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Research Article

Effective Flow of Micropolar Fluid through a Thin or Long Pipe

Igor Pažanin

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

Correspondence should be addressed to Igor Pažanin, pazanin@math.hr

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The aim of this paper is to present the result about asymptotic approximation of the micropolar fluid flow through a thin (or long) straight pipe with variable cross section. We assume that the flow is governed by the prescribed pressure drop between pipe's ends. Such model has relevance to some important industrial and engineering applications. The asymptotic behavior of the flow is investigated via rigorous asymptotic analysis with respect to the small parameter, being the ratio between pipe's thickness and its length. In the case of circular pipe, we obtain the explicit formulae for the approximation showing explicitly the effects of microstructure on the flow. We prove the corresponding error estimate justifying the obtained asymptotic model.

1. Introduction

The Navier-Stokes model of classical hydrodynamics has a drastic limitation: it does not take into account the microstructure of the fluid. One of the best-established theories of fluids with microstructure is the theory of micropolar fluids, introduced by Eringen [1]. The mathematical model of micropolar fluid enables us to study many physical phenomenae arising from the local structure and micromotions of the fluid particles. It describes the behavior of numerous real fluids (such as polymeric suspensions, liquid crystals, muddy fluids, and animal blood) better than the classical Navier-Stokes model, especially when the characteristic dimensions of the flow (e.g., diameter of the pipe) become small. Due to its importance in industrial and engineering applications, there are large number of papers on micropolar fluid flow, mostly in the engineering literature (see, e.g., [2–7]). The monograph [8] provides a unified picture of the mathematical theory underlying the applications of this particular model. We would also like to point out two recent papers of Dupuy et al. [9, 10] in which the authors rigorously derive asymptotic models for two-dimensional micropolar flow through a periodically constricted tube and a thin curvilinear channel. It is important to emphasize that 2D setting (in which the microrotation is a scalar function) has often been employed, especially in blood motion modeling. However, in the present paper, our aim is to study 3D flow describing the real physical situation.

We consider one important application of micropolar fluids: laminar flow in a straight pipe with variable cross section. We suppose that the flow is stationary and governed by the prescribed pressure drop between pipe's ends. It is well-known that the stationary Navier-Stokes system describing the viscous flow in straight pipe with impermeable walls governed by the prescribed pressure drop has a solution in the form of the Poiseuille flow, which in case of pipe with constant circular cross section reads

$$v(r) = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right), \qquad \begin{cases} v\text{: velocity,} \\ \mu\text{ : viscosity,} \\ \Delta p\text{: pressure drop,} \\ L, R\text{: pipe's length and radius.} \end{cases} \tag{1.1}$$

However, Poiseuille formula gives an exact solution only in case of laminar flow of Newtonian fluid through a pipe with constant cross section. If the pipe has a variable cross section or it is curved, one can only derive the approximation of the solution by a singular perturbation techniques (see, e.g., [11–14]). Here we deal with the micropolar fluid model (representing the generalization of the Navier-Stokes model) which introduces a new vector field, the angular velocity field of rotation of particles (microrotation). Correspondingly, one new vector equation is added to Navier-Stokes system, expressing the conservation of the angular momentum. Naturally, one cannot hope to obtain the exact solution of such (coupled) system of equations so our goal is to derive an asymptotic approximation of the solution and evaluate the difference between the exact solution of the governing problem (which we cannot find) and the asymptotic one. Generally, there are several methods that enables us to find the asymptotic behavior of the flow. By taking the average over the cross section of the pipe, we can obtain simple one-dimensional approximation, based on the assumption that, in case of very thin (or very long) pipe, the variations of the solution on the cross section are of no relevance for the global flow. However, obtained approximation would have low order of accuracy and gives no information about flow profile in the pipe. Another approach, which we use here, is based on the rigorous asymptotic analysis with respect to the small parameter ε, introduced as the ratio between pipe's thickness and its length. It relies on two-scale asymptotic expansions in powers of small parameter which, in our case, have the form

$$\begin{gathered} \mathbf{U}_\varepsilon(x) = \varepsilon^2 \mathbf{U}_0\left(x_1, \frac{x'}{\varepsilon}\right) + \varepsilon^3 \mathbf{U}_1\left(x_1, \frac{x'}{\varepsilon}\right) + \cdots \ \text{(velocity)}, \\ \mathbf{W}_\varepsilon(x) = \varepsilon^2 \mathbf{W}_0\left(x_1, \frac{x'}{\varepsilon}\right) + \varepsilon^3 \mathbf{W}_1\left(x_1, \frac{x'}{\varepsilon}\right) + \cdots \ \text{(microrotation)}, \\ P_\varepsilon(x) = P_0(x_1) + \varepsilon P_1\left(x_1, \frac{x'}{\varepsilon}\right) + \cdots \ \text{(pressure)}. \end{gathered} \tag{1.2}$$

The variable x_1 is directed along the pipe, while $x' = (x_2, x_3)$ describes the cross section. The role of dilated (fast) variable $y' = x'/\varepsilon$ is to capture the fast changes of the solution on the pipe's cross section. Plugging the above expansions in the governing system and collecting the terms with equal powers of ε, lead us to the recursive sequence of linear problems. Assuming that the pipe's cross section is circular (which is the most common case in real-life situations), we are in position to solve those problems explicitly and to clearly observe the influence of the microstructure on the effective flow. The main difficulty arises from the fact that the governing system is coupled so we have to simultaneously solve boundary-value problems for velocity and for microrotation. Furthermore, in some thin layer in the vicinity of pipe's ends we have some influence of the boundary condition for the microrotation which cannot be captured by the formal (interior) expansion, so we have to construct the appropriate boundary-layer correctors to fix our approximation.

The paper is organized as follows: in Section 2, we describe the geometry of our three dimensional domain and present the governing system of equations describing the fluid motion. After discussing its solvability, in Section 3, we write the problem in rescaled domain (independent of small parameter ε) and construct an asymptotic expansion of the solution in terms of the pipe's thickness. The last section is devoted to rigorous justification of the derived asymptotic model. After deriving some *a priori* bounds for the original solution, we prove the error estimates in the appropriate norm. It turns out that our asymptotic solution approximates the flow with an error of order $\varepsilon^3\sqrt{\varepsilon}$ for the velocity and with an error of order ε^4 for the microrotation.

2. Position of the Problem

2.1. The Geometry

In order to describe the thin pipe with a small parameter ε appearing explicitly, we first introduce

$$\Omega = \left\{ \left(x_1, y'\right) \in \mathbf{R}^3 : 0 < x_1 < \ell,\ y' = \left(y_2, y_3\right) \in B(x_1) \right\}, \tag{2.1}$$

where the family of bounded domains $\{B(x_1)\}_{x_1 \in [0,\ell]} \subset \mathbf{R}^2$ is chosen such that Ω is locally Lipschitz. Now, we define our thin pipe with variable cross section $B(x_1)$ and length ℓ by

$$\Omega_\varepsilon = \left\{ x = \left(x_1, x'\right) \in \mathbf{R}^3 : 0 < x_1 < \ell,\ x' = (x_2, x_3) \in \varepsilon\ B(x_1) \right\}. \tag{2.2}$$

We are particularly interested in the case when the pipe Ω_ε has circular cross section, that is, when

$$B(x_1) = \left\{ y' \in \mathbf{R}^2 : \left|y'\right| < R(x_1) \right\}, \tag{2.3}$$

with R being a strictly positive bounded function defined on $\mathbf{R}$. Finally, we denote the ends of the pipe by $\Sigma_\varepsilon^i = \varepsilon B(i)$, $i = 0, \ell$, while its lateral boundary is given by

$$\Gamma_\varepsilon = \left\{ x = \left(x_1, x'\right) \in \mathbf{R}^3 : 0 < x_1 < \ell,\ x' = (x_2, x_3) \in \varepsilon \partial B(x_1) \right\}. \tag{2.4}$$

2.2. The Governing Equations

The governing system of equations expresses the balance of momentum, mass, and angular momentum, which in stationary regime reads

$$-\mu\Delta\mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon\nabla)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = a \operatorname{rot}\mathbf{w}_\varepsilon + \mathbf{f}, \tag{2.5}$$

$$\operatorname{div}\mathbf{u}_\varepsilon = 0, \quad \text{in } \Omega_\varepsilon, \tag{2.6}$$

$$-\alpha\Delta\mathbf{w}_\varepsilon + (\mathbf{u}_\varepsilon\nabla)\mathbf{w}_\varepsilon - \beta\nabla \operatorname{div}\mathbf{w}_\varepsilon + \gamma\mathbf{w}_\varepsilon = a \operatorname{rot}\mathbf{u}_\varepsilon + \mathbf{g}. \tag{2.7}$$

The unknown functions are $\mathbf{u}_\varepsilon$, $\mathbf{w}_\varepsilon$ and p_ε standing for the velocity, the microrotation and the pressure of the fluid, respectively. The fields $\mathbf{f} = \mathbf{f}(x_1)$, $\mathbf{g} = \mathbf{g}(x_1)$ represent given external forces and moments, respectievly and we assume $\mathbf{f}, \mathbf{g} \in C^1([0,\ell])^3$. Viscosity coefficients read $\mu = \nu+\nu_r$, $a = 2\nu_r$, $\alpha = c_a+c_d$, $\beta = c_0+c_d-c_a$, $\gamma = 4\nu_r$, where ν, ν_r, c_0, c_a, c_d are the given positive constants (ν is the usual Newtonian viscosity, ν_r is microrotation viscosity, c_0, c_a, c_d are the coefficients of angular viscosities). Observe that if we put ν_r to be equal zero, then the system becomes decoupled and (2.5)-(2.6) reduce to classical Navier-Stokes equations. We refer the reader to [8] for a rigorous derivation of the above system from general conservation laws.

We complete the system (2.5)–(2.7) with the following boundary conditions

$$\mathbf{u}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \tag{2.8}$$

$$\mathbf{e}_1 \times \mathbf{u}_\varepsilon = 0, \quad p_\varepsilon = q_i \quad \text{on } \Sigma_\varepsilon^i,\ i = 0, \ell, \tag{2.9}$$

$$\mathbf{w}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{2.10}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denotes the standard Cartesian basis.

Remark 2.1. By prescribing constant pressures q_0, q_ℓ on Σ_ε^i, we assure that the fluid flow is governed by a pressure drop between pipe's ends. Condition (2.8) is the classical no-slip boundary condition for the velocity. Imposing that the tangential component of the velocity $\mathbf{e}_1 \times \mathbf{u}_\varepsilon$ equals to zero is not a serious restriction since the only part that counts is the normal part, due to the Saint-Venant's principle for thin domains (see, e.g., [11]). The boundary conditions for the velocity and pressure as in (2.8), (2.9) are physically clear and justified. On the other hand, there exists no general agreement about the type of the boundary condition one should set for microrotation. The most commonly used throughout the literature is the one as in (2.10), although we can also find other types of boundary conditions (see, e.g., [15, 16]). Nevertheless, it must be emphasized that not much has been done in proving the well-posedeness of the corresponding boundary-value problems, except in the case of the classical Dirichlet condition (2.10).

In [8, Chapter 2, pages 60–69], the homogeneous Dirichlet boundary-value problem for an incompressible micropolar fluid is considered, with velocity prescribed on the whole boundary. Using fixed-point argument, the existence of its weak solution is proved (Theorem 1.1.1). Furthermore, such solution is shown to be unique if the viscosity μ is large enough (Theorem 1.1.2). In our setting (2.5)–(2.10), the only difference is that we prescribe the value of pressures at pipe's ends in order to consider the situation naturally arising in the applications. Pressure boundary condition $(2.9)_2$ should be considered in view of the corresponding variational formulation: find $\mathbf{u}_\varepsilon \in V_\varepsilon = \{\mathbf{v} \in H^1(\Omega_\varepsilon)^3 : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_\varepsilon, \mathbf{v} = 0$ on Γ_ε, $\mathbf{e}_1 \times \mathbf{v} = 0$ on Σ_ε^i, $i = 0, \ell\}$, $w_\varepsilon \in H_0^1(\Omega_\varepsilon)^3$, such that

$$\mu \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon \nabla \mathbf{v} + \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \nabla) \mathbf{u}_\varepsilon \ \mathbf{v} = q_0 \int_{\Sigma_\varepsilon^0} \mathbf{v} \cdot \mathbf{e}_1 - q_\ell \int_{\Sigma_\varepsilon^\ell} \mathbf{v} \cdot \mathbf{e}_1 + a \int_{\Omega_\varepsilon} \operatorname{rot} \mathbf{w}_\varepsilon \cdot \mathbf{v} + \int_{\Omega_\varepsilon} \mathbf{f} \mathbf{v}, \tag{2.11}$$

for any $\mathbf{v} \in V_\varepsilon$. As we can see, the nonlinear term in (2.11) does not vanish, causing the absence of the energy equality. Such technical difficulty can be elegantly overcome by prescribing dynamic (Bernoulli) pressure $p + (1/2)|\mathbf{u}|^2$ (which has no physical justification in the case of viscous fluid), or by restricting to the case of small boundary data. Indeed, from (2.11), it follows that we do not actually impose the value of the pressure at pipe's ends $x_1 = 0, \ell$, but only the pressure drop $q_0 - q_\ell$. Following the approach first proposed in [17] (see also [18] for details) and supposing that the pressure drop is reasonably small, one can easily adapt the proof of Theorems 1.1.1 and 1.1.2 from [8] to our situation and prove that the velocity $\mathbf{u}_\varepsilon$ is unique in some ball $B_{R_0} = \{\mathbf{v} \in V_\varepsilon; \|\nabla \mathbf{v}\|_{L^2(\Omega_\varepsilon)} \le R_0\}$, with radius R_0 remaining bounded as $\varepsilon \to 0$. This fact is crucial for proving the a priori estimate for the velocity since it enable us to control the inertial term in (2.5) (see Section 4, Proposition 4.2).

3. Asymptotic Analysis

3.1. Rescaling of the Domain

Our main goal is to find the asymptotic behavior of the flow, as the thickness $\varepsilon \to 0$. To accomplish that, we first need to rescale the domain, that is, to write the governing problem on Ω instead of Ω_ε. Introducing the new functions

$$\mathbf{U}_\varepsilon(x_1, y') = \mathbf{u}_\varepsilon(x_1, \varepsilon y'), \qquad P_\varepsilon(x_1, y') = p_\varepsilon(x_1, \varepsilon y'), \qquad \mathbf{W}_\varepsilon(x_1, y') = \mathbf{w}_\varepsilon(x_1, \varepsilon y'), \tag{3.1}$$

we can write the equations (2.5)–(2.7) in the following form:

$$\begin{aligned} &-\mu\left(\frac{\partial^2 \mathbf{U}_\varepsilon}{\partial x_1^2} + \frac{1}{\varepsilon^2}\Delta_{y'}\mathbf{U}_\varepsilon\right) + U_\varepsilon^1 \frac{\partial \mathbf{U}_\varepsilon}{\partial x_1} + \frac{1}{\varepsilon}\left(\mathbf{U}_\varepsilon \nabla_{y'}\right)\mathbf{U}_\varepsilon + \frac{\partial P_\varepsilon}{\partial x_1}\mathbf{e}_1 + \frac{1}{\varepsilon}\nabla_{y'} P_\varepsilon \\ &\quad = \frac{a}{\varepsilon}\left(\frac{\partial W_\varepsilon^3}{\partial y_2} - \frac{\partial W_\varepsilon^2}{\partial y_3}\right)\mathbf{e}_1 + \frac{a}{\varepsilon}\operatorname{rot}_{y'} W_\varepsilon^1 - a\frac{\partial W_\varepsilon^3}{\partial x_1}\mathbf{e}_2 + a\frac{\partial W_\varepsilon^2}{\partial x_1}\mathbf{e}_3 + \mathbf{f}(x_1), \end{aligned} \tag{3.2}$$

$$\frac{\partial U_\varepsilon^1}{\partial x_1} + \frac{1}{\varepsilon}\operatorname{div}_{y'}\mathbf{U}_\varepsilon = 0, \quad \text{in } \Omega, \tag{3.3}$$

$$\begin{aligned} &-\alpha\left(\frac{\partial^2 \mathbf{W}_\varepsilon}{\partial x_1^2} + \frac{1}{\varepsilon^2}\Delta_{y'}\mathbf{W}_\varepsilon\right) + U_\varepsilon^1 \frac{\partial \mathbf{W}_\varepsilon}{\partial x_1} + \frac{1}{\varepsilon}(\mathbf{U}_\varepsilon \nabla_{y'})\mathbf{W}_\varepsilon \\ &\qquad - \beta\left(\frac{\partial^2 W_\varepsilon^1}{\partial x_1^2}\mathbf{e}_1 + \frac{1}{\varepsilon}\frac{\partial}{\partial x_1}(\operatorname{div}_{y'}\mathbf{W}_\varepsilon)\mathbf{e}_1 + \frac{1}{\varepsilon}\nabla_{y'}\left(\frac{\partial W_\varepsilon^1}{\partial x_1}\right) + \frac{1}{\varepsilon^2}\nabla_{y'}(\operatorname{div}_{y'}\mathbf{W}_\varepsilon)\right) + \gamma \mathbf{W}_\varepsilon \\ &\quad = \frac{a}{\varepsilon}\left(\frac{\partial U_\varepsilon^3}{\partial y_2} - \frac{\partial U_\varepsilon^2}{\partial y_3}\right)\mathbf{e}_1 + \frac{a}{\varepsilon}\operatorname{rot}_{y'} U_\varepsilon^1 - a\frac{\partial U_\varepsilon^3}{\partial x_1}\mathbf{e}_2 + a\frac{\partial U_\varepsilon^2}{\partial x_1}\mathbf{e}_3 + \mathbf{g}(x_1). \end{aligned} \tag{3.4}$$

Here and in the sequel, we use the following notations for the formal partial differential operators:

$$\operatorname{div}_{y'}\mathbf{V} = \frac{\partial V^2}{\partial y_2} + \frac{\partial V^3}{\partial y_3}, \quad \Delta_{y'}\mathbf{V} = \frac{\partial^2 \mathbf{V}}{\partial y_2^2} + \frac{\partial^2 \mathbf{V}}{\partial y_3^2}, \quad V^i = \mathbf{V}\cdot\mathbf{e}_i,$$
$$\nabla_{y'}v = \frac{\partial v}{\partial y_2}\mathbf{e}_2 + \frac{\partial v}{\partial y_3}\mathbf{e}_3, \quad \operatorname{rot}_{y'}v = \frac{\partial v}{\partial y_3}\mathbf{e}_2 - \frac{\partial v}{\partial y_2}\mathbf{e}_3. \tag{3.5}$$

3.2. Asymptotic Expansions

In this section, we construct the formal asymptotic expansion of the solution in powers of small parameter ε. As mentioned in Introduction, we expand as follows:

$$\mathbf{U}_\varepsilon(x_1,y') = \varepsilon^2\mathbf{U}_0(x_1,y') + \varepsilon^3\mathbf{U}_1(x_1,y') + \cdots,$$
$$\mathbf{W}_\varepsilon(x_1,y') = \varepsilon^2\mathbf{W}_0(x_1,y') + \varepsilon^3\mathbf{W}_1(x_1,y') + \cdots,$$
$$P_\varepsilon(x_1,y') = P_0(x_1) + \varepsilon P_1(x_1,y') + \cdots. \tag{3.6}$$

3.2.1. First-Order Approximation

Substituting the expansions (3.6) into the rescaled equations (3.2)–(3.4), after collecting the terms with equal powers of ε, we obtain the following problems for first-order approximation $(\mathbf{U}_0, P_0, \mathbf{W}_0)$:

$$1: -\mu\Delta_{y'}\mathbf{U}_0 + \frac{dP_0}{dx_1}\mathbf{e}_1 + \nabla_{y'}P_1 = \mathbf{f}(x_1) \quad \text{in } \Omega,$$
$$\varepsilon: \operatorname{div}_{y'}\mathbf{U}_0 = 0 \quad \text{in } \Omega,$$
$$\mathbf{U}_0 = 0 \quad \text{on } \Gamma, \tag{3.7}$$

$$1: -\alpha\Delta_{y'}\mathbf{W}_0 - \beta\nabla_{y'}(\operatorname{div}_{y'}\mathbf{W}_0) = \mathbf{g}(x_1) \quad \text{in } \Omega,$$
$$\mathbf{W}_0 = 0 \quad \text{on } \Gamma. \tag{3.8}$$

Here, we denote $\Gamma = \{(x_1,y') \in \mathbf{R}^3 : 0 < x_1 < \ell,\ y' \in \varepsilon\partial B(x_1)\}$. Notice that the problems for the velocity and the microrotation are, at this stage, decoupled. The system (3.7) can be solved by taking

$$\mathbf{U}_0(x_1,y') = \frac{1}{\mu}\chi(x_1,y')\left(f^1(x_1) - \frac{dP_0}{dx_1}(x_1)\right)\mathbf{e}_1, \qquad P_1(x_1,y') = f^2(x_1)y_2 + f^3(x_1)y_3, \tag{3.9}$$

where $f^i = \mathbf{f}\cdot\mathbf{e}_i$ and $\chi(x_1,y')$ denotes the solution of the auxiliary problem posed on the cross section $B(x_1)$:

$$-\Delta_{y'}\chi(x_1,\cdot) = 1 \quad \text{in } B(x_1), \qquad \chi(x_1,\cdot) = 0 \quad \text{on } \partial B(x_1). \tag{3.10}$$

If the pipe has circular cross section (2.3), we can compute χ explicitly from (3.10):

$$\chi(x_1, y') = \frac{1}{4}\left(R(x_1)^2 - |y'|^2\right). \tag{3.11}$$

We still have to determine $P_0(x_1)$. The next term in $(3.7)_2$ implies

$$\varepsilon^2 : \frac{\partial U_0^1}{\partial x_1} + \operatorname{div}_{y'} \mathbf{U}_1 = 0 \quad \text{in } \Omega. \tag{3.12}$$

Integration over $B(x_1)$ with respect to y' yields

$$\frac{\partial}{\partial x_1}\left(\int_{B(x_1)} U_0^1 \, dy'\right) = 0 \Longrightarrow \int_{B(x_1)} U_0^1 dy' = C_1 = \text{const}. \tag{3.13}$$

Introducing

$$\theta(x_1) = \int_{B(x_1)} \chi(x_1, y') dy', \tag{3.14}$$

from $(3.9)_1$, we deduce

$$\theta(x_1)\left(f^1(x_1) - \frac{dP_0}{dx_1}(x_1)\right) = C_1. \tag{3.15}$$

It follows

$$P_0(x_1) = -C_1 \int_0^{x_1} \frac{d\xi}{\theta(\xi)} + \int_0^{x_1} f^1(\xi) d\xi + C_2, \tag{3.16}$$

with C_2 being an arbitrary constant. Taking into account the pressure boundary condition $(2.9)_2$, we get

$$C_1 = \left(\int_0^{\ell} \frac{d\xi}{\theta(\xi)}\right)^{-1} \left(q_0 - q_\ell + \int_0^{\ell} f^1(\xi) d\xi\right), \quad C_2 = q_0. \tag{3.17}$$

Therefore, in the case of circular pipe, we have

$$P_0(x_1) = q_0 + \int_0^{x_1} f^1(\xi) d\xi - \left(q_0 - q_\ell + \int_0^{\ell} f^1(\xi) d\xi\right)\left(\int_0^{\ell} \frac{d\xi}{R(\xi)^4}\right)^{-1} \int_0^{x_1} \frac{d\xi}{R(\xi)^4}, \tag{3.18}$$

$$\mathbf{U}_0(x_1, y') = \frac{R(x_1)^2 - |y'|^2}{4\mu R(x_1)^4}\left(\int_0^{\ell} \frac{d\xi}{R(\xi)^4}\right)^{-1} \left(q_0 - q_\ell + \int_0^{\ell} f^1(\xi) d\xi\right)\mathbf{e}_1. \tag{3.19}$$

Similarly, it can be verified that the problem (3.8) for microrotation will be satisfied for

$$W_0^1(x_1,y') = \frac{1}{\alpha}\,\chi(x_1,y')g^1(x_1) = \frac{1}{4\alpha}\left(R(x_1)^2 - |y'|^2\right)g^1(x_1),$$
$$W_0^2(x_1,y') = \frac{2}{2\alpha+\beta}\,\chi(x_1,y')g^2(x_1) = \frac{1}{2(2\alpha+\beta)}\left(R(x_1)^2 - |y'|^2\right)g^2(x_1), \tag{3.20}$$
$$W_0^3(x_1,y') = \frac{2}{2\alpha+\beta}\,\chi(x_1,y')g^3(x_1) = \frac{1}{2(2\alpha+\beta)}\left(R(x_1)^2 - |y'|^2\right)g^3(x_1),\ g^i = \mathbf{g}\cdot\mathbf{e}_i.$$

3.2.2. Correctors

Now, we compute the correctors. The $\mathcal{O}(\varepsilon)$ term from momentum equation (3.2) gives

$$\varepsilon : -\mu\Delta_{y'}\mathbf{U}_1 + \nabla_{y'}P_2 = a\left(\frac{\partial W_0^3}{\partial y_2} - \frac{\partial W_0^2}{\partial y_3}\right)\mathbf{e}_1 + a\,\mathrm{rot}_{y'}W_0^1 - \frac{\partial P_1}{\partial x_1}\mathbf{e}_1. \tag{3.21}$$

The system is not decoupled anymore, so the effects of the microstructure on the fluid velocity occur. Inserting the expressions for P_1 and $\mathbf{W}_0$ derived for circular pipe, we get the following problem for the first component:

$$\mu\Delta_{y'}U_1^1 = \left(\frac{df^2}{dx_1} + \frac{ag^3}{2(2\alpha+\beta)}\right)y_2 + \left(\frac{df^3}{dx_1} + \frac{ag^2}{2(2\alpha+\beta)}\right)y_3 \quad \text{in } \Omega,$$
$$U_1^1 = 0 \quad \text{on } \Gamma. \tag{3.22}$$

Let us introduce $\chi_i(x_1,y')$, $i = 2,3$ as the solutions of the following two problems posed on $B(x_1)$:

$$\Delta_{y'}\chi_i(x_1,y') = y_i \quad \text{in } B(x_1), \qquad \chi_i(x_1,\cdot) = 0 \quad \text{on } \partial B(x_1). \tag{3.23}$$

Taking into account (2.3) and using the polar coordinates yield

$$\chi_2(x_1,y') = \frac{1}{8}\left(|y'|^2 - R(x_1)^2\right)y_2, \qquad \chi_3(x_1,y') = \frac{1}{8}\left(|y'|^2 - R(x_1)^2\right)y_3. \tag{3.24}$$

We seek the solution of system (3.22) in the form

$$U_1^1 = \frac{1}{\mu}\left[\chi_2\left(\frac{df^2}{dx_1} + \frac{ag^3}{2(2\alpha+\beta)}\right) + \chi_3\left(\frac{df^3}{dx_1} + \frac{ag^2}{2(2\alpha+\beta)}\right)\right], \tag{3.25}$$

implying

$$U_1^1(x_1,y') = \frac{1}{8\mu}\left(|y'|^2 - R(x_1)^2\right)\left[y_2\left(\frac{df^2}{dx_1}(x_1) + \frac{ag^3(x_1)}{2(2\alpha+\beta)}\right) + y_3\left(\frac{df^3}{dx_1}(x_1) + \frac{ag^2(x_1)}{2(2\alpha+\beta)}\right)\right]. \tag{3.26}$$

For the other two velocity components from (3.21) and (3.3), we obtain

$$\begin{gathered}
\varepsilon : -\mu\Delta_{y'}\mathbf{U}_1' + \nabla_{y'}P_2 = a\,\mathrm{rot}_{y'}W_0^1 = -\frac{a}{2\alpha}\,g_1(x_1)(y_3\mathbf{e}_2 - y_2\mathbf{e}_3) \quad \text{in } \Omega,\\
\varepsilon^2 : \mathrm{div}_{y'}\mathbf{U}_1' = -\frac{\partial U_0^1}{\partial x_1} \neq 0 \quad \text{in } \Omega,\\
\mathbf{U}_1' = 0 \quad \text{on } \Gamma,
\end{gathered} \tag{3.27}$$

where $\mathbf{U}_1' = (U_1^2, U_1^3)$. Because $\mathbf{U}_1'$ is not divergence-free, it is not likely that the above system can be explicitly solved. However, it is important to emphasize that, since $\int_{B(x_1)}(\partial U_0^1/\partial x_1) = 0$, such problem admits a unique solution $(\mathbf{U}_1', P_2) \in H^1(\Omega)\times L^2(\Omega)$ (see Theorem IV.6.1. from [19]).

It remains to construct the corrector for the microrotation. From (3.4), we deduce

$$\varepsilon : -\alpha\Delta_{y'}\mathbf{W}_1 - \beta\nabla_{y'}(\mathrm{div}_{y'}\mathbf{W}_1) = \beta\left(\frac{\partial}{\partial x_1}(\mathrm{div}_{y'}\mathbf{W}_0)\,\mathbf{e}_1 + \nabla_{y'}\left(\frac{\partial W_0^1}{\partial x_1}\right)\right) + a\,\mathrm{rot}_{y'}U_0^1. \tag{3.28}$$

If we write the above system by the components and take into account (3.19) and (3.20), we get

$$\begin{gathered}
\Delta_{y'}W_1^1 = \frac{\beta}{\alpha(2\alpha+\beta)}\left(y_2\frac{dg^2}{dx_1} + y_3\frac{dg^3}{dx_1}\right),\\
\alpha\Delta_{y'}W_1^2 + \beta\frac{\partial}{\partial y_2}(\mathrm{div}_{y'}\mathbf{W}_1) = \frac{\beta}{2\alpha}\frac{dg^1}{dx_1}y_2 + \frac{a}{2\mu R^4}\left(\int_0^\ell \frac{d\xi}{R(\xi)^4}\right)^{-1}\left(q_0 - q_\ell + \int_0^\ell f^1(\xi)d\xi\right)y_3,\\
\alpha\Delta_{y'}W_1^3 + \beta\frac{\partial}{\partial y_3}(\mathrm{div}_{y'}\mathbf{W}_1) = -\frac{a}{2\mu R^4}\left(\int_0^\ell \frac{d\xi}{R(\xi)^4}\right)^{-1}\left(q_0 - q_\ell + \int_0^\ell f^1(\xi)d\xi\right)y_2 + \frac{\beta}{2\alpha}\frac{dg^1}{dx_1}y_3,\\
W_1^1 = 0 \quad \text{on } \Gamma.
\end{gathered} \tag{3.29}$$

Similarly as for U_1^1, we obtain

$$W_1^1(x_1,y') = \frac{\beta}{8\alpha(2\alpha+\beta)}\left(|y'|^2 - R(x_1)^2\right)\left(y_2\frac{dg^2}{dx_1} + y_3\frac{dg^3}{dx_1}\right). \tag{3.30}$$

The problem satisfied by the other two components W_1^2, W_1^3 must be solved carefully as a system implying

$$W_1^2(x_1,y') = \left(|y'|^2 - R(x_1)^2\right)\left[y_2\frac{\beta}{16\alpha(\alpha+\beta)}\frac{dg^1}{dx_1}(x_1) + y_3\frac{a\left(q_0 - q_\ell + \int_0^\ell f^1(\xi)d\xi\right)}{16\mu\alpha R(x_1)^4\int_0^\ell\left(d\xi/R(\xi)^4\right)}\right], \tag{3.31}$$

$$W_1^3(x_1,y') = \left(R(x_1)^2 - |y'|^2\right)\left[y_2\frac{a\left(q_0 - q_\ell + \int_0^\ell f^1(\xi)d\xi\right)}{16\mu\alpha R(x_1)^4\int_0^\ell\left(d\xi/R(\xi)^4\right)} + y_3\frac{\beta}{16\alpha(\alpha+\beta)}\frac{dg^1}{dx_1}(x_1)\right]. \tag{3.32}$$

3.2.3. Boundary Layers for Microrotation

It is important to notice that our approximation $\mathbf{w}_\varepsilon^1(x) = \varepsilon^2\mathbf{W}_0(x_1,x'/\varepsilon) + \varepsilon^3\mathbf{W}_1(x_1,x'/\varepsilon)$ was computed to satisfy the governing equations and the boundary condition on Γ_ε, while the boundary conditions on pipe's ends were not taken into account. Consequently, the traces of $\mathbf{w}_\varepsilon^1$ on Σ_ε^i $(i = 0,\ell)$ may be different from 0. Thus, before proving convergence, we need to correct our interior expansion in the boundary layer near $x_1 = 0$ and $x_1 = \ell$.

Near $x_1 = 0$, we introduce the boundary layer correctors $\mathbf{B}_i$ $(i = 0,1)$ depending on the dilated variable $(y_1,y') = x/\varepsilon$, as the solutions of the following Dirichlet boundary-value problems posed in the semi-infinite strip $\mathcal{G} = \{(y_1,y') \in \mathbf{R}^3 : y_1 > 0,\ y' = (y_2,y_3) \in B(y_1)\}$:

$$\begin{gathered}-\alpha\Delta\mathbf{B}_i - \beta\nabla\operatorname{div}\mathbf{B}_i = 0 \quad \text{in } \mathcal{G},\\ \mathbf{B}_i = 0 \quad \text{on } \omega,\\ \mathbf{B}_i(0,y') = -\mathbf{W}_i(0,y'),\end{gathered} \tag{3.33}$$

for $i = 0,1$ and $\omega = \{(y_1,y') \in \mathbf{R}^3 : y_1 > 0,\ y' = (y_2,y_3) \in \partial B(y_1)\}$. Using the standard techniques (see [19, Chapter XI.4, pages 252–262] or [20, Appendix]), it can be proved that the system (3.33) admits a unique solution $B_0 \in H^1(\mathcal{G})^3$ which is exponentially decaying to zero as $y_1 \to +\infty$ (see, e.g., [19]). Analogously, the boundary layer correctors $\mathbf{H}_i$ $(i = 0,1)$ corresponding to the opposite side $x_1 = \ell$ are constructed as the unique solutions of the following problems:

$$\begin{gathered}-\alpha\Delta\mathbf{H}_i - \beta\nabla\operatorname{div}\mathbf{H}_i = 0 \quad \text{in } \mathcal{N},\\ \mathbf{H}_i = 0 \quad \text{on } \sigma,\\ \mathbf{H}_i\left(0,y'\right) = -\mathbf{W}_i(\ell,y'),\end{gathered} \tag{3.34}$$

for $i = 0,1$, $\mathcal{N} = \{(y_1,y') \in R^3 : y_1 < 0,\ y' = (y_2,y_3) \in B(y_1)\}$, $\sigma = \{(y_1,y') \in \mathbf{R}^3 : y_1 < 0,\ y' = (y_2,y_3) \in \partial B(y_1)\}$ and its exponential decay to zero at infinity follows as well.

3.2.4. Asymptotic Approximation

To conclude this section, let us write the obtained asymptotic approximation. For the microrotation, it has the following form:

$$\begin{aligned}\mathbf{w}_\varepsilon^1(x) = \varepsilon^2\left[\mathbf{W}_0\left(x_1,\frac{x'}{\varepsilon}\right)+\mathbf{B}_0\left(\frac{x_1}{\varepsilon},\frac{x'}{\varepsilon}\right)+\mathbf{H}_0\left(\frac{x_1-\ell}{\varepsilon},\frac{x'}{\varepsilon}\right)\right]\\ +\varepsilon^3\left[\mathbf{W}_1\left(x_1,\frac{x'}{\varepsilon}\right)+\mathbf{B}_1\left(\frac{x_1}{\varepsilon},\frac{x'}{\varepsilon}\right)+\mathbf{H}_1\left(\frac{x_1-\ell}{\varepsilon},\frac{x'}{\varepsilon}\right)\right],\end{aligned} \tag{3.35}$$

where $\mathbf{W}_0=\sum_i W_0^i\mathbf{e}_i$ and $\mathbf{W}_1=\sum_i W_1^i\mathbf{e}_i$ are given by the explicit formulae (3.20) and (3.30)–(3.32), respectively. On the other hand, the approximation for the velocity/pressure reads:

$$\mathbf{u}_\varepsilon^1(x) = \varepsilon^2\mathbf{U}_0\left(x_1,\frac{x'}{\varepsilon}\right)+\varepsilon^3\mathbf{U}_1\left(x_1,\frac{x'}{\varepsilon}\right), \tag{3.36}$$

$$q_\varepsilon(x) = P_0(x_1)+\varepsilon P_1\left(x_1,\frac{x'}{\varepsilon}\right). \tag{3.37}$$

The first term in the expansion $\mathbf{u}_\varepsilon^0(x) = \varepsilon^2 U_0^1(x_1,x'/\varepsilon)\ \mathbf{e}_1$, given by (3.19) is, in fact, the Poiseuille solution and we do not observe the effects of microstructure here. The Poiseuille flow is, therefore, corrected by a lower-order term which contains those effects (see (3.26) and (3.27)).

4. Error Estimates

In this section, we rigorously justify the obtained asymptotic approximation. The first step is to derive the a priori estimates for the original solution. We start by a technical result.

Lemma 4.1. *There exists a constant $C>0$, independent of ε, such that*

$$\|\varphi\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}, \tag{4.1}$$

$$\|\varphi\|_{L^4(\Omega_\varepsilon)} \le C\varepsilon^{1/4}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}, \tag{4.2}$$

for any $\varphi\in H^1(\Omega_\varepsilon)$, such that $\varphi=0$ on Γ_ε.

The above estimates can be verified by a simple change of variables (see, e.g., [11, Lemmas 7, 8]).

Proposition 4.2. *Let $(\mathbf{u}_\varepsilon,p_\varepsilon,\mathbf{w}_\varepsilon)$ be the solution of the problem* (2.5)–(2.10)*, then there exists $C>0$, independent of ε, such that*

$$\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^3, \qquad \|\nabla\mathbf{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^2. \tag{4.3}$$

Proof. Multiplying the equation (2.7) by $\mathbf{w}_\varepsilon$ and integrating over Ω_ε gives

$$\alpha\int_{\Omega_\varepsilon}|\nabla\mathbf{w}_\varepsilon|^2+\int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\nabla)\mathbf{w}_\varepsilon\mathbf{w}_\varepsilon+\beta\int_{\Omega_\varepsilon}(\operatorname{div}\mathbf{w}_\varepsilon)^2+\gamma\int_{\Omega_\varepsilon}|\mathbf{w}_\varepsilon|^2=a\int_{\Omega_\varepsilon}\operatorname{rot}\mathbf{u}_\varepsilon\mathbf{w}_\varepsilon+\int_{\Omega_\varepsilon}\mathbf{g}\mathbf{w}_\varepsilon. \tag{4.4}$$

First, we deduce

$$\int_{\Omega_\varepsilon^\alpha}(\mathbf{u}_\varepsilon\nabla)\mathbf{w}_\varepsilon\mathbf{w}_\varepsilon=\frac{1}{2}\int_{\Omega_\varepsilon^\alpha}\mathbf{u}_\varepsilon\cdot\nabla|\mathbf{w}_\varepsilon|^2=\frac{1}{2}\int_{\Omega_\varepsilon^\alpha}\operatorname{div}\left(|\mathbf{w}_\varepsilon|^2\mathbf{u}_\varepsilon\right)-\frac{1}{2}\int_{\Omega_\varepsilon^\alpha}|\mathbf{w}_\varepsilon|^2\operatorname{div}\mathbf{u}_\varepsilon=0. \tag{4.5}$$

Using Poincaré's inequality (4.1), we get

$$\left|\int_{\Omega_\varepsilon}\operatorname{rot}\mathbf{u}_\varepsilon\cdot\mathbf{w}_\varepsilon\right|\le\|\operatorname{rot}\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\mathbf{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\le C\varepsilon\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{4.6}$$

$$\left|\int_{\Omega_\varepsilon}\mathbf{g}\mathbf{w}_\varepsilon\right|\le C\ |\Omega_\varepsilon|^{1/2}\|\mathbf{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\le C\varepsilon^2\|\nabla\mathbf{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{4.7}$$

Applying (4.5)–(4.7) into (4.4) implies

$$\|\nabla\mathbf{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\le C\varepsilon\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^2. \tag{4.8}$$

Now, we multiply (2.5) by $\mathbf{u}_\varepsilon$ and, after integrating over Ω_ε, we obtain

$$\mu\int_{\Omega_\varepsilon}|\nabla\mathbf{u}_\varepsilon|^2+\int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\nabla)\mathbf{u}_\varepsilon\mathbf{u}_\varepsilon=q_0\int_{\Sigma_\varepsilon^0}\mathbf{u}_\varepsilon\cdot\mathbf{e}_1-q_\ell\int_{\Sigma_\varepsilon^\ell}\mathbf{u}_\varepsilon\cdot\mathbf{e}_1+a\int_{\Omega_\varepsilon}\operatorname{rot}\mathbf{w}_\varepsilon\mathbf{u}_\varepsilon+\int_{\Omega_\varepsilon}\mathbf{f}\mathbf{u}_\varepsilon. \tag{4.9}$$

Employing the inequality (4.2), we have

$$\left|\int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\nabla)\mathbf{u}_\varepsilon\mathbf{u}_\varepsilon\right|\le\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\mathbf{u}_\varepsilon\|^2_{L^4(\Omega_\varepsilon)}\le C\sqrt{\varepsilon}\|\nabla\mathbf{u}_\varepsilon\|^3_{L^2(\Omega_\varepsilon)}. \tag{4.10}$$

Taking into account (4.1), we obtain

$$\begin{aligned}\left|q_0\int_{\Sigma_\varepsilon^0}\mathbf{u}_\varepsilon\cdot\mathbf{e}_1-q_\ell\int_{\Sigma_\varepsilon^\ell}\mathbf{u}_\varepsilon\cdot\mathbf{e}_1\right|&=\left|\int_{\Omega_\varepsilon}\operatorname{div}\left(\left(q_0+\frac{q_\ell-q_0}{\ell}x_1\right)\mathbf{u}_\varepsilon\right)\right|\le\ C\varepsilon^2\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\ \left|\int_{\Omega_\varepsilon}\operatorname{rot}\mathbf{w}_\varepsilon\mathbf{u}_\varepsilon\right|&=\left|\int_{\Omega_\varepsilon}\mathbf{w}_\varepsilon\operatorname{rot}\mathbf{u}_\varepsilon\right|\le C\varepsilon\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.\end{aligned} \tag{4.11}$$

From the last assertion, in view of (4.8), we conclude that

$$\left|\int_{\Omega_\varepsilon} \operatorname{rot} \mathbf{w}_\varepsilon \mathbf{u}_\varepsilon\right| \leq C\varepsilon^2 \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + C\varepsilon^3 \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{4.12}$$

Finally, as in (4.7) we get

$$\left|\int_{\Omega_\varepsilon} \mathbf{f}\mathbf{u}_\varepsilon\right| \leq C\varepsilon^2 \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{4.13}$$

Inserting the obtained estimates (4.10)–(4.13) in (4.9) yields

$$\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq C\sqrt{\varepsilon}\ \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^3 + C\varepsilon^3 \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{4.14}$$

Due to pressure boundary condition, the velocity $\mathbf{u}_\varepsilon$ is unique in the ball B_{R_0} with radius R_0 (see the discussion at the end of Section 2.2.). For that reason it is sufficient to choose ε such that $CR_0\sqrt{\varepsilon} < 1/2$ in order to deduce $(4.3)_1$ from (4.14). The estimate $(4.3)_2$ follows immediately from (4.8). □

The main result of this section can be stated as follows.

Theorem 4.3. *Let $\mathbf{w}_\varepsilon^1$, $\mathbf{u}_\varepsilon^1$, and q_ε be defined by* (3.35), (3.36), *and* (3.37), *respectively. Then the following estimates hold:*

$$|\Omega_\varepsilon|^{-1/2} \left\|\mathbf{w}_\varepsilon - \mathbf{w}_\varepsilon^1\right\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}\left(\varepsilon^4\right), \tag{4.15}$$

$$|\Omega_\varepsilon|^{-1/2} \left\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1\right\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}\left(\varepsilon^{7/2}\right), \tag{4.16}$$

$$|\Omega_\varepsilon|^{-1/2} \|p_\varepsilon - q_\varepsilon\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}\left(\varepsilon^{3/2}\right). \tag{4.17}$$

Remark 4.4. Since our domain Ω_ε is shrinking, the convergence in the norm $\|\cdot\|_{L^2(\Omega_\varepsilon)}$ would be worthless. Indeed, any L^∞-bounded function would converge to zero in such norm. Therefore, we express the error estimates in the rescaled norm $|\Omega_\varepsilon|^{-1/2}\ \|\cdot\|_{L^2(\Omega_\varepsilon)}$, where $|\Omega_\varepsilon| = \mathcal{O}(\varepsilon^2)$ stands for the Lebesgue measure of Ω_ε.

Proof. The function $\mathbf{w}_\varepsilon^1$ satisfies the following equation:

$$-\alpha\Delta \mathbf{w}_\varepsilon^1 + \left(\mathbf{u}_\varepsilon^1\nabla\right)\mathbf{w}_\varepsilon^1 - \beta\nabla \operatorname{div} \mathbf{w}_\varepsilon^1 + \gamma \mathbf{w}_\varepsilon^1 = a \operatorname{rot} \mathbf{u}_\varepsilon^0 + \mathbf{g} + \mathbf{h}_\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{4.18}$$

where $\|\mathbf{h}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2)$, that is, $\|\mathbf{h}_\varepsilon\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^3)$. Now, we introduce

$$\mathbf{s}_\varepsilon = \mathbf{w}_\varepsilon - \mathbf{w}_\varepsilon^1 \tag{4.19}$$

as the difference between the original solution and our asymptotic approximation. Subtracting the equations (2.7) and (4.18) gives

$$-\alpha\Delta\mathbf{s}_\varepsilon + (\mathbf{u}_\varepsilon\nabla)\mathbf{s}_\varepsilon + \left(\left(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1\right)\nabla\right)\mathbf{w}_\varepsilon^1 - \beta\nabla\operatorname{div}\mathbf{s}_\varepsilon + \gamma\mathbf{s}_\varepsilon = a\operatorname{rot}\left(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^0\right) - \mathbf{h}_\varepsilon \quad \text{in } \Omega_\varepsilon. \tag{4.20}$$

Multiplying the above equation by $\mathbf{s}_\varepsilon$ and integrating over Ω_ε lead to

$$\begin{aligned}\alpha\int_{\Omega_\varepsilon}|\nabla\mathbf{s}_\varepsilon|^2 + \int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\nabla)\mathbf{s}_\varepsilon\mathbf{s}_\varepsilon + \beta\int_{\Omega_\varepsilon}(\operatorname{div}\mathbf{s}_\varepsilon)^2 + \gamma\int_{\Omega_\varepsilon}|\mathbf{s}_\varepsilon|^2 \\ = a\int_{\Omega_\varepsilon}\operatorname{rot}\left(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^0\right)\mathbf{s}_\varepsilon - \int_{\Omega_\varepsilon}\left(\left(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1\right)\nabla\right)\mathbf{w}_\varepsilon^1\mathbf{s}_\varepsilon - \int_{\Omega_\varepsilon}\mathbf{h}_\varepsilon\mathbf{s}_\varepsilon.\end{aligned} \tag{4.21}$$

As in (4.5), we have $\int_{\Omega_\varepsilon}(u_\varepsilon\nabla)\mathbf{s}_\varepsilon\mathbf{s}_\varepsilon = 0$. Now, we carefully estimate each term on the right-hand side of (4.21)

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}\operatorname{rot}\left(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^0\right)\mathbf{s}_\varepsilon\right| &\le C\varepsilon\left\|\nabla(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^0)\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{s}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\le C\varepsilon\left\{\left\|\nabla\left(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1\right)\right\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla\left(\mathbf{u}_\varepsilon^1 - \mathbf{u}_\varepsilon^0\right)\right\|_{L^2(\Omega_\varepsilon)}\right\}\|\nabla\mathbf{s}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\le C\varepsilon\left\|\nabla(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1)\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{s}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^4\|\nabla\mathbf{s}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\ \left|\int_{\Omega_\varepsilon}\left(\left(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1\right)\nabla\right)\mathbf{w}_\varepsilon^1\mathbf{s}_\varepsilon\right| &\le \left\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1\right\|_{L^4(\Omega_\varepsilon)}\left\|\nabla\mathbf{w}_\varepsilon^1\right\|_{L^2(\Omega_\varepsilon)}\|\mathbf{s}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \\ &\le C\varepsilon^{5/2}\left\|\nabla(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1)\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{s}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\ \left|\int_{\Omega_\varepsilon}\mathbf{h}_\varepsilon\mathbf{s}_\varepsilon\right| &\le \|\mathbf{h}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\mathbf{s}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^4\|\nabla\mathbf{s}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.\end{aligned} \tag{4.22}$$

Taking into account the obtained estimates (4.22), from (4.21), we obtain

$$\|\nabla\mathbf{s}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon\left\|\nabla(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^1)\right\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^4. \tag{4.23}$$

The problem satisfied by $(\mathbf{u}_\varepsilon^1, q_\varepsilon)$ is the following:

$$\begin{aligned}-\mu\Delta\mathbf{u}_\varepsilon^1 + \left(\mathbf{u}_\varepsilon^1\nabla\right)\mathbf{u}_\varepsilon^1 + \nabla q_\varepsilon = a\operatorname{rot}\mathbf{w}_\varepsilon^0 + \mathbf{f} + \mathbf{E}_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div}\mathbf{u}_\varepsilon^1 = \pi_\varepsilon \quad \text{in } \Omega_\varepsilon,\end{aligned} \tag{4.24}$$

where $\mathbf{w}_\varepsilon^0(x) = \varepsilon^2\mathbf{W}_0(x_1, x'/\varepsilon)$ and $\|\mathbf{E}_\varepsilon\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^3)$, $\|\pi_\varepsilon\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^4)$. Before proceeding, it is important to notice that the norm of π_ε is not small enough to obtain satisfactory error estimate. Therefore, we need to construct the divergence correction. Since $\pi_\varepsilon = \varepsilon^3(\partial U_1^1/\partial x_1)$, we define $\Psi(x_1, y')$ as the solution of the problem

$$\operatorname{div}_{y'}\Psi = \frac{\partial \Psi_2}{\partial y_2} + \frac{\partial \Psi_3}{\partial y_3} = \frac{\partial U_1^1}{\partial x_1}(x_1, y') \quad \text{in } \Omega, \qquad \Psi = 0 \quad \text{on } \Gamma, \tag{4.25}$$

(here x_1 is treated only as a parameter). Taking into account (3.26), by a simple integration one can easily verify that $\int_B (\partial U_1^1/\partial x_1)(x_1, y')dy' = 0$, $0 < x_1 < l$ implying that such Ψ exists. Now we define our divergence correction as

$$\Psi_\varepsilon(x) = \varepsilon^4 \sum_{i=2}^{3} \Psi_i\left(x_1, \frac{x'}{\varepsilon}\right)\mathbf{e}_i \tag{4.26}$$

and put

$$\mathbf{v}_\varepsilon^1 = \mathbf{u}_\varepsilon^{\mathbf{1}} - \Psi_\varepsilon. \tag{4.27}$$

Such $\mathbf{v}_\varepsilon^0$ is divergence-free. Moreover, Ψ_ε is chosen such that it keeps the estimate for $\mathbf{E}_\varepsilon$, that is, $\|\widetilde{\mathbf{E}}_\varepsilon\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^3)$. Denoting

$$\mathbf{R}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon^1, \qquad r_\varepsilon = p_\varepsilon - q_\varepsilon, \tag{4.28}$$

we have

$$-\mu\Delta\mathbf{R}_\varepsilon + (\mathbf{u}_\varepsilon\nabla)\mathbf{R}_\varepsilon + (\mathbf{R}_\varepsilon\nabla)\mathbf{v}_\varepsilon^1 + \nabla r_\varepsilon = a \ \operatorname{rot}\left(\mathbf{w}_\varepsilon - \mathbf{w}_\varepsilon^0\right) - \widetilde{\mathbf{E}}_\varepsilon \quad \text{in } \Omega_\varepsilon, \qquad \operatorname{div}\mathbf{R}_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \qquad \mathbf{R}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon. \tag{4.29}$$

Now we introduce $\mathbf{d}_\varepsilon$ as the solution of the problem

$$\operatorname{div}\mathbf{d}_\varepsilon = r_\varepsilon \quad \text{in } \Omega_\varepsilon, \qquad \mathbf{d}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{4.30}$$

If we suppose $\int_{\Omega_\varepsilon} r_\varepsilon = 0$, such problem has at least one solution which satisfies

$$\|\nabla\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{4.31}$$

(see, e.g., Lemma 9 from [11]) Multiplying $(4.29)_1$ by $\mathbf{d}_\varepsilon$ and integrating over Ω_ε, we obtain

$$\begin{aligned}\|r_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \mu\int_{\Omega_\varepsilon}\nabla\mathbf{R}_\varepsilon\nabla\mathbf{d}_\varepsilon + \int_{\Omega_\varepsilon}\Big((\mathbf{u}_\varepsilon\nabla)\mathbf{R}_\varepsilon + (\mathbf{R}_\varepsilon\nabla)\mathbf{v}^1_\varepsilon\Big)\mathbf{d}_\varepsilon \\ - a\int_{\Omega_\varepsilon}\operatorname{rot}\Big(\mathbf{w}_\varepsilon - \mathbf{w}^0_\varepsilon\Big)d_\varepsilon + \int_{\Omega_\varepsilon}\widetilde{\mathbf{E}}_\varepsilon\mathbf{d}_\varepsilon.\end{aligned} \tag{4.32}$$

We estimate the terms on the right-hand side in (4.32) using a priori estimates and Lemma 4.1,

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon}\nabla\mathbf{R}_\varepsilon\nabla\mathbf{d}_\varepsilon\right| &\le \|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\varepsilon}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
\left|\int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\nabla)\mathbf{R}_\varepsilon\mathbf{d}_\varepsilon\right| &\le \|\mathbf{u}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\mathbf{d}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \le C\varepsilon^{5/2}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
\left|\int_{\Omega_\varepsilon}(\mathbf{R}_\varepsilon\nabla)\mathbf{v}^1_\varepsilon\mathbf{d}_\varepsilon\right| &\le \|\mathbf{R}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\left\|\nabla\mathbf{v}^1_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\mathbf{d}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \le C\varepsilon^{3/2}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
\left|\int_{\Omega_\varepsilon}\widetilde{\mathbf{E}}_\varepsilon\ \mathbf{d}_\varepsilon\right| &\le \left\|\widetilde{\mathbf{E}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^3\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
\left|\int_{\Omega_\varepsilon}\operatorname{rot}\Big(\mathbf{w}_\varepsilon - \mathbf{w}^0_\varepsilon\Big)\mathbf{d}_\varepsilon\right| &\le C\varepsilon\left\|\nabla\left(\mathbf{w}_\varepsilon - \mathbf{w}^0_\varepsilon\right)\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon\Big\{\left\|\nabla\left(\mathbf{w}_\varepsilon - \mathbf{w}^1_\varepsilon\right)\right\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla\left(\mathbf{w}^1_\varepsilon - \mathbf{w}^0_\varepsilon\right)\right\|_{L^2(\Omega_\varepsilon)}\Big\}\|\nabla\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\le C\|\nabla s_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{5/2}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{5/2}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)}.
\end{aligned} \tag{4.33}$$

Applying (4.33) into (4.32), we get

$$\|r_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\varepsilon}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{5/2}. \tag{4.34}$$

Now, we multiply the equation $(4.29)_1$ by $\mathbf{R}_\varepsilon$ and integrate over Ω_ε to obtain

$$\mu\|\nabla\mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \int_{\Omega_\varepsilon}\Big((\mathbf{u}_\varepsilon\nabla)\mathbf{R}_\varepsilon + (\mathbf{R}_\varepsilon\nabla)\mathbf{v}^1_\varepsilon\Big)\mathbf{R}_\varepsilon = a\int_{\Omega_\varepsilon}\operatorname{rot}\Big(\mathbf{w}_\varepsilon - \mathbf{w}^0_\varepsilon\Big)\mathbf{R}_\varepsilon - \int_{\Omega_\varepsilon}\widetilde{\mathbf{E}}_\varepsilon\mathbf{R}_\varepsilon. \tag{4.35}$$

Analogously, we have

$$\begin{gathered}
\left|\int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\nabla)\mathbf{R}_\varepsilon\mathbf{R}_\varepsilon\right| \le \|\mathbf{u}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\mathbf{R}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \le C\varepsilon^{7/2}\|\nabla\mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}, \\
\left|\int_{\Omega_\varepsilon}(\mathbf{R}_\varepsilon\nabla)\mathbf{v}^1_\varepsilon\mathbf{R}_\varepsilon\right| \le \|\mathbf{R}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\left\|\nabla\mathbf{v}^1_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\mathbf{R}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \le C\varepsilon^{5/2}\ \|\nabla\mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}, \\
\left|\int_{\Omega_\varepsilon}\widetilde{\mathbf{E}}_\varepsilon\mathbf{R}_\varepsilon\right| \le \left\|\widetilde{\mathbf{E}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^4\ \|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
\left|\int_{\Omega_\varepsilon}\operatorname{rot}\left(\mathbf{w}_\varepsilon-\mathbf{w}^0_\varepsilon\right)\mathbf{R}_\varepsilon\right| \le C\varepsilon^2\|\nabla\mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^{7/2}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)},
\end{gathered} \tag{4.36}$$

implying

$$\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{7/2}. \tag{4.37}$$

The estimates for the velocity (4.16) and pressure (4.17) now follow directly from Poincaré's inequality (4.1) and (4.34). The estimate (4.15) for the microrotation then follows from (4.23) and the theorem is completely proved. □

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