_{3i}^{(s)} K_{3j}^{(s)}}, \quad s = 1,2,3. \tag{2.14}$$

Obtained system of the three fractional order differential equations (2.14) present three uncoupled fractional order differential equations independent along normal coordinates ξ_s, $s = 1,2,3$ of the considered fractional order model of the gear transmission dynamics. All three fractional order differential equations are of the same type and each presents one mode of the fractional order mode vibrations. Analytical solution is easy to obtain by using one of [58] or [42] or [69] or [62]. Solutions of here fractional order differential equation is possible to solve by using the approach presented in the Appendix A. (It is possible to solve these fractional order differential equation by using the approach presented in the Appendix A).

Then, for the solutions of the each fractional order differential equations (2.13), we can write the following expressions:

$$\xi_s(t) = \xi_{0s} \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha s}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha s}^{2j} t^{-\alpha j}}{\omega_s^{2j} \Gamma(2k+1-\alpha j)}$$
$$+ \dot{\xi}_{0s} \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha s}^{2k} t^{2k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha s}^{-2j} t^{-\alpha j}}{\omega_s^{2j} \Gamma(2k+2-\alpha j)}, \quad s = 1,2,3, \tag{2.15}$$

where $\xi_s(0) = \xi_{0s}$ and $\dot{\xi}_s(0) = \dot{\xi}_{0s}$ are initial values of these main coordinates defined by initial conditions. Expressions (2.15) for main system coordinates present fractional order models like one frequency vibration modes.

Now, we can separate three sets of the two fractional order time components $\eta_s(t)$ and $\zeta_s(t)$, $s = 1,2,3$ and in the expression of the solutions along normal coordinates of the governing system of fractional differential equations describing our second model of the gear transmission fractional order dynamics we can write in the following forms:

$$\eta_s(t) = \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha s}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha s}^{2j} t^{-\alpha j}}{\omega_s^{2j} \Gamma(2k+1-\alpha j)}, \quad s = 1,2,3, \tag{2.16}$$

$$\zeta_s(t) = \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha s}^{2k} t^{2k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha s}^{-2j} t^{-\alpha j}}{\omega_s^{2j} \Gamma(2k+2-\alpha j)}, \quad s = 1,2,3. \tag{2.17}$$

These three series of the two fractional order time components $\eta_s(t)$ and $\zeta_s(t)$, $s = 1,2,3$ present series of the six fractional order modes like one frequency modified cos as well as sin vibration mode components.

Then the solution of the basic system of the fractional order differential equations (2.7) along generalized coordinates ϑ_i, $i = 1,2,3$ contain sixth time functions in the forms (2.16) and (2.17). Finally for the solution of the basic system of the fractional order differential equations (2.7) describing dynamics of the fractional order two-step gear transmission it is possible to express in the following form:

$$\vartheta_k(t) = \sum_{s=1}^{s=3} K_{3k}^{(s)} \xi_s(t) = \sum_{s=1}^{s=3} K_{3k}^{(s)} \xi_{0s} \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha s}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha s}^{2j} t^{-\alpha j}}{\omega_s^{2j} \Gamma(2k+1-\alpha j)}$$
$$+ \sum_{s=1}^{s=3} K_{3k}^{(s)} \dot{\xi}_{0s} \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha s}^{2k} t^{2k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha s}^{-2j} t^{-\alpha j}}{\omega_s^{2j} \Gamma(2k+2-\alpha j)}, \quad k = 1,2,3. \tag{2.18}$$

2.5. *Numerical Analysis of the Solutions of the Governing System of Fractional Order Differential Equations of Two-Step Gear Transmission Dynamics, with Fractional Order Tooth Coupling*

We can see that for fractional order model of the double gear transmission vibrations was transformed by eigen normal coordinates ξ_s, $s = 1,2,3$ of the corresponding linear system

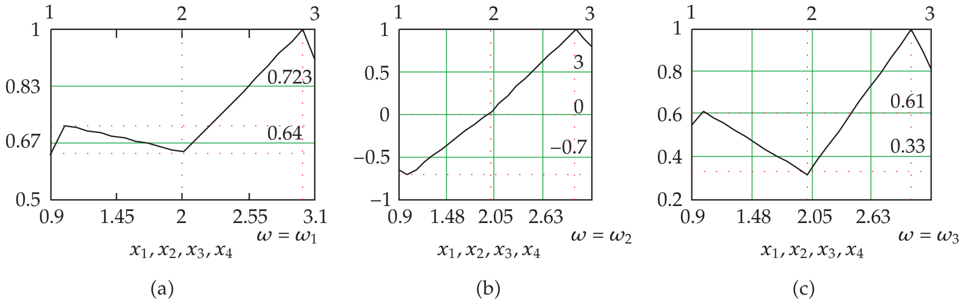


Figure 2: Relations between eigen amplitudes of eigen main normal modes of corresponding system of the basic linear differential equations (2.12), (a) for first, (b) for second, and (c) for third mode.

into three separate independent fractional order oscillators, each with one degree of freedom, and each fractional order differential equation contain only one main coordinate of the system dynamics.

Relations between eigen amplitudes of eigen main normal modes of corresponding system of the basic linear differential equations (2.8) are given on Figure 2(a) for first, 2(b) for second and 2(c) for third mode.

By using different numerical values of the kinetic and geometrical parameters of the two-step gear transmission model, the series of the graphical presentation of the three sets of the two-time components $\eta_s(t)$ and $\zeta_s(t)$, $s = 1, 2, 3$ of the solutions, by using expressions (2.15) and (2.16), are obtained. In the series Figures 3–7 are presented characteristic modes for different values of the α coefficient of the fractional order of the used standard light fractional order element for describing teeth coupling between gears (see Appendix B).

In Figure 3, first eigen fractional order time components $\eta_1(t)$ and $\zeta_1(t)$ for different system kinetic and geometric parameter values are presented.

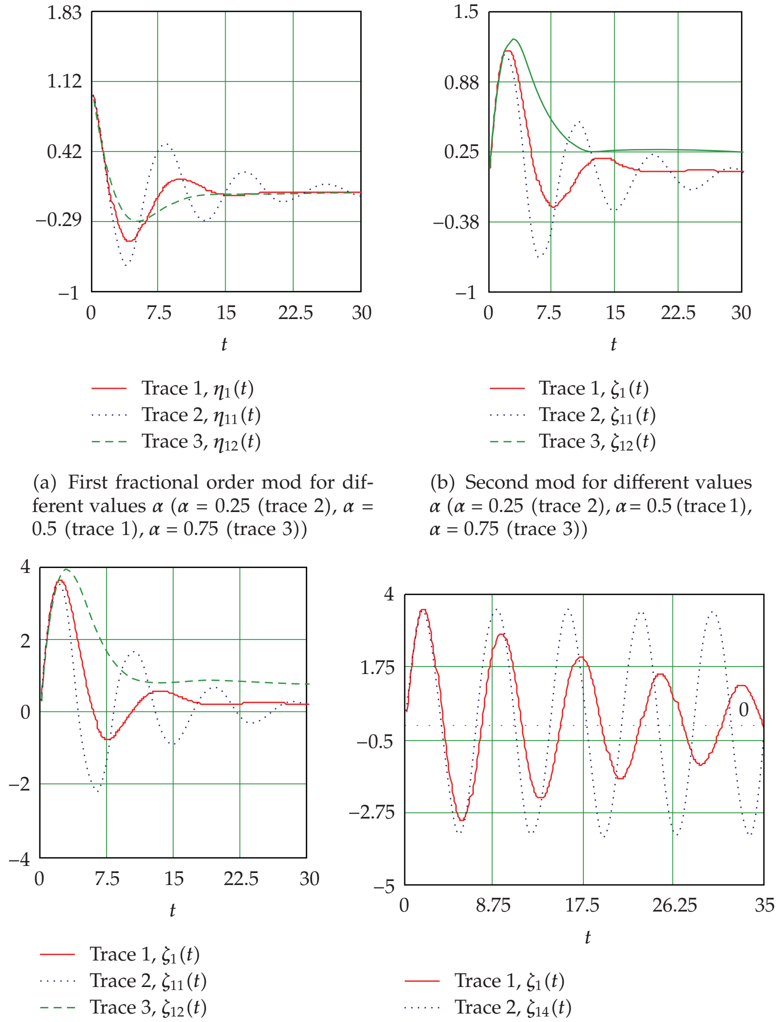
In Figure 4, first eigen fractional order mode $\xi_1(t)$ with corresponding first eigen fractional order time components $\eta_1(t)$ and $\zeta_1(t)$ for different system kinetic and geometric parameter values are presented. First eigen fractional order mode is like one frequency vibration mode similar to first single frequency eigen mode of the corresponding linear system.

In Figure 5, second eigen fractional mode $\xi_2(t)$ with corresponding second fractional order time components $\eta_2(t)$ and $\zeta_2(t)$ for different system kinetic and geometric parameter values, are presented. Second eigen fractional order mode is like one frequency vibration mode similar to second single frequency eigen mode of the corresponding linear system.

In Figure 6, third eigen fractional mode $\xi_3(t)$ with corresponding third fractional order time components $\eta_3(t)$ and $\zeta_3(t)$ for different system kinetic and geometric parameter values are presented. Third eigen fractional order mode is like one frequency vibration mode similar to third single frequency eigen mode of the corresponding linear system.

In Figure 7, first and second eigen fractional modes, $\xi_1(\alpha, t)$ and $\xi_2(\alpha, t)$ are presented by surfaces with corresponding first and second fractional order time components, $\eta_1(\alpha, t)$—surfaces in left column and $\eta_2(\alpha, t)$—surfaces in right column for same system kinetic and geometric parameter values are presented.

The third eigen fractional mode $\xi_3(\alpha, t)$ is not presented by surfaces with corresponding third fractional order time components $\eta_3(\alpha, t)$ by the reason that corresponding surfaces qe similar as two previous first and second eigen fractional modes, $\xi_1(\alpha, t)$ and $\xi_2(\alpha, t)$



(a) First fractional order mod for different values α ($\alpha = 0.25$ (trace 2), $\alpha = 0.5$ (trace 1), $\alpha = 0.75$ (trace 3))

(b) Second mod for different values α ($\alpha = 0.25$ (trace 2), $\alpha = 0.5$ (trace 1), $\alpha = 0.75$ (trace 3))

(c) The first coordinate for diferent values α ($\alpha = 0.25$ (trace 2), $\alpha = 0.5$ (trace 1), $\alpha = 0.75$ (trace 3))

(d) The first coordinate for diferent values α ($\alpha = 0$ (trace 2) and $\alpha = 0.2$ (trace 1))

Figure 3: First eigen fractional order time components $\eta_1(t)$ and $\zeta_1(t)$ for different system kinetic and geometric parameter values.

presented in Figure 7, and some characteristic properties are visible in the graph presented in Figure 6.

3. Concluding Remarks

Two approaches to the models of the gear transmission system dynamics with possibility of investigate different properties of the very complex dynamics of the corresponding real gear transmission system are possible.

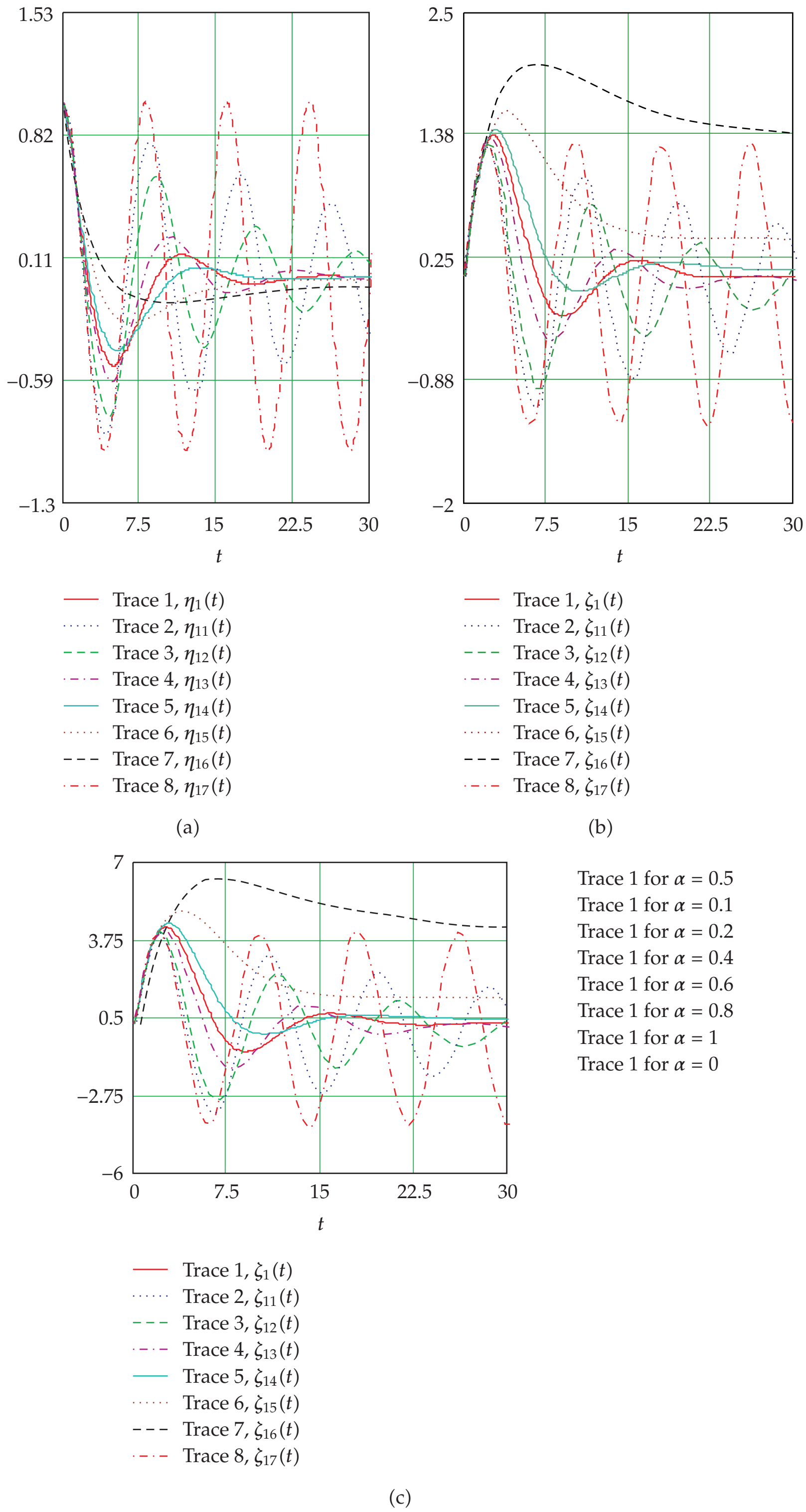


Figure 4: First eigen fractional mode $\xi_1(t)$ with corresponding first fractional order time components $\eta_1(t)$ and $\zeta_1(t)$ for different system kinetic and geometric parameter values. Eigen fractional order mode is like one frequency vibration mode similar to first single frequency eigen mode of the corresponding linear system.

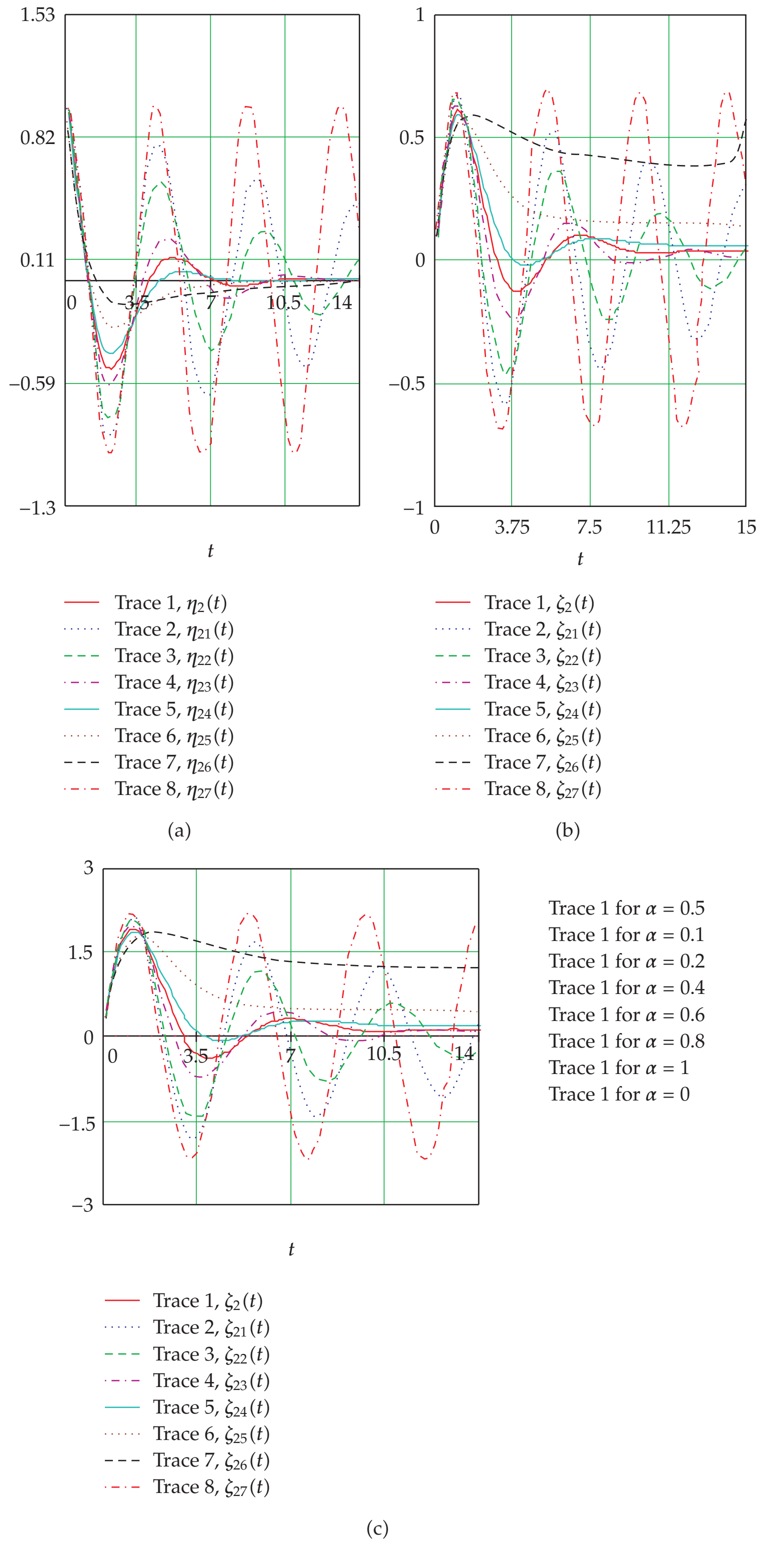


Figure 5: Second eigen fractional mode $\xi_2(t)$ with corresponding second fractional order time components $\eta_2(t)$ and $\zeta_2(t)$ for different system kinetic and geometric parameter values. Eigen fractional order mode is like one frequency vibration mode similar to second single frequency eigen mode of the corresponding linear system.

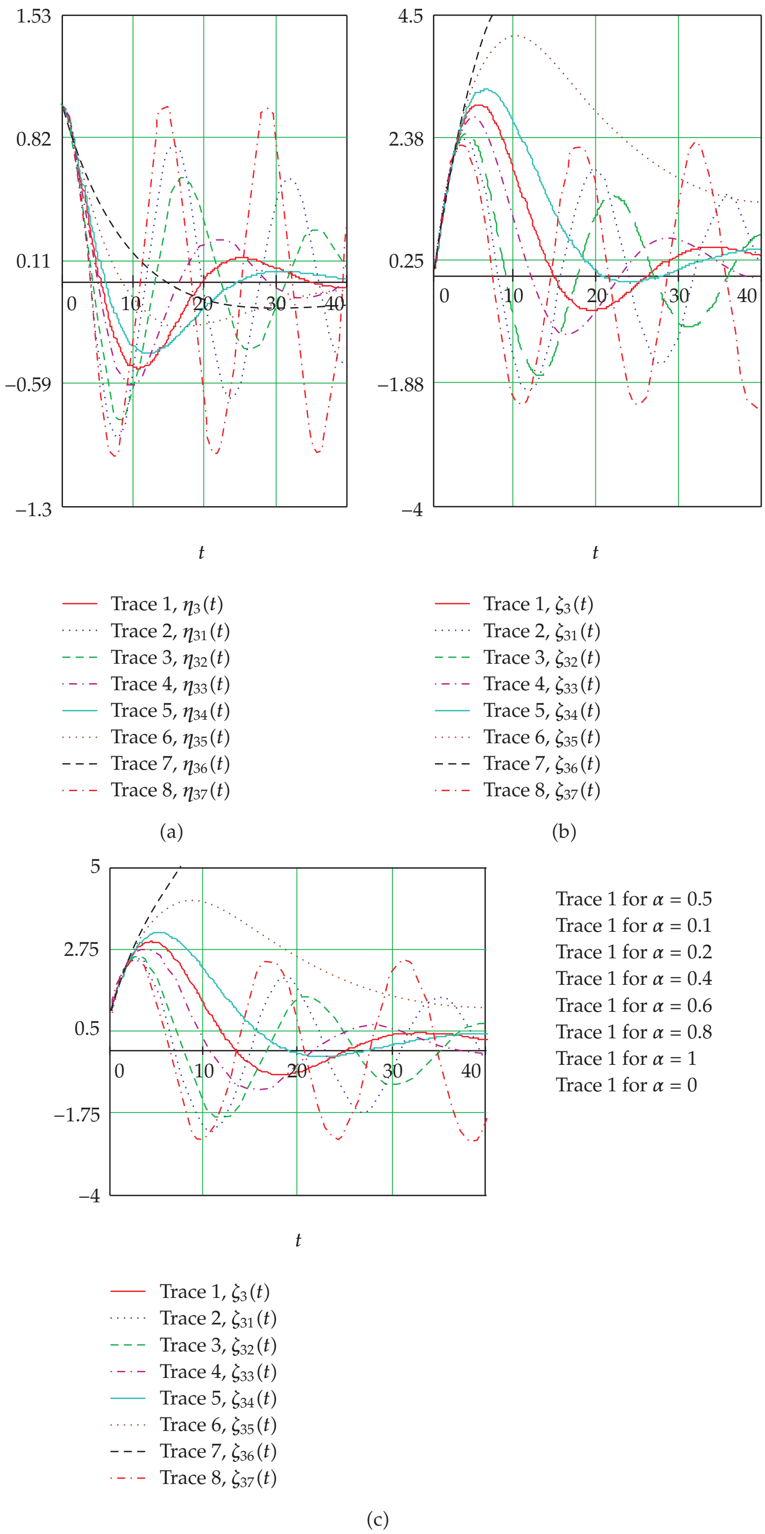
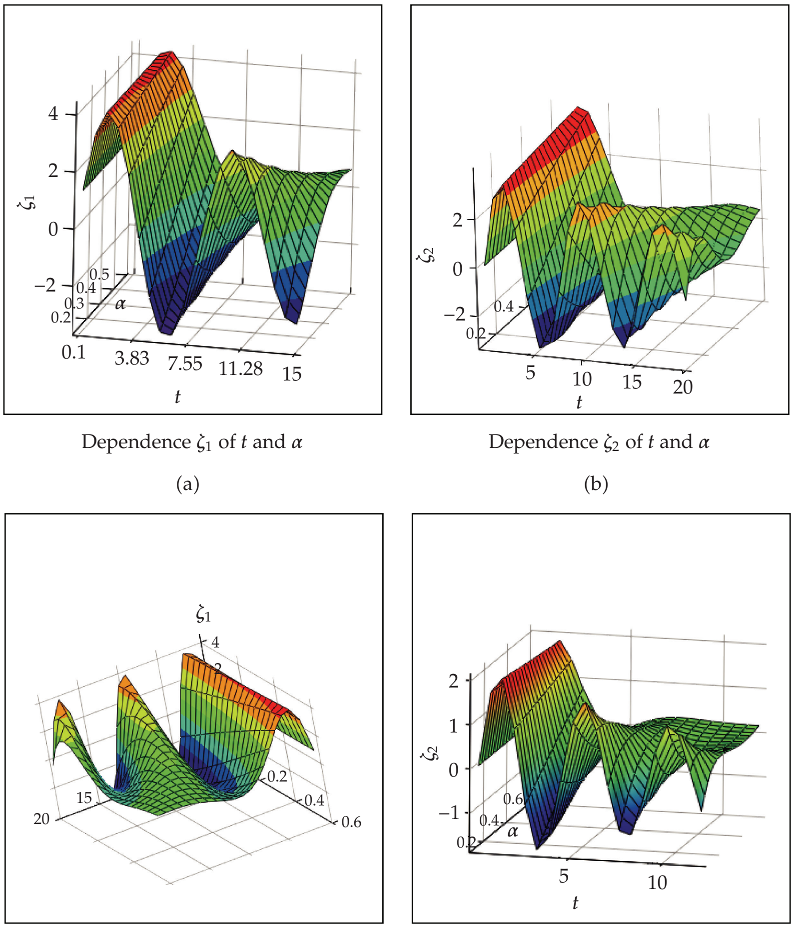


Figure 6: Third eigen fractional mode $\xi_3(t)$ with corresponding third fractional order time components $\eta_3(t)$ and $\zeta_3(t)$ for different system kinetic and geometric parameter values. Eigen fractional order mode is like one frequency vibration mode similar to third single frequency eigen mode of the corresponding linear system.



Dependence ζ_1 of t and α

(a)

Dependence ζ_2 of t and α

(b)

Dependence ζ_1 of t and α (for $c_\alpha/c = 1$)

(c)

Dependence ζ_2 of t and α (for $c_\alpha/c = 1$)

(d)

Figure 7: First and second eigen fractional modes, $\xi_1(\alpha, t)$ and $\xi_2(\alpha, t)$ presented by surfaces with corresponding first and second fractional order time components $\eta_1(\alpha, t)$—surfaces in left column and $\eta_2(\alpha, t)$—surfaces in right column for same system kinetic and geometric parameter values.

First approach give a model based on the rigid rotors coupled with rigid gear teeth, with mass distributions not balanced and in the form of the mass particles as the series of the mass debalances of the gears in multistep gear transmission. By very simple model is possible and useful investigation of the nonlinear dynamics of the multistep gear transmission and nonlinear phenomena in free and forced dynamics. This model is suitable to explain source of vibrations and big noise, as well as no stability in gear transmission dynamics. Layering of the homoclinic orbits in phase plane is source of a sensitive dependence nonlinear type of regime of gear transmission system dynamics.

Second approach give a model based on the two-step gear transmission taking into account deformation and creeping and also visco-elastic teeth gears coupling. Our investigation was focused to a new model of the fractional order dynamics of the gear transmissiont. For this model we obtain analytical expressions for the corresponding fractional order modes like one frequency eigen vibrational modes. Generalization of this model to the similar model of the multistep gear transmission is very easy.

Appendices

A. Solution of a Fractional Order Differential Equation of a Fractional Order Creep Oscillator with Single Degree of Freedom

The fractional order differential equations from all three (79) obtained and considered cases of eigen fractional order partial-particular oscillators of the hybrid fractional order gear transmission system are in mathematical analogy same type of fractional order differential equation with corresponding unknown time-function, $\xi_s(t)$, $s = 1, 2, 3$. For all these time functions $\xi_s(t)$, $s = 1, 2, 3$, we can use notation $T(t)$ and all previous derived fractional order differential equations (79) of eigen fractional order partial oscillators with one degree of freedom, correspond to the fractional order model dynamics of the gear transmission system dynamics with three degree of freedom, we can rewrite it in the following form:

$$\ddot{T}(t) \pm \omega_\alpha^2 T^{(\alpha)}(t) + \omega_0^2 T(t) = 0. \tag{A.1}$$

This fractional order differential equation (A.1) on unknown time-function $T(t)$, can be solved by applying Laplace transforms (see [42, 58] or [67, 69]). Upon that fact Laplace transform of solution is in the form

$$\boldsymbol{\mathcal{T}}(p) = \boldsymbol{\mathcal{L}}[T(t)] \frac{pT(0) + \dot{T}(0)}{p^2 + \omega_0^2\left[1 \pm \left(\omega_\alpha^2/\omega_0^2\right)\mathbf{R}(p)\right]}, \tag{A.2}$$

where $\boldsymbol{\mathcal{L}}[\mathfrak{D}_\alpha^t[T(t)]] = \mathbf{R}(p)\boldsymbol{\mathcal{L}}[T(t)]$ is Laplace transform of a fractional derivative $d^\alpha T(t)/dt^\alpha$ for $0 \le \alpha \le 1$. For creep rheological material those Laplace transforms are of the form:

$$\boldsymbol{\mathcal{L}}\left[\mathfrak{D}_\alpha^t[T(t)]\right] = \mathbf{R}(p)\boldsymbol{\mathcal{L}}[T(t)] - \frac{d^{\alpha-1}}{dt^{\alpha-1}}T(0) = p^\alpha \boldsymbol{\mathcal{L}}[T(t)] - \frac{d^{\alpha-1}}{dt^{\alpha-1}}T(0) \tag{A.3}$$

where the initial value are

$$\left.\frac{d^{\alpha-1}T(t)}{dt^{\alpha-1}}\right|_{t=0} = 0, \tag{A.4}$$

so, in that case Laplace transform of time-function is given by the following expression:

$$\boldsymbol{\mathcal{L}}\{T(t)\} = \frac{pT_0 + \dot{T}_0}{\left[p^2 \pm \omega_\alpha^2 p^\alpha + \omega_0^2\right]}. \tag{A.5}$$

Table 1: The datas of gear box.

	Pinion	Middle 1	Middle 2	Output gear
Number of the teeth	51	72	19	73
Modulus, mm	1,405	1,405	2,2175	2,2175
Face whith, mm	22,5	29	20	20
Inertias	0 01837	0 03837	0 00071	0 1740
Contact ratio	1,60		1,7	
Mean stiffness	$4,24 \times 10^9$		$3,45 \times 10^9$	
Mesh Phasing		0 257		
Torque T, Nm	100	0	0	258,4

For boundary cases, when material parameters α take the following values: $\alpha = 0$ and $\alpha = 1$ we have the two special simple cases, whose corresponding fractional-differential equations and solutions are known. In these cases fractional-differential equations are:

$$(1^*) \quad \ddot{T}(t) \pm \tilde{\omega}_{0\alpha}^2 T^{(0)}(t) + \omega_0^2 T(t) = 0 \quad \text{for } \alpha = 0, \tag{A.6}$$

where $T^{(0)}(t) = T(t)$, and

$$(2^*) \quad \ddot{T}(t) \pm \omega_{1\alpha}^2 T^{(1)}(t) + \omega_0^2 T(t) = 0 \quad \text{for } \alpha = 1, \tag{A.7}$$

where $T^{(1)}(t) = \dot{T}(t)$.

The solutions to equations (C.6) and (C.7) are

$$(1^*) \quad T(t) = T_0 \cos t\sqrt{\omega_0^2 \pm \tilde{\omega}_{0\alpha}^2} + \frac{\dot{T}_0}{\sqrt{\omega_0^2 \pm \tilde{\omega}_{0\alpha}^2}} \sin t\sqrt{\omega_0^2 \pm \tilde{\omega}_{0\alpha}^2} \tag{A.8}$$

for $\alpha = 0$.

$$\left(2^{*(a)}\right) \quad T(t) = e^{\mp(\omega_1^2/2)t}\left\{T_0 \cos t\sqrt{\omega_0^2 - \frac{\omega_{1\alpha}^4}{4}} + \frac{\dot{T}_0}{\sqrt{\omega_0^2 - \omega_{1\alpha}^4/4}} \sin t\sqrt{\omega_0^2 - \frac{\omega_{1\alpha}^4}{4}}\right\} \tag{A.9}$$

for $\alpha = 1$ and for $\omega_0 > (1/2)\omega_{1\alpha}^2$, (for soft creep) or for strong creep:

$$\left(2^{*(b)}\right) \quad T(t) = e^{\mp(\omega_{1\alpha}^2/2)t}\left\{T_0 Ch\, t\sqrt{\frac{\omega_{1\alpha}^4}{4} - \omega_0^2} + \frac{\dot{T}_0}{\sqrt{\omega_{1\alpha}^4/4 - \omega_0^2}} Sh\, t\sqrt{\frac{\omega_{1\alpha}^4}{4} - \omega_0^2}\right\} \tag{A.10}$$

for $\alpha = 1$ and for $\omega_0 < (1/2)\omega_{1\alpha}^2$.

For kritical case

$$\left(2^{*(c)}\right) \quad T(t) = e^{\mp(\omega_{1\alpha}^2/2)t}\left\{T_0 + \frac{2\dot{T}_0}{\omega_{1\alpha}^2} t\right\} \text{za } \alpha = 1, \quad \text{za } \omega_0 = \frac{1}{2}\omega_{1\alpha}^2. \tag{A.11}$$

Fractional-differential equation (A.1) for the general case, when α is real number from interval $0 < \alpha < 1$ can be solved by using Laplace's transformation. By using that is

$$\mathcal{L}\left\{\frac{d^{\alpha}T(t)}{dt^{\alpha}}\right\} = p^{\alpha}\mathcal{L}\{T(t)\} - \left.\frac{d^{\alpha-1}T(t)}{dt^{\alpha-1}}\right|_{t=0} = p^{\alpha}\mathcal{L}\{T(t)\}, \tag{A.12}$$

and by introducing for initial conditions of fractional derivatives in the form (A.3), and after taking Laplace's transform of (A.1) we obtain the equation (A.2) with respect to the Laplace transform of solution, or in the following form:

$$\mathcal{L}\{T(t)\} = \frac{pT_{0i} + \dot{T}_{0i}}{2\left(p^2 \pm \omega_{\alpha}^2 p^{\alpha} + \omega_0^2\right)}. \tag{A.13}$$

For the case when $\omega_0^2 \neq 0$, the Laplace transform of the solution can be developed into series by following way:

$$\begin{aligned}\mathcal{L}\{T(t)\} &= \frac{pT_0 + \dot{T}_0}{p^2\left[1 + \left(\omega_{\alpha}^2/p^2\right)\left(\pm p^{\alpha} + \omega_0^2/\omega_{\alpha}^2\right)\right]} \\ &= \left(T_0 + \frac{\dot{T}_0}{p}\right)\frac{1}{p}\frac{1}{1 + \left(\omega_{\alpha}^2/p^2\right)\left(\pm p^{\alpha} + \omega_0^2/\omega_{\alpha}^2\right)},\end{aligned} \tag{A.14}$$

$$\begin{aligned}\mathcal{L}\{T(t)\} &= \left(T_0 + \frac{\dot{T}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_{\alpha}^{2k}}{p^{2k}}\left(\pm p^{\alpha} + \frac{\omega_0^2}{\omega_{\alpha}^2}\right)^k, \\ \mathcal{L}\{T(t)\} &= \left(T_0 + \frac{\dot{T}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_{\alpha}^{2k}}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j p^{\alpha j}\omega_{\alpha}^{2(j-k)}}{\omega_o^{2j}}.\end{aligned} \tag{A.15}$$

In writing (A.15) it is assumed that expansion leads to convergent series. The inverse Laplace transform of previous Laplace transform of solution (A.15) in term-by-term steps is based on known theorems, and yield the following solution of differential equation (A.1) of time function in the following form of time series:

$$\begin{aligned}T(t) = \mathcal{L}^{-1}\mathcal{L}\{T(t)\} &= T_0\sum_{k=0}^{\infty}(-1)^k\omega_{\alpha}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j\omega_{\alpha}^{2j}t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+1-\alpha j)} \\ &+ \dot{T}_0\sum_{k=0}^{\infty}(-1)^k\omega_{\alpha}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j\omega_{\alpha}^{-2j}t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+2-\alpha j)}.\end{aligned} \tag{A.16}$$

B. Example of Numerical Experiment

See Table 1.

Acknowledgments

Parts of this research were supported by the Ministry of Sciences of Republic Serbia through Mathematical Institute SANU Belgrade Grants no. ON144002 Theoretical and Applied Mechanics of the Rigid and Solid Body. Mechanics of Materials, and also through the Faculty of Mechanical Engineering University of Niš and State University of Novi Pazar.

References

[1] M. Benton and A. Seireg, "Influecing instability and resonances in geared systems," *Journal of Mechanical Design*, vol. 103, no. 2, pp. 372–378, 1981.

[2] M. Blagojević, V. Nikolić-Stanojević, N. Marjanović, and L. J. Veljović, "Analysis of cycloid drive dynamic behavior," *Scientific Technical Rewiew*, no. 1, pp. 52–56, 2009.

[3] J. G. Bollinger and R. J. Harker, "Instability potencijal of high speed gearing," *Journal of the Industrial Mathematics*, vol. 17, pp. 35–39, 1967.

[4] D. Dimitrijevic and V. Nikolić, "Eigenfreuqency analysis of the spur gear pair with moving excentric masses on the body of one of the gears," *FME Transactions*, vol. 35, no. 3, pp. 157–163, 2007.

[5] K. Hedrih (Stevanović), "Interpretation of the motion of a heavy body around a stationary axis and kinetic pressures on bearing by means of the mass moment vectors for the pole and the axis," *Theoretical and Applied Mechanics*, no. 20, pp. 69–88, 1994.

[6] K. Hedrih (Stevanović), "Comments to the fundamental theorem and extension of the rotation tensor of the deformable body," *Tensor*, vol. 61, no. 3, pp. 290–303, 1999.

[7] K. Hedrih (Stevanović), "Leonhard Euler (1707–1783) and rigid body dynamics," *Scientific Technical Review*, vol. 57, no. 3-4, pp. 3–12, 2008.

[8] K. Hedrih (Stevanović), "Contribution to the coupled rotor nonlinear dynamics," in *Advances in Nonlinear Sciences*, Monograph, pp. 229–259, Academy of Nonlinear Sciences, Belgrade, Serbia, 2004.

[9] K. Hedrih (Stevanović), "Homoclinic orbits layering in the coupled rotor nonlinear dynamics and 2chatic clock models: a paradigm for vibrations and noise in machines," in *Proceedings of the 21st International Congress of Theoretical and Applied Mechanics (ICTAM '04)*, p. 320, Warsaw, Poland, August 2004.

[10] K. Hedrih (Stevanović), "Phase portraits and homoclinic orbits visualization of nonlinear dynamics of multiple step reductor/multiplier," in *Proceedings of the 11th World Congress in Mechanism and Machine Sciences (IFToMM '04)*, vol. 2, pp. 1508–1512, China Machine Press, Tianjin, China, April 2004.

[11] K. Hedrih (Stevanović), "Homoclinic orbits layering in the coupled rotor nonlinear dynamics and chaotic clock models," in *Proceedings of the 21st International Congress of Theoretical and Applied Mechanics (ICTAM '04)*, W. Gutkowski and T. A. Kowalewski, Eds., vol. 2, p. 421, Springer, Warsaw, Poland, August 2004.

[12] K. Hedrih (Stevanović), R. Knežević, and R. Cvetković, "Dynamics of planetary reductoe with turbulent damping," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 2, no. 3, pp. 265–277, 2001.

[13] K. Hedrih (Stevanović) and R. Knežević, "Structural stability of the planetary reductor nonlinear dynamics phase portrait," *Facta Universitatis, Series Mechanical Engineering*, vol. 1, no. 7, pp. 911–923, 2000.

[14] H. Vinayak and R. Singh, "Multi-body dynamics and modal analysis of compliant gear bodies," *Journal of Sound and Vibration*, vol. 210, no. 2, pp. 171–212, 1998.

[15] F. C. Moon, "Chaotic clock models: a paradigm for noise in machines," in *Proceedings of the IUTAM Symposium on Chaotic Dynamics and Control of Systems and Processes in Mechanics*, vol. 39, pp. 41–44, Dipartimento di Ingegneria Strutturale e Geotecnica, Università di Roma " La Sapienza", Roma, Italy, 2003.

[16] S. Leela, "A new unified concept of stability," *Facta Universitates, Series Mechanics, Automatic Control and Robotics*, vol. 4, no. 16, pp. 33–42, 2004.

[17] K. Hedrih (Stevanović), *Vector Method of the Heavy Rotor Kinetic Parameter Analysis and Nonlinear Dynamics*, Monograph, University of Niš, 2001.

[18] K. Hedrih (Stevanović), "The analogy between the stress state model, the strain state model and the mass inertia moment state model," *Facta Universitatis, Series Mechanics, Automatic Control and Robotics*, vol. 1, no. 1, pp. 105–120, 1991.

[19] K. Hedrih (Stevanović), "Nonlinear dynamics of a gyrorotor and sensitive dependence on initial conditions of a heavy gyrorotor forced vibration/rotation motion," in *Proceednings of the 2nd International Conference on Control of Oscillations and Chaos (COC '00)*, F. L. Chernousko and A. I. Fradkov, Eds., vol. 2, pp. 259–266, IEEE, CSS, IUTAM, SPICS, St. Petersburg, Institute for Problems of mechanical Engineering of Russian Academy of Sciences and St. Petersburg State University, 2000, Semi-Plenary Invited Lecture.
[20] K. Hedrih (Stevanović), "Nonlinear dynamics of a gyrorotor with turbulent damping," *Nauka Tehnika Bezbednost*, vol. 1, pp. 109–127, 2001, Serbian.
[21] K. Hedrih (Stevanović), "Axoids (cones) in the nonlinear dynamics of the heavy rotors with many axes of the rotation (gyrorotors)," in *Proceedings of the International Congress of Nonlinear Analysis and It's Applications*, p. 130, Moscow, Russia, September 1998.
[22] K. Hedrih (Stevanović) and G.. Janevski, "Nonlinear dynamics of a gyro-disc-rotor and structural dependence of a phase portrait on the initial conditions," in *Proceedings of Dynamics of Machine*, pp. 81–88, Institute of Thermomechanics, Czech Committee of the European Mechanics Society, Prague, Czech Republic, 2000.
[23] K. Hedrih (Stevanović) and L. Simonovió, "Phase portraits and homoclinic orbits: visualization of nonlinear dynamics of reductor," *Journal of Politechnica University Timisoara, Romonia, Transaction on Mechanical Engineering*, vol. 47, no. 61, pp. 76–86, 2008.
[24] K. Hedrih (Stevanović) and L. Veljović, "Nonlinear dynamics of the heavy gyro-rotor with two skew rotating axes," *Journal of Physics: Conference Series*, vol. 96, no. 1, Article ID 012221, 2008.
[25] K. Hedrih (Stevanović) and L. Veljović, "Nonlinear dynamic of heavy gyro- rotor with two rotating axes," *Facta Universitates Series Mechanics, Automatic Control and Robotics*, vol. 14, no. 16, pp. 55–68, 2004.
[26] J. Lin and R. G. Parker, "Analatical characterization od the unique properties of planetarly gear free vibration," *Journal of Vibrattion and Acoustics*, vol. 121, pp. 316–321, 1999.
[27] S. Y.T. Lang, "Graph-theoretic modelling of epicyclic gear systems," *Mechanism and Machine Theory*, vol. 40, no. 5, pp. 511–529, 2005.
[28] T. Sun and H. Hu, "Nonlinear dynamics of a planetary gear system with multiple clearances," *Mechanism and Machine Theory*, vol. 38, no. 12, pp. 1371–1390, 2003.
[29] K. Hedrih (Stevanović), "WEB Presentation of the Lectures on Mechanics III—Dynamics," 2006-2007, http://www.hm.co.yu/mehanika/.
[30] K. Hedrih (Stevanović), "Vectors of the body mass moments: topics from mathematics and mechanics," *Zbornik radova*, vol. 8, no. 16, pp. 45–104, 1998.
[31] K. Hedrih (Stevanović), "The mass moment vectors at n-dimensional coordinate system," *Tensor*, vol. 54, pp. 83–87, 1993.
[32] K. Hedrih (Stevanović), "Derivatives of the mass moment vectors at the dimensional coordinate system N," *Facta Universitatis. Series: Mathematics and Informatics*, vol. 13, no. 1, pp. 139–150, 2001.
[33] K. Hedrih (Stevanović), "Derivatives of the mass moments vectors with applications," in *Proceedings of the 5th National Congress on Mechanics*, pp. 694–705, Ioannina, Greece, 1998, invited lecture.
[34] K. Hedrih (Stevanović), "Vector method of the kinetic parameters analysis of the rotor with many axes and nonlinear dynamics," in *Proceedings of the 3rd International Conference on Nonlinear Mechanics (ICNM '98)*, pp. 42–47, Shanghai, China, August 1998, Parallel General Lecture.
[35] K. Hedrih (Stevanović), "On some interpretations of the rigid bodies kinetic parameters," in *Proceedings of the 18th International Congress of Theoretical and Applied Mechanics (ICTAM '92)*, pp. 73–74, Haifa, Israel, August 1992.
[36] K. Hedrih (Stevanović), "For optimal time of study: vector and tensor methods in classical mechanics," *Nonlinear Dynamics*, pp. 98–107, 2008.
[37] A. A. Andronov, A. A Vitt, and S. E. Haykin, *Teoriya Kolebaniy*, Nauka, Moscow, Russia, 1981.
[38] D. Rašković and K. Hedrih (Stevanović), "Ubrzanje drugog reda (trzaj) pri obrtanju tela oko nepomične tačke," *Zbornik radova Tehničkog fakulteta, Niš*, pp. 93–100, 1966-1967.
[39] K. Hedrih (Stevanović), "Dynamics of coupled systems," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 2, pp. 310–334, 2008.
[40] K. Hedrih (Stevanović), "Vectorial method of the kinetic parameters analysis of the rotor with two rotation axes and nonlinear dynamics in the field of the turbulent damping," in *Proceedings of the 3rd International Symposium on Classical and Celestial Mechanics*, p. 163, Velikiye Luki, Russia, 1998.

[41] M. Ajmi and P. Velex, “A model for simulating the quasi-static and dynamic behaviour of solid wide-faced spur and helical gears,” *Mechanism and Machine Theory*, vol. 40, no. 2, pp. 173–190, 2005.
[42] B. S. Bačlić and T. M. Atanacković, “Stability and creep of a fractional derivative order viscoelastic rod,” *Bulletin, Classe des Sciences Mathématiques et Natturalles Sciences Mathématiques*, no. 25, pp. 115–131, 2000.
[43] D. Rašković, *Theory of Vibrations*, Naučna knjiga, Beograd, Serbia, 1965.
[44] D. Rašković, *Mehanika III, Dinamika*, Naučna knjiga, Beograd, Serbia, 1972.
[45] K. Hedrih (Stevanović), “Partial fractional order differential equations of transversal vibrations of creep-connected double plate systems,” in *Fractional Differentiation and Its Applications*, A. Le Mahaute, J. A. Tenreiro Machado, J. C. Trigeassou, and J. Sabatier, Eds., vol. 2 of *Monograph*, pp. 289–302, 2005.
[46] K. Hedrih (Stevanović), “Interpretation of the motion of a heavy body around a stationary axis in the field with turbulent damping and kinetic pressures on bearing by means of the mass moment vector for the pole and the axis,” *Facta Universitatis, Series Mechanics, Automatic Control and Robotics*, vol. 1, no. 4, pp. 519–538, 1994.
[47] K. Hedrih (Stevanović), “Some interpretations of the rigid body kinetic parameters,” *Tehnika, Mašinstvo*, vol. 45, pp. M8–M13, 1996.
[48] K. Hedrih (Stevanović), “Vector method and phase plane method, approach for the analysis of kinetic parameters and hmoclinic orbits of coupled rotor dynamics,” in *Proceedings of the International Symposium on Analysis, Manifolds and Mechanics*, Calcutta, India, 2003, in Honor of Professor M. C. Chaki’s 90th Birth Anniversary.
[49] K. Hedrih (Stevanović), “Nonlinear dynamics of rotor with a vibrating axis and sensitive dependence on initial conditions of forced vibration/rotation/stochasticlikechaoticlike motion of a heavy rotor,” in *Proceedings of the 3rd Bogoliubov Readings: Asymptotic and Qualitative Methods of Nonlinear Mechanics (ASYM ’97)*, pp. 73–74, Tezi dopovidey, Mezhdunarodnaya konferenciya, Inst. Math. NANU, Kiev, Russia, September 1997.
[50] K. Hedrih (Stevanović), “Nonlinear dynamics of a rotor with a vibrating axis, and sensitive dependence on the initial conditions of the forced vibration of a heavy rotor,” *Nelīnīīnī Kolivannya* , vol. 3, no. 1, pp. 129–145, 2000.
[51] K. Hedrih (Stevanović), “Nonlinear Dynamics of a Rotor with a vibrating axis, and sensitive dependence on the initial conditions of the forced vibration of a heavy rotor,” *International Journal Nonlinear Oscillations*, vol. 3, no. 1, pp. 129–145, 2000.
[52] Ph. Holmes, “Nonlinear oscillations and the smale horseshoe map,” in *Proceedings of Symposium in Applied Mathematics*, vol. 39, pp. 81–88, American Mathematical Society, Providence, RI, USA, 1989.
[53] J. J. Stoker, *Nonlinear Vibrations*, Interscience, New York, NY, USA, 1950.
[54] K. Hedrih (Stevanović), “Homoclinic Orbits of Nonlinear Dynamics of Multi-step Reductor,” *Nauka Tehnika Bezbednost*, vol. 12, no. 1, pp. 45–65, 2002.
[55] V. Nikolić, *Mechanical Analysis of the Gear Transmission*, Monograph, Mašinski fakultet u Kragujevcu, 1999.
[56] J. Posa and S. Wojciech, “Dynamics of systems with changing configuration and with flexible beam-like links,” *Mechanism and Machine Theory*, vol. 35, no. 11, pp. 1515–1534, 2000.
[57] M. Enelund, *Fractional Calculus and Linear Viscoelasticity in Structural Dynamics*, Division of Solid Mechanics, Chalmers Tekniska Hogskola, Goteborg, Sweden, 1996.
[58] R. Gorenflo and F. Mainardi, “Fractional calculus, integral and differential equations of fractional order,” in *Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996)*, vol. 378 of *CISM Courses and Lectures*, pp. 223–276, Springer, Vienna, Austria, 1997.
[59] O. A Goroško and K. Hedrih (Stevanović), “Construction of the Lagrange’s mechanics of the hereditary systems,” *Facta Universitatis, Series Mechanics, Automatic Control and Robotics*, vol. 6, no. 1, pp. 1–22, 2007.
[60] O. A. Goroško and K. Hedrih (Stevanović), “Advances in development of the analytical dynamics of the hereditary discrete systems,” *Journal of Physics: Conference Series*, vol. 96, no. 1, Article ID 012143, 2008.
[61] K. Hedrih (Stevanović), “Discrete continuum method,” in *Proceedings of the 6th World Congress on Computational Mechanics (WCCM VI) in Conjunction with the 2nd Asian-Pacific Congress on Computational Mechanics (APCOM ’04)*, vol. 1, pp. 1–11, Tsinghua University Press & Springer, Beijing, China, September 2004.
[62] K. Hedrih (Stevanović), “Modes of the homogeneous chain dynamics,” *Signal Processing*, vol. 86, no. 10, pp. 2678–2702, 2006.

[63] O. A Goroško and K. Hedrih (Stevanović), *Analytical Dynamics (Mechanics) of Discrete Hereditary Systems*, Monograph, University of Niš, 2001.
[64] K. Hedrih (Stevanović), "Creep vibrations of a fractional derivative order constititive relation deformable bodies," in *Proceedings of the 8th American Congress of Applied Mechanics (PACAM '04)*, vol. 10 of *Applied Mechanics in Americas*, pp. 548–551, La Habana, Cuba, 2004.
[65] K. Hedrih (Stevanović), "Transversal creep vibrations of a beam with fractional derivative constitutive relation order, first part: partial fractional-differential equation . part second: stochastic stability of the beam dynamic shape, under axial bounded noise excitation," in *Proceedings of 4th International Conference on Nonlinear Mechanics (ICNM '04)*, W. Z. Chien, Ed., vol. 10 of *Applied Mechanics in Americas*, pp. 584–595, Shanghai, China, August 2002.
[66] K. Hedrih (Stevanović), "The transversal creeping vibrations of a fractional derivative order constitutive relation of nonhomogeneous beam," *Mathematical Problems in Engineering*, vol. 2006, no. 5, Article ID 46236, 18 pages, 2006.
[67] K. Hedrih (Stevanović), "Partial fractional order differential equations of transversal vibrations of creep connected double plates systems," in *Proceedings of 1st IFAC Workshop on Fractional Differentiation and Its Application (FDA ' 04)*, vol. 10, pp. 299–304, Bordeaux, France, July 2004.
[68] K. Hedrih (Stevanović), "The longitudinal and transversal creep vibrations of a fractional order constitutive relation beams," *Scientific Bulletin of the Politehnica, University of Timisoara, Transaction on Mechanics*, vol. 48, no. 62, pp. 5–12, 13–22, 2003.
[69] K. Hedrih (Stevanović) and A. Filipovski, "Longitudinal vibration of a fractional derivative order rheological rod with variable cross section," *Facta Universitatis, Series Mechanics, Automatic Control and Robotics*, vol. 3, no. 12, pp. 327–350, 2000.
[70] K. Hedrih (Stevanović), "Dynamics of multipendulum systems with fractional order creep elements," *Journal of Theoretical and Applied Mechanics*, vol. 46, no. 3, pp. 483–509, 2008.
[71] K. Hedrih (Stevanović), "Hybrid systems and hybrid dynamics: theory and applications," in *Proceedings of the 8th HSTAM International Congress on Mechanics*, N. Bazwos, D. L. Karabalis, D. Polyzos, D. E. Beskos, and J. T. Katsikadelis, Eds., vol. 1, pp. 77–86, Patras, Greece, July 2007, invited plernary lecture.