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Research Article

A Fully Discrete Galerkin Method for a Nonlinear Space-Fractional Diffusion Equation

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The spatial transport process in fractal media is generally anomalous. The space-fractional advection-diffusion equation can be used to characterize such a process. In this paper, a fully discrete scheme is given for a type of nonlinear space-fractional anomalous advection-diffusion equation. In the spatial direction, we use the finite element method, and in the temporal direction, we use the modified Crank-Nicolson approximation. Here the fractional derivative indicates the Caputo derivative. The error estimate for the fully discrete scheme is derived. And the numerical examples are also included which are in line with the theoretical analysis.

1. Introduction

The normal diffusive motion is modeled to describe the standard Brownian motion. The relation between the flow and the divergence of the particle displacement represents

$$J(x,t) = -a\frac{\partial c}{\partial x} + bc, \tag{1.1}$$

where J is the diffusive flow. Inserting the above equation into the equation of mass conservation

$$\frac{\partial J}{\partial x} = -\frac{\partial c}{\partial t}, \tag{1.2}$$

we obtain the standard convection-diffusion equation. From the viewpoint of physics, it means that during the method of time random walkers, the overall particle displacement up to time t can be represented as a sum of independent random steps, in the case that both the mean-squared displacement per step and the mean time needed to perform a step are finite. The measured variance growth in the direction of flow of tracer plumes is typically at a Fickian rate, $\langle (c-\overline{c})^2 \rangle \sim t$.

The transport process in fractal media cannot be described with the normal diffusion. The process is nonlocal and it does not follow the classical Fickian law. It depicts a particle in spreading tracer cloud which has a standard deviation, and which grows like $t^{2\alpha}$ for some $0 < \alpha < 1$, excluding the Fickian case $\alpha = 1/2$. The description of anomalous diffusion means that the measure variance growth in the direction of flow has a deviation from the Fickian case, it follows the super-Fickian rate $\langle (c-\overline{c})^2 \rangle \sim t^{2\alpha}$ when $\alpha > 1/2$, or does the subdiffusion rate $\langle (c-\overline{c})^2 \rangle \sim t^{2\alpha}$ if $0 < \alpha < 1/2$. With the help of the continuous time random walk and the Fourier transform, the governing equation with space fractional derivative can be derived as follows

$$\frac{\partial u}{\partial t} = D\left(a(u)\,{}_aD_x^{\beta}u\right) + b(u)Du + f(x,t,u), \quad 0 < \beta < 1, \tag{1.3}$$

where D denotes integer derivative respect to x, and D^{β} is fractional derivative. There are some authors studying the spacial anomalous diffusion equation in theoretical analysis and numerical simulations [1–10]. Now the fractional anomalous diffusion becomes a hot topic because of its widely applications in the evolution of various dynamical systems under the influence of stochastic forces. For example, it is a well-suited tool for the description of anomalous transport processes in both absence and presence of external velocities or force fields. Since the groundwater velocities span many orders of magnitude and give rise to diffusion-like dispersion (a term that combines molecular diffusion and hydrodynamic dispersion), the fractional diffusion is an important process in hydrogeology. It can be used to describe the systems with reactions and diffusions across a wide range of applications including nerve cell signaling, animal coat patterns, population dispersal, and chemical waves. In general, fractional anomalous diffusions have numerous applications in statistical physics, biophysics, chemistry, hydrogeology, and biology [4, 11–20].

In this paper, we mainly study one kind of typical nonlinear space-fractional partial differential equations by using the finite element method, which reads in the following form:

$$\begin{gathered} \frac{\partial u}{\partial t} = D\left(a(u)\,{}_aD_x^{\beta}u\right) + b(u)Du + f(x,t,u), \quad x \in \Omega,\ t \in (0,T], \\ u|_{t=0} = \varphi(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = g, \quad t \in (0,T], \end{gathered} \tag{1.4}$$

where Ω is a spacial domain with boundary $\partial\Omega$, D^{β} is the βth ($0 < \beta < 1$) order fractional derivative with respect to the space variable x in the Caputo sense (which will be introduced later on), a, b, f are functions of x, t, u, φ and g are known functions which satisfy the conditions requested by the theorem of error estimations.

The rest of this paper is constructed as follows. In Section 2 the fractional integral, fractional derivative, and the fractional derivative spaces are introduced. The error estimates of the finite element approximation for (1.4) are studied in Section 3, and in Section 4, numerical examples are taken to verify the theoretical results derived in Section 3.

2. Fractional Derivative Space

In this section, we firstly introduce the fractional integral (or Riemann-Liouville integral), the Caputo fractional derivative, and their corresponding fractional derivative space.

Definition 2.1. The αth order left and right Riemann-Liouville integrals of function $u(x)$ are defined as follows

$$
\begin{aligned}
{}_aI_x^{\alpha}u(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}u(s)ds, \\
{}_xI_b^{\alpha}u(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}u(s)ds,
\end{aligned}
\tag{2.1}
$$

where $\alpha > 0$, and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. The αth order Caputo derivative of function $u(x)$ is defined as,

$$
\begin{aligned}
{}_aD_x^{\alpha}u(x) &= {}_aI_x^{n-\alpha}\frac{d^n u(x)}{dx^n}, \quad n-1<\alpha<n\in Z^+, \\
{}_xD_b^{\alpha}u(x) &= (-1)^n {}_xI_b^{n-\alpha}\frac{d^n u(x)}{dx^n}, \quad n-1<\alpha<n\in Z^+.
\end{aligned}
\tag{2.2}
$$

The αth order Riemann-Liouville derivative of function $u(x)$ is defined by changing the order of integration and differentiation.

Lemma 2.3 (see [8]). *If $u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0$, then the Caputo fractional derivative is equal to the Riemann-Liouville derivative.*

Definition 2.4. The fractional derivative space $J^{\alpha}(\Omega)$ is defined as follows:

$$
J^{\alpha}(\Omega) = \left\{u \in L^2(\Omega) : {}_aD_x^{\alpha}u \in L^2(\Omega),\ n-1 \le \alpha < n\right\},
\tag{2.3}
$$

endowed with the seminorm

$$
|u|_{J^{\alpha}} = \| {}_aD_x^{\alpha}u\|_{L^2(\Omega)},
\tag{2.4}
$$

and the norm

$$
\|u\|_{J^{\alpha}} = \left(|u|_{J^{\alpha}}^2 + \sum_{k\le[\alpha]} \left\|D^k u\right\|^2\right)^{1/2}.
\tag{2.5}
$$

Let $J_0^\alpha(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the above norm and seminorm.

Definition 2.5. Define the seminorm

$$|u|_{H^\alpha} = \left\| |iw|^\alpha F(u) \right\|_{L^2(\Omega)}, \tag{2.6}$$

and the norm

$$\|u\|_{H^\alpha} = \left(|u|_{H^\alpha}^2 + \sum_{k \le [\alpha]} \left\| D^k u \right\|^2 \right)^{1/2}, \tag{2.7}$$

where i is the imaginary unit, and F is the Fourier transform, and which can define another fractional derivative space $H^\alpha(\Omega)$.

Let $H_0^\alpha(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm and seminorm.

Definition 2.6. The fractional space $J_s^\alpha(\Omega)$ is defined below

$$J_s^\alpha(\Omega) = \left\{ u \in L^2(\Omega) \ : \ {}_aD_x^\alpha u \in L^2(\Omega), \ {}_xD_b^\alpha u \in L^2(\Omega), \ n-1 \le \alpha < n \right\}, \tag{2.8}$$

endowed with the seminorm

$$|u|_{J_s^\alpha} = \left| \left({}_aD_x^\alpha u, {}_xD_b^\alpha u \right)^{1/2} \right|_{L^2(\Omega)}, \tag{2.9}$$

and the norm

$$\|u\|_{J_s^\alpha} = \left(\sum_{k \le [\alpha]} \left\| D^k u \right\|^2 + |u|_{J_s^\alpha}^2 \right)^{1/2}. \tag{2.10}$$

Theorem 2.7 (see [3, 6]). *J_s^α, J^α, and H^α are equal with equivalent seminorm and norm.*

The following are some useful results.

Lemma 2.8 (see [3]). *For $u \in J_0^\alpha(\Omega)$, $0 < \beta < \alpha$, then*

$${}_aD_x^\alpha u(x) = {}_aD_x^{\alpha-\beta} \, {}_aD_x^\beta u. \tag{2.11}$$

Lemma 2.9 (see [2]). *For $u \in H_0^\alpha(\Omega)$, one has*

$$\|u\|_{L^2(\Omega)} \le c|u|_{H_0^\alpha}. \tag{2.12}$$

For $0 < \beta < \alpha$,

$$|u|_{H_0^\beta(\Omega)} \le c|u|_{H_0^\alpha}. \tag{2.13}$$

Since J_s^α, J^α, and H^α are equal with equivalent seminorm and norm, the norms with each space which will be used following are without distinction, and the notations are used seminorm $|\cdot|_\alpha$ and norm $\|\cdot\|_\alpha$.

3. Finite Element Approximation

Let $\Omega = [a,b]$, and $0 \le \beta < 1$. Define $\alpha = (1+\beta)/2$. In this section, we will formulate a fully discrete Galerkin finite element method for a type of nonlinear anomalous diffusion equation as follows.

Problem 1 (Nonlinear spacial anomalous diffusion equation). We consider equations of the form

$$\begin{gathered}\frac{\partial u}{\partial t} = D\left(a(u)\,{}_aD_x^\beta u\right) + b(u)Du + f(x,t,u), \quad (x,t) \in \Omega \times (0,T], \\ u(x,t) = \phi(x,t), \quad x \in \partial\Omega \times (0,T], \\ u(x,0) = g(x), \quad x \in \overline{\Omega}.\end{gathered} \tag{3.1}$$

We always assume that

$$0 < m < a(u) < M, \quad 0 < m < b(u) < M, \quad 0 < m < f(u) < M. \tag{3.2}$$

The algorithm and analysis in this paper are applicable for a large class of linear and nonlinear functions (including polynomials and exponentials) in the unknown variables. Throughout the paper, we assume the following mild Lipschitz continuity conditions on a, b, and f: there exist positive constants L and c such that for $x \in \Omega, t \in (0,T]$, and $s,t \in R$,

$$|a(x,t,s) - a(x,t,r)| \le L|s-r|, \tag{3.3}$$

$$|b(x,t,s) - b(x,t,r)| \le L|s-r|, \tag{3.4}$$

$$\left|f(x,t,s) - f(x,t,r)\right| \le L|s-r|. \tag{3.5}$$

In order to derive a variational form of Problem 1, we suppose that u is a sufficiently smooth solution of Problem 1. Multiplying an arbitrary $v \in H_0^\alpha(\Omega)$ in both sides yields

$$\int_\Omega \frac{\partial u}{\partial t} v\,dx = \int_\Omega D\left(a(u)\,{}_aD_x^\beta u\right) v\,dx + \int_\Omega b(u)Duv\,dx + \int_\Omega f(x,t,u)v\,dx. \tag{3.6}$$

Rewriting the above expression yields

$$\int_\Omega \frac{\partial u}{\partial t} v\,dx + \int_\Omega a(u)\,{}_aD_x^\beta u Dv\,dx - \int_\Omega b(u)Duv\,dx = \int_\Omega f(x,t,u)v\,dx. \tag{3.7}$$

We define the associated bilinear form $A : J_0^{\alpha}(\Omega) \times J_0^{\alpha}(\Omega) \rightarrow R$ as

$$A(u,v) = \left(a(u)\,{}_aD_x^{\beta}u, Dv\right) - (b(u)Du, v), \tag{3.8}$$

where $(\cdot,\cdot)$ denotes the inner product on $L^2(\Omega)$ and $J_0^{\alpha}(\Omega)$.

For given $f \in J^{-\alpha}(\Omega)$, we define the associated function $F : J_0^{\alpha}(\Omega) \rightarrow R$ as

$$F(v) = \langle f, v\rangle. \tag{3.9}$$

Definition 3.1. A function $u \in J_0^{\alpha}(\Omega)$ is a variational solution of Problem 1 provided that

$$\left(\frac{\partial u}{\partial t}, v\right) + A(u,v) = F(v), \quad \forall v \in J_0^{\alpha}(\Omega). \tag{3.10}$$

Now we are ready to describe a fully discrete Galerkin finite element method to solve nonlinear Problem 1. In our new scheme, the finite element trial and test spaces for Problem 1 are chosen to be same.

For a positive integer N, let $\prod^t = \{t_n\}_{n=0}^{N}$ be a uniform partition of the time interval $(0,T]$ such that $t_n = n\tau$, where $\tau = T/N$, and let $t_{n-1/2} = t_n - \tau/2$. Throughout the paper, we use the following notation for a function ϕ:

$$\phi^n = \phi(t_n), \overline{\partial}_t\phi^n = \frac{\phi^n - \phi^{n-1}}{\tau}, \qquad \overline{\phi}^n = \frac{\phi^n + \phi^{n-1}}{2}, \qquad \tilde{\phi}^n = \frac{3\phi^{n-1} - \phi^{n-2}}{2}. \tag{3.11}$$

Let $\mathcal{K}_h = \{K\}$ be a partition of spatial domain Ω. Define h_k as the diameter of the element K and $h = \max_{K\in\mathcal{K}_h} h_K$. And let S_h be a finite element space

$$S_h = \left\{v \in H_0^{\alpha}(\Omega) : v|_K \in P_{r-1}(K), K \in \mathcal{K}_h\right\}, \tag{3.12}$$

where $P_{r-1}(K)$ is the set of polynomials of degree $r-1$ on a given domain K. And the functions in S_h are continuous on Ω. Our fully discrete quadrature scheme to solve Problem 1 is to find u_h: for $v \in S_h$ such that

$$\left(\overline{\partial}_t u_h^n, v\right) + \left(a(\tilde{u}_h^n)\,{}_aD_x^{\beta}\overline{u}_h^n, Dv\right) - \left(b(\tilde{u}_h^n)D\overline{u}_h^n, v\right) = \langle f(\tilde{u}_h^n), v\rangle. \tag{3.13}$$

The linear systems in the above equation requires selecting the value of u_h^0 and u_h^1. Given u_h^0 depending on the initial data $g(x)$, we select u_h^1 by solving the following predictor-corrector linear systems:

$$\begin{gathered}\left(\frac{u_h^{1,0}-u_h^0}{\tau},v\right)+\left(a\left(u_h^0\right)\,{}_aD_x^{\beta}\frac{u_h^{1,0}+u_h^0}{2},Dv\right)-\left(b\left(u_h^0\right)D\frac{u_h^{1,0}+u_h^0}{2},v\right)=\left\langle f\left(u_h^0\right),v\right\rangle,\\ \left(\frac{u_h^{1}-u_h^0}{\tau},v\right)+\left(a\left(\frac{u_h^{1,0}+u_h^0}{2}\right){}_aD_x^{\beta}\frac{u_h^{1}+u_h^0}{2},Dv\right)-\left(b\left(\frac{u_h^{1,0}+u_h^0}{2}\right)D\frac{u_h^{1}+u_h^0}{2},v\right)\\ =\left\langle f\left(\frac{u_h^{1,0}+u_h^0}{2}\right),v\right\rangle.\end{gathered}\tag{3.14}$$

Lemma 3.2. *For $u,v,w\in J_{s,0}^{\alpha}(\Omega)$, $0<m\le a(u)\le M$, $\alpha=(1+\beta)/2$, there exist constants γ_1,γ_2 such that*

$$\left(a(u)_aD_x^{\beta}u,Dv\right)\le\gamma_1\|u\|_{\alpha}\cdot\|v\|_{\alpha},\qquad\left(a(w)_aD_x^{\beta}v,Dv\right)\ge\gamma_2\|v\|_{\alpha}^2.\tag{3.15}$$

Proof. With the assumption of $a(u)$ in (3.3) and the property of dual space

$$\begin{aligned}\left(a(w)_aD_x^{\beta}u,Dv\right)&\le\left\|a(w)_aD_x^{\beta}u\right\|_{1-\alpha}\cdot\|Dv\|_{-(1-\alpha)}\\ &\le Mc\|u\|_{1-\alpha+\beta}\cdot\|v\|_{-(1-\alpha)+1}\le\gamma_1\|u\|_{\alpha}\cdot\|v\|_{\alpha},\\ \left(a(w)_aD_x^{\beta}v,Dv\right)&=-\left(Da(w)_aD_x^{\beta}v,v\right)\\ &=-\left({}_aD_x^{(1-\beta)/2}a(w)_aD_x^{\beta}v,\ {}_xD_b^{(1+\beta)/2}v\right)\ge m|v|_{J_s^{\alpha}}^2\ge\gamma_2\|v\|_{\alpha}^2.\end{aligned}\tag{3.16}$$

□

Lemma 3.3 (see [2]). *For $\Omega\subset R^n,\alpha>n/4,\ v,w\in H_0^{\alpha}(\Omega),\varepsilon>0$, one has*

$$(vb(w),\nabla v)\le c_0\frac{(q\varepsilon)^{-p/q}}{p}\|\nabla b(w)\|^p\cdot\|v\|^2+\varepsilon\|v\|_{\alpha}^2,\tag{3.17}$$

where $p=4\alpha/(4\alpha-n),q=4\alpha/n$.

Theorem 3.4. *Let u_h^n be bounded, then for a sufficiently small step τ, there exists a unique solution $u_h^n\in S_h$ satisfying scheme* (3.13).

Proof. As scheme represents a finite system of problem, the continuity and coercivity of $(\overline{u}_h^n, \overline{\omega}_h^n)/\tau + A(\overline{u}_h^n, \overline{\omega}_h^n)$ is the sufficient and essential condition for the existence and uniqueness of u_h^n. Let $v = \overline{u}_h^n,\ w = \overline{\omega}_h^n$, then

$$\begin{aligned}\frac{(v,v)}{\tau} + A(v,v) &= \frac{(v,v)}{\tau} + \left(a(w)\,{}_aD_x^{\beta}v, Dv\right) - (b(w)Dv, v)\\ &\geq \frac{\|v\|^2}{\tau} + \gamma_2\|v\|_{\alpha} - c_0\|Db(w)\|^2\|v\|^2 - \varepsilon\|v\|_{\alpha}^2\\ &= (\gamma_2 - \varepsilon)\|v\|_{\alpha}^2 + \left(\tau^{-1} - c_0\|Db(w)\|^2\right)\|v\|^2\\ &\geq c\|v\|_{\alpha}^2.\end{aligned} \tag{3.18}$$

For the chosen sufficiently small τ, the above inequality holds.

$$\begin{aligned}\frac{(v,w)}{\tau} + A(v,w) &= \frac{(v,w)}{\tau} + \left(a(u)\,{}_aD_x^{\beta}v, Dw\right) + (Db(u)v, Dw)\\ &\leq \frac{\|u\|\cdot\|w\|}{\tau} + \gamma_1\|v\|_{\alpha}\|w\|_{\alpha} + \|v\|\cdot\|D(b(u)w)\|\\ &\leq \frac{\|u\|\cdot\|w\|}{\tau} + \gamma_1\|v\|_{\alpha}\|w\|_{\alpha} + M\frac{\|v\|\cdot\|w\|}{h}\\ &\leq c\|v\|_{\alpha}\|w\|_{\alpha}.\end{aligned} \tag{3.19}$$

Hence, the scheme (3.13) is uniquely solvable for u_h^n.

Let $\rho^n = P_hu^n - u^n$, and $\theta^n = u_h^n - P_hu^n$, then

$$u_h^n - u^n = u_h^n - P_hu^n + P_hu^n - u^n = \theta^n + \rho^n, \tag{3.20}$$

where P_hu^n is a Rits-Galerkin projection operator defined as follows:

$$\begin{aligned}\left(a(w)\,{}_aD_x^{\beta}(u^n - P_hu^n), Dv\right) &= 0,\\ \left(a(u_0)\,{}_aD_x^{\beta}(u^n - P_hu^n), Dv\right) &= 0.\end{aligned} \tag{3.21}$$

□

Lemma 3.5. *Let $a(u)$, $b(u)$ be smooth functions on Ω, $0 < m \leq a(u)$, $b(u) \leq M$, and P_hu^n is defined as above, then*

$$\begin{aligned}\|\,{}_aD_x^{\alpha}(u^n - P_hu^n)\| &\leq ch^{k+1-\alpha}\|u\|_{k+1},\\ \|(P_hu^n - u^n)\| &\leq ch^{k+1}\|u\|_{k+1}.\end{aligned} \tag{3.22}$$

Proof. Using the definition of $P_h u^n$, one gets

$$\begin{aligned} \|{}_aD_x^\alpha(P_h u^n - u^n)\|^2 &= |({}_aD_x^\alpha(P_h u^n - u^n), {}_aD_x^\alpha(P_h u^n - u^n))| \\ &\le c\|{}_aD_x^\alpha(P_h u^n - u^n)\| \cdot \|{}_aD_x^\alpha(\chi - u^n)\|, \end{aligned} \tag{3.23}$$

where $\chi \in S_h$. Utilizing the interpolation of $I_h u^n$ leads to

$$\|{}_aD_x^\alpha(P_h u^n - u^n)\| \le \inf_{\chi \in S_h} c\|\chi - u\|_\alpha \le c\|I_h u^n - u^n\|_\alpha \le ch^{k+1-\alpha}\|u\|_{k+1}. \tag{3.24}$$

Next we estimate $\|P_h u^n - u^n\|$. For all $\phi \in L^2(\Omega)$, w is the solution of the following equation:

$$\begin{aligned} -{}_aD_x^{2\alpha} w = \phi, \quad w \in \Omega, \\ w = 0, \quad w \in \partial\Omega. \end{aligned} \tag{3.25}$$

So we have

$$\|w\|_{2\alpha} \le \gamma_3\|\phi\|. \tag{3.26}$$

For all $\chi \in S_h$, with the help of approximation properties of S_h and the weak form, we can obtain

$$\begin{aligned} (P_h u^n - u^n, \phi) &= -\left(P_h u^n - u^n, {}_aD_x^{2\alpha} w\right) = -({}_xD_b^\alpha(P_h u^n - u^n), {}_aD_x^\alpha w) \\ &= -({}_xD_b^\alpha(P_h u^n - u^n), {}_aD_x^\alpha(w - \chi)) \le \|P_h u^n - u^n\|_\alpha \|w - \chi\|_\alpha \\ &\le \|P_h u^n - u^n\|_\alpha \inf_{\chi \in S_h} \|w - \chi\|_\alpha \\ &\le ch^{r-\alpha}\|u\|_r h^\alpha \|w\|_{2\alpha} = ch^r\|u\|_r\|\phi\|, \\ \|P_h u^n - u^n\| &= \sup_{0 \ne \phi \in L^2(\Omega)} \frac{(P_h u^n - u^n, \phi)}{\|\phi\|} \le ch^r\|u\|_r. \end{aligned} \tag{3.27}$$

□

Lemma 3.6 (see [21]). *Let T_h, $0 < h \le 1$, denote a quasiuniform family of subdivisions of a polyhedral domain $\Omega \subset R^d$. Let (K', P, N) be a reference finite element such that $P \subset W^{l,p}(K') \cap W^{m,q}(K')$ is a finite-dimensional space of functions on K', N is a basis for P', where $1 \le p \le \infty, 1 \le p \le \infty$, and $0 \le m \le l$. For $K \in T_h$, let (K, P_K, N_K) be the affine equivalent element, and $V_h = v : v$ is measurable and $v|_K \in P_K$, for all $K \in T_h$. Then there exists a constant $C = C(l, p, q)$ such that*

$$\left[\sum_{k \in T_h} \|v\|^2_{W^{l,p}(K)}\right]^{1/p} \le Ch^{m-l+\min(0, d/p-d/q)} \cdot \left[\sum_{k \in T_h} \|v\|^q_{W^{m,q}(K)}\right]^{1/q}. \tag{3.28}$$

The following Gronwall's lemma is useful for the error analysis later on.

Lemma 3.7 (see [2]). *Let $\Delta t, H$ and a_n, b_n, c_n, γ_n (for integer $n \geq 0$) be nonnegative numbers such that*

$$a_N + \Delta t\sum_{n=0}^{N} b_n \leq \Delta t\sum_{n=0}^{N}\gamma_n a_n + \Delta t\sum_{n=0}^{N} c_n + H, \tag{3.29}$$

for $N \geq 0$. Suppose that $\Delta t\gamma_n < 1$, for all n, and set $\sigma_n = (1-\Delta t\gamma_n)^{-1}$. Then

$$a_N + \Delta t\sum_{n=0}^{N} b_n \leq \exp\left(\Delta t\sum_{n=0}^{N}\sigma_n\gamma_n\right)\left\{\Delta t\sum_{n=0}^{N} c_n + H\right\}, \tag{3.30}$$

for $N \geq 0$.

The following norms are also used in the analysis:

$$\begin{aligned} |||v|||_{\infty,k} &= \max_{0\leq n\leq N}\|v^n\|_k, \\ |||v|||_{0,k} &= \left[\sum_{n=0}^{N}\tau\|v^n\|_k^2\right]^{1/2}. \end{aligned} \tag{3.31}$$

Theorem 3.8. *Assume that Problem 1 has a solution u satisfying $u_{tt}, u_{ttt} \in L^2(0,T,L^2(\Omega))$ with $u, u_t \in L^2(0,T,H^{k+1})$. If $\Delta t \leq ch$, then the finite element approximation is convergent to the solution of Problem 1 on the interval (0,T], as $\Delta t, h \to 0$. The approximation u_h also satisfies the following error estimates*

$$\begin{aligned} \|u-u_h\|_{0,\alpha} \leq C\Big(& h^{k+1}\|u_t\|_{0,k+1} + h^{k+1-\alpha}\|u\|_{0,k+1} + \tau^2\|u_{tt}\|_{0,0} \\ &+\tau h^{k+1-\alpha}\|u_{tt}\|_{0,k+1} + \tau^2\|u_{ttt}\|_{0,0}\Big), \end{aligned} \tag{3.32}$$

$$\begin{aligned} \|u-u_h\|_{\infty,0} \leq C\Big(& h^{k+1}\|u_t\|_{0,k+1} + h^{k+1-\alpha}\|u\|_{0,k+1} + \tau^2\|u_{ttt}\|_{0,0} \\ &+\tau h^{k+1-\alpha}\|u_{tt}\|_{0,k+1} + \tau^2\|u_{tt}\|_{0,0} + h^{k+1}\|u\|_{\infty,k+1}^2\Big). \end{aligned} \tag{3.33}$$

Proof. For $t = t_n - \tau/2 = t_{n-1/2}, n = 0, 1, \ldots, N$, find $u^{n-1/2}$ such that

$$\left(\partial_t u^{n-1/2}, v\right) + \left(a\left(u^{n-1/2}\right){}_aD_x u^{n-1/2}, Dv\right) - \left(b\left(u^{n-1/2}\right)Du^{n-1/2}, v\right) = \left\langle f\left(u^{n-1/2}\right), v\right\rangle. \tag{3.34}$$

Subtracting the above equation from the fully discrete scheme (3.13), and substituting $u_h^n - u^n = (u_h^n - P_h u^n) + (P_h u^n - u^n) = \theta^n + \rho^n$ into it, we obtain the following error formulation relating to θ^n and ρ^n:

$$\begin{aligned}
&\left(\overline{\partial}_t \theta^n, v\right) + \left(a(\tilde{u}_h^n)\ {}_aD_x^\beta \overline{\theta}^n, Dv\right) - \left(b(\tilde{u}_h^n) D\overline{\theta^n}, v\right) \\
&\quad = \left(a(\tilde{u}_h^n)\, {}_aD_x^\beta \overline{I_h u}^n, Dv\right) + \left(b(\tilde{u}_h^n)\, {}_aD_x^\beta \overline{I_h u}^n, v\right) + \left(\partial_t u^{n-1/2}, v\right) - \left(\overline{\partial}_t I_h u^n, v\right) \\
&\qquad + \left(a\left(u^{n-1/2}\right) {}_aD_x^\beta u^{n-1/2}, Dv\right) - \left(b\left(u^{n-1/2}\right) D u^{n-1/2}, v\right) + \left(f(\tilde{u}_h^n), v\right) - \left(f\left(u^{n-1/2}\right), v\right) \\
&\quad = -\left(a(\tilde{u}_h^n)\, {}_aD_x^\beta \overline{\rho}^n, Dv\right) + \left\{\left(a\left(u^{n-1/2}\right) {}_aD_x^\beta u^{n-1/2} - a(\tilde{u}_h^n)\, {}_aD_x^\beta \overline{u^n}, Dv\right)\right\} \\
&\qquad + \left(b(\tilde{u}_h^n) D\overline{\rho}^n, v\right) + \left\{\left(b(\tilde{u}_h^n) D\overline{u}^n - \left(b\left(u^{n-1/2}\right) D u^{n-1/2}, Dv\right)\right\}\right. \\
&\qquad + \left\{\left(f(\tilde{u}_h^n) - f\left(u^{n-1/2}\right), v\right)\right\} + \left\{\left(\partial_t u^{n-1/2} - \overline{\partial}_t I_h u^n, v\right)\right\} \\
&\quad = R_1(v) + R_2(v) + R_3(v) + R_4(v) + R_5(v) + R_6(v).
\end{aligned} \tag{3.35}$$

Setting $v = \overline{\theta^n}$, we obtain

$$\begin{aligned}
&\left(\overline{\partial}_t \theta^n, \overline{\theta^n}\right) + \left(a(\tilde{u}^n)_a D_x^\beta \overline{\theta}^n, D\overline{\theta}^n\right) - \left(b(\tilde{u}^n) D\overline{\theta}^n, \overline{\theta}^n\right) \\
&\quad = R_1\left(\overline{\theta}^n\right) + R_2\left(\overline{\theta}^n\right) + R_3\left(\overline{\theta}^n\right) + R_4\left(\overline{\theta}^n\right) + R_5\left(\overline{\theta}^n\right) + R_6\left(\overline{\theta}^n\right).
\end{aligned} \tag{3.36}$$

Note that

$$\left(\overline{\partial}_t \theta^n, \overline{\theta^n}\right) = \left(\frac{\theta^n - \theta^{n-1}}{\tau}, \frac{\theta^n + \theta^{n-1}}{2}\right) = \frac{1}{2\tau}\left(\|\theta^n\|^2 - \left\|\theta^{n-1}\right\|^2\right). \tag{3.37}$$

According to (3.2) and Lemma 3.2, we have

$$\left(a(\tilde{u}^n)_a D_x^\beta \overline{\theta}^n, D\overline{\theta}^n\right) \geq m\left|\overline{\theta}^n\right|_\alpha^2 \geq c\left(|\theta^n|_\alpha^2 + \left|\theta^{n-1}\right|_\alpha^2\right). \tag{3.38}$$

From Lemma 3.3, the following inequality can be derived:

$$\begin{aligned}
\left(b(\tilde{u}^n)\overline{\theta}^n, D\overline{\theta}^n\right) &\leq c_0 \varepsilon_2^{-c_1} \|Db(\tilde{u}^n)\|^{c^2} \left\|\overline{\theta}^n\right\|^2 + \varepsilon_3 \left\|\overline{\theta}^n\right\|_\alpha^2 \\
&= c_0 \varepsilon_2^{-c_1} \|Db(\tilde{u}^n)\|^{c^2} \left\|\frac{\theta^n + \theta^{n-1}}{2}\right\|^2 + \varepsilon_3 \left\|\frac{\theta^n + \theta^{n-1}}{2}\right\|_\alpha^2 \\
&\leq c_3 \varepsilon_2^{-c_1} \|Db(\tilde{u}^n)\|^{c^2} \left(\|\theta^n\|^2 + \left\|\theta^{n-1}\right\|^2\right) + c_4 \varepsilon_3 \left(\|\theta^n\|_\alpha^2 + \left\|\theta^{n-1}\right\|_\alpha^2\right).
\end{aligned} \tag{3.39}$$

Substituting (3.37)–(3.39) into (3.36) then multiplying (3.36) by 2τ, summing from $n = 1$ to N, we have

$$\begin{aligned}
&\|\theta^n\|^2 - \left\|\theta^2\right\|^2 + \tau\sum_{n=1}^{N}(2mc - 2c_4\varepsilon_3)\left(\|\theta^n\|_\alpha^2 + \left\|\theta^{n-1}\right\|_\alpha^2\right)\\
&\quad\le 2\tau\sum_{n=1}^{N}c_3\varepsilon_2^{-c_1}\left\|Db(\widetilde{u}_h^n)\right\|^{c_2}\left(\|\theta^n\|^2 + \left\|\theta^{n-1}\right\|^2\right)\\
&\qquad + 2\tau\sum_{n=3}^{N}\left[R_1\left(\overline{\theta}^n\right) + R_2\left(\overline{\theta}^n\right) + R_3\left(\overline{\theta}^n\right) + R_4\left(\overline{\theta}^n\right) + R_5\left(\overline{\theta}^n\right) + R_6\left(\overline{\theta}^n\right)\right].
\end{aligned}\tag{3.40}$$

We now estimate R_1 to R_6 in the right hand of (3.40),

$$\begin{aligned}
R_1\left(\overline{\theta}^n\right) &= \left({}_aD_x^{1-\alpha}\left(a(\widetilde{u}_h^n)\,{}_aD_x^{\beta}\overline{\rho}^n\right),\ {}_aD_x^{\alpha}\overline{\theta}^n\right)\\
&\le M\left({}_aD_x^{\alpha}\overline{\rho}^n,\ {}_aD_x^{\alpha}\overline{\theta}^n\right) \le M\left\|{}_aD_x^{\alpha}\overline{\rho}^n\right\|\left\|{}_aD_x^{\alpha}\overline{\theta}^n\right\|\\
&\le \varepsilon_4\left\|\overline{\theta}^n\right\|_\alpha^2 + \frac{c_5^2}{4\varepsilon_4}\left\|\overline{\rho}^n\right\|_\alpha^2\\
&= \frac{\varepsilon_4}{2}\left\|\rho^n + \rho^{n-1}\right\|_\alpha^2 + \frac{c_5^2}{16\varepsilon_4}\left\|\theta^n + \theta^{n-1}\right\|_\alpha^2\\
&\le \varepsilon_4 c_6\left(\|\theta^n\|_\alpha^2 + \left\|\theta^{n-1}\right\|_\alpha^2\right) + \frac{c_7}{\varepsilon_4}\left(\|\rho^n\|_\alpha^2 + \left\|\rho^{n-1}\right\|_\alpha^2\right).
\end{aligned}\tag{3.41}$$

Secondly, we deduce the estimation of R_2,

$$\begin{aligned}
R_2\left(\overline{\theta}^n\right) &= \left(-a(\widetilde{u}_h^n)\,{}_aD_x^{\beta}\overline{u}^n, D\overline{\theta}^n\right) + \left(a\left(u^{n-1/2}\right){}_aD_x^{\beta}u^{n-1/2}, D\overline{\theta}^n\right)\\
&= \left(\left(a\left(u^{n-1/2}\right) - a(\widetilde{u}_h^n)\right){}_aD_x^{\beta}\overline{u}^n, D\overline{\theta}^n\right) + \left(a\left(u^{n-1/2}\right)\left({}_aD_x^{\beta}u^{n-1/2} - {}_aD_x^{\beta}\overline{u}^n\right), D\overline{\theta}^n\right)\\
&= R_{21} + R_{22},
\end{aligned}\tag{3.42}$$

where

$$\begin{aligned}
R_{21} &= \left(\left[a\left(u^{n-1/2}\right) - a(\widetilde{u}^n)\right]{}_aD_x^{\beta}\widetilde{u}^n\right), D\overline{\theta}^n\Big)\\
&\le \frac{c_8}{4\varepsilon_5}\left\|\left[a\left(u^{n-1/2}\right) - a(\widetilde{u}^n)\right]{}_aD_x^{\beta}\widetilde{u}^n\right\|_{1-\alpha}^2 + \varepsilon_5\left\|D\overline{\theta}^n\right\|_{\alpha-1}^2\\
&\le c_9\left\|a\left(u^{n-1/2}\right) - a(\widetilde{u}^n)\right\|\left\|{}_aD_x^{\beta}\widetilde{u}^n\right\|_{1-\alpha}^2 + \varepsilon_5\left\|\overline{\theta}^n\right\|_\alpha^2\\
&\le c_9 L\left\|u^{n-1/2} - \widetilde{u}^n\right\| + \varepsilon_5\left\|\overline{\theta}^n\right\|_\alpha^2,
\end{aligned}$$

$$\begin{aligned}
R_{22} &= \left(a\left(u^{n-1/2}\right)\left({}_aD_x^{\beta}u^{n-1/2} - {}_aD_x^{\beta}\overline{u}^{n}\right), D\overline{\theta}^{n}\right) \\
&\le \frac{c_{10}}{4\varepsilon_6}\left\|a\left(u^{n-1/2}\right)\left[{}_aD_x^{\beta}u^{n-1/2} - {}_aD_x^{\beta}\overline{u}^{n}\right]\right\|_{1-\alpha}^2 + \varepsilon_6\left\|D\overline{\theta}^{n}\right\|_{-1+\alpha}^2 \\
&\le c_{10}\left\|a\left(u^{n-1/2}\right)\right\|^2\left\|{}_aD_x^{\beta}u^{n-1/2} - {}_aD_x^{\beta}\overline{u}^{n}\right\|_{1-\alpha}^2 + \varepsilon_6\left\|\overline{\theta}^{n}\right\|_{\alpha}^2 \\
&\le c_{10}M^2\left\|u^{n-1/2} - \overline{u}^{n}\right\|_{\alpha}^2 + \varepsilon_6 c\left(\|\theta^n\|_{\alpha}^2 + \left\|\theta^{n-1}\right\|_{\alpha}^2\right).
\end{aligned} \tag{3.43}$$

The estimations of $\|\widetilde{u}^n - u^{n-1/2}\|$ and $\|\overline{u}^n - u^{n-1/2}\|_{\alpha}$ can be derived as follows:

$$\begin{aligned}
\left\|\widetilde{u}^n - u^{n-1/2}\right\| &= \left\|\frac{3}{2}\left[u^{n-1/2} - \frac{\tau}{2}u_t^{n-1/2} + \frac{u_{tt}^{n-1/2}}{2!}\left(\frac{\tau}{2}\right)^2 + O\left(\tau^3\right)\right]\right. \\
&\qquad \left. -\frac{1}{2}\left[u^{n-1/2} - \frac{3\tau}{2}u_t^{n-1/2} + \frac{u_{tt}^{n-1/2}}{2!}\left(\frac{3\tau}{2}\right)^2 + O\left(\tau^3\right)\right] - u^{n-1/2}\right\| \\
&\le c_{11}\tau^2\|u_{tt}(t_{n-1/2})\| \le c_{11}\tau^2\int_{t_{n-1}}^{t_n}\|u_{tt}(\cdot, s)\|ds, \\
\left\|\overline{u}^n - u^{n-1/2}\right\|_{\alpha} &= \left\|\tau^{-1}\left\{\int_{t_{n-1/2}}^{t_n}(s - t_n)^2u_{tt}(s)ds + \int_{t_{n-1}}^{t_{n-1/2}}(s - t_{n-1})^2u_{tt}(s)ds\right\}\right\|_{\alpha} \\
&\le c_{12}\tau\left\|\int_{t_{n-1}}^{t_n}u_{tt}(s)ds\right\|_{\alpha} \\
&\le c_{12}\tau\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|_{\alpha}ds \\
&\le c_{12}\tau h^{k+1-\alpha}\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|_{k+1}ds.
\end{aligned} \tag{3.44}$$

Thirdly, it is turn to consider R_3,

$$\begin{aligned}
R_3\left(\overline{\theta}^n\right) &= \left(b(\widetilde{u}_h^n)D\overline{\rho}^n, \overline{\theta}^n\right) \le \|b(\widetilde{u}_h^n)D\overline{\rho}^n\|_{-\alpha}\left\|\overline{\theta}^n\right\|_{\alpha} \\
&\le \frac{c_{13}}{4\varepsilon_7}\|b(\widetilde{u}_h^n)\|^2\|\overline{\rho}^n\|_{1-\alpha}^2 + \varepsilon_7\left\|\overline{\theta}^n\right\|_{\alpha}^2 \\
&\le \frac{c_{14}}{4\varepsilon_7}\left(\|\rho^n\|_{1-\alpha}^2 + \left\|\rho^{n-1}\right\|_{1-\alpha}^2\right) + \varepsilon_7 c_{15}\left(\|\theta^n\|_{\alpha}^2 + \left\|\theta^{n-1}\right\|_{\alpha}^2\right).
\end{aligned} \tag{3.45}$$

Next,

$$\begin{aligned}
R_4\left(\overline{\theta}^n\right) &= \left(b(\tilde{u}_h^n)D\overline{u}^n,\overline{\theta}^n\right) - \left(b\left(u^{n-1/2}\right)Du^{n-1/2},\overline{\theta}^n\right) \\
&= \left(\left(b(\tilde{u}_h^n)D\overline{u}^n - b\left(u^{n-1/2}\right)D\overline{u}^n\right),\overline{\theta}^n\right) + \left(b\left(u^{n-1/2}\right)\left(D\overline{u}^n - Du^{n-1/2}\right),\overline{\theta}^n\right) \\
&= R_{41} + R_{42},
\end{aligned} \tag{3.46}$$

where

$$\begin{aligned}
R_{41} &\leq \frac{c_{16}}{4\varepsilon_8}\left\|\left\{b(\tilde{u}_h^n) - b\left(u^{n-1/2}\right)\right\}D\overline{u}^n\right\|_{1-\alpha}^2 + \varepsilon_8\left\|\overline{\theta}^n\right\|_\alpha^2 \\
&\leq \frac{c_{16}L}{4\varepsilon_8}\left\|\tilde{u}_h^n - u^{n-1/2}\right\|^2\|D\overline{u}^n\|_{1-\alpha}^2 + \varepsilon_8\left\|\overline{\theta}^n\right\|_\alpha^2 \\
&= \frac{c_{16}L}{4\varepsilon_8}\left\|\tilde{u}_h^n - \tilde{u}^n + \tilde{u}^n - u^{n-1/2}\right\|^2\|D\overline{u}^n\|_{1-\alpha}^2 + \varepsilon_8\left\|\overline{\theta}^n\right\|_\alpha^2 \\
&\leq c_{17}\|\tilde{u}_h^n - \tilde{u}^n\|^2 + c_{17}\left\|\tilde{u}^n - u^{n-1/2}\right\|^2 + \varepsilon_8\left\|\overline{\theta}^n\right\|_\alpha^2 \\
&\leq c_{17}\left\|\tilde{\theta}^n + \tilde{\rho}^n\right\|^2 + c_{17}\left\|\tilde{u}^n - u^{n-1/2}\right\|^2 + \varepsilon_8\left\|\overline{\theta}^n\right\|_\alpha^2 \\
&\leq c_{18}\left(\left\|\tilde{\theta}^n\right\|^2 + \|\tilde{\rho}^n\|^2\right) + c_{17}\left\|\tilde{u}^n - u^{n-1/2}\right\|^2 + \varepsilon_8\left\|\overline{\theta}^n\right\|_\alpha^2.
\end{aligned} \tag{3.47}$$

Rewriting R_{42} by the aid of (3.20), we have

$$R_{42} \leq \frac{c_{19}}{4\varepsilon_9}\left\|\overline{u}^n - u^{n-1/2}\right\|^2 + \varepsilon_9\left\|\overline{\theta}^n\right\|_\alpha^2. \tag{3.48}$$

The estimation of R_5 is deduced as follows:

$$\begin{aligned}
R_5\left(\overline{\theta^n}\right) &\leq \left\|f(\tilde{u}_h^n) - f\left(u^{n-1/2}\right)\right\|\left\|\overline{\theta}^n\right\| \\
&\leq L\left\|\tilde{u}_h^n - u^{n-1/2}\right\|\left\|\overline{\theta}^n\right\| \\
&\leq \frac{Lc_{20}}{4\varepsilon_{10}}\left\|\tilde{u}_h^n - u^{n-1/2}\right\|^2 + \varepsilon_{10}\left\|\overline{\theta}^n\right\|^2 \\
&\leq Lc_{21}\left(\left\|\tilde{\theta}^n + \tilde{\rho}^n\right\|^2 + \left\|\tilde{u}^n - u^{n-1/2}\right\|^2\right) + \varepsilon_{10}\left\|\overline{\theta}^n\right\|^2 \\
&\leq c_{22}\left(\left\|\tilde{\theta}^n\right\|^2 + \|\tilde{\rho}^n\|^2\right) + Lc_{21}\left\|\tilde{u}^n - u^{n-1/2}\right\|^2 + \varepsilon_{10}\left\|\overline{\theta}^n\right\|^2.
\end{aligned} \tag{3.49}$$

Last, we estimate R_6,

$$\begin{aligned}
R_6\left(\overline{\theta}^n\right) &= \left(\partial_t u^{n-1/2}, \overline{\theta}^n\right) - \left(\overline{\partial}_t P_h u^n, \overline{\theta}^n\right) \\
&= \left(\partial_t u^{n-1/2} - \overline{\partial} u^n, \overline{\theta}^n\right) + \left(\overline{\partial}_t u^n - \overline{\partial}_t P_h u^n, \overline{\theta}^n\right) \\
&= \left(\partial_t u^{n-1/2} - \overline{\partial}_t u^n, \overline{\theta}^n\right) + \left(\overline{\partial}_t \rho^n, \overline{\theta}^n\right) \\
&\le \left\|\partial_t u^{n-1/2} - \overline{\partial}_t u^n\right\| \|\theta^n\| + \left\|\overline{\partial}_t \rho^n\right\| \|\theta^n\|,
\end{aligned} \tag{3.50}$$

where

$$\begin{aligned}
\left\|\partial_t u^{n-1/2} - \overline{\partial} u^n\right\| &= (2\tau)^{-1} c_{23} \left\| \int_{t_{n-1/2}}^{t_n} (s - t_n)^2 u_{ttt}(s) ds + \int_{t_{n-1}}^{t_{n-1/2}} (s - t_{n-1})^2 u_{ttt}(s) ds \right\| \\
&\le c_{23} \tau \left\| \int_{t_{n-1}}^{t_n} u_{ttt}(s) ds \right\| \\
&\le c_{23} \tau \int_{t_{n-1}}^{t_n} \|u_{ttt}(s)\| ds, \\
\left\|\overline{\partial}_t \rho^n\right\| &= \left\| \frac{\rho^n - \rho^{n-1}}{\tau} \right\| \le \tau^{-1} \left\| \int_{t_{n-1}}^{t_n} \rho_t^n(s) ds \right\| \\
&\le \tau^{-1} \int_{t_{n-1}}^{t_n} \|u_t(s)\| ds \le \tau^{-1} \int_{t_{n-1}}^{t_n} 1 ds \int_{t_{n-1}}^{t_n} \|u_t(s)\| ds \\
&= \int_{t_{n-1}}^{t_n} \|u_t(s)\| \le h^{k+1} \int_{t_{n-1}}^{t_n} \|u_t(s)\|_{k+1} ds.
\end{aligned} \tag{3.51}$$

The $\|\theta^2\|$ should be estimated with (3.14). Let $n = 1$ then subtracting (3.34) from the two equations of (3.14), respectively, one gets

$$\begin{aligned}
&\left(\overline{\partial}_t \theta^{1,0}, v\right) + \left(a\left(u_h^0\right) {}_a D_x^\beta \overline{\theta}^{1,0}, Dv\right) - b\left(\left(u_h^0\right) D\overline{\theta}^{1,0}, v\right) \\
&\quad = -\left(a\left(u_h^0\right) {}_a D_x^\beta \overline{\rho}^{1,0}, Dv\right) - \left\{\left(a\left(u^{1/2}\right) {}_a D_x^\beta u^{1/2} - a\left(u_h^0\right) {}_a D_x^\beta \overline{u}^{1,0}, Dv\right)\right\} \\
&\qquad + \left(b\left(u_h^0\right) D\overline{\rho}^{1,0}, v\right) + \left\{\left(b\left(u_h^0\right) D\overline{u}^{1,0} - b\left(u^{1/2}\right) Du^{1/2}, Dv\right)\right\} \\
&\qquad + \left\{\left(f\left(u_h^0\right) - f\left(u^{1/2}\right), v\right)\right\} + \left\{\left(\partial_t u^{1/2} - \overline{\partial}_t I_h u^{1,0}, v\right)\right\} \\
&\quad = R_1(v) + R_2(v) + R_3(v) + R_4(v) + R_5(v) + R_6(v),
\end{aligned}$$

$$\left(\overline{\partial}_t\theta^1, v\right) + \left(a\left(u_h^0\right) {}_aD_x^{\beta}\overline{\theta}^1, Dv\right) - b\left(\left(u_h^0\right)D\overline{\theta}^1, v\right)$$
$$= -\left(a\left(\frac{u_h^0 + u_h^{1,0}}{2}\right) {}_aD_x^{\beta}\overline{\rho}^1, Dv\right) - \left\{\left(a\left(u^{1/2}\right) {}_aD_x^{\beta}u^{1/2} - a\left(\frac{u_h^0 + u_h^{1,0}}{2}\right) {}_aD_x^{\beta}\overline{u}^1, Dv\right)\right\}$$
$$+ \left(b\left(\frac{u_h^0 + u_h^{1,0}}{2}\right)D\overline{\rho}^1, v\right) + \left\{\left(b\left(\frac{u_h^0 + u_h^{1,0}}{2}\right)D\overline{u}^1 - \left(b\left(u^{1/2}\right)Du^{1/2}, Dv\right)\right\}$$
$$+ \left\{\left(f\left(\frac{u_h^0 + u_h^{1,0}}{2}\right) - f\left(u^{1/2}\right), v\right)\right\} + \left\{\left(\partial_t u^{1/2} - \overline{\partial}_t I_h u^1, v\right)\right\}$$
$$= R_1(v) + R_2(v) + R_3(v) + R_4(v) + R_5(v) + R_6(v). \tag{3.52}$$

Setting $v = \overline{\theta^{1,0}}$, and using the similar estimation (see (3.40)), one has

$$\left\|\theta^{1,0}\right\|^2 \le c\left\{\tau^2\int_{t_0}^{t_1}\|u_{tt}(s)\|_{\alpha}^2 ds + h^{2(k+1-\alpha)}\|u\|_{k+1}^2 + \tau^2\int_{t_0}^{t_1}\|u_{ttt}(s)\|^2 ds + ch^{2(k+1)}\int_{t_0}^{t_1}\|u_t(s)\|_{k+1}^2 ds\right\}. \tag{3.53}$$

Letting $v = \overline{\theta^1}$, applying the above result of $\theta^{1,0}$, and using the similar estimation (see (3.53)), we get

$$\left\|\theta^{1}\right\|^2 \le c\left\{\tau^2\int_{t_0}^{t_1}\|u_{tt}(s)\|_{\alpha}^2 ds + h^{2(k+1-\alpha)}\|u\|_{k+1}^2 + \tau^2\int_{t_0}^{t_1}\|u_{ttt}(s)\|^2 ds + ch^{2(k+1)}\int_{t_0}^{t_1}\|u_t(s)\|_{k+1}^2 ds\right\}. \tag{3.54}$$

Using $T = N\tau$ and Gronwall's lemma, we get

$$|||\theta|||_{0,\alpha}^2 = \sum_{n=0}^{N}\tau\|\theta\|_{\alpha}^2. \tag{3.55}$$

Hence, using the interpolation property and

$$|||u - u_h|||_{0,\alpha} \le |||\theta|||_{0,\alpha} + \left|\left|\left|\rho\right|\right|\right|_{0,\alpha}, \tag{3.56}$$

the estimate (3.32) holds.

Also using the interpolation property, Gronwall's lemma, and the approximation properties, we get

$$\begin{aligned} |||u - u_h|||_{\infty,0} &\le |||\theta|||_{\infty,0} + \big|\big|\big|\rho\big|\big|\big|_{\infty,0} \\ &\le \max_{0\le n\le N} |||\theta^n|||^2 + h^{2k+1}|||u|||^2_{\infty,k+1}, \end{aligned} \tag{3.57}$$

which is just the estimate (3.33). □

4. Numerical Examples

In this section, we present the numerical results which confirm the theoretical analysis in Section 3.

Let K denote a uniform partition on $[0, a]$, and S_h the space of continuous piecewise linear functions on K, that is, $k = 1$.. In order to implement the Galerkin finite element approximation, we adapt finite element discrete along the space axis, and finite difference scheme along the time axis. We associate shape function of space X_h with the standard basis of hat functions on the uniform grid of size $h = 1/n$. We have the predicted rates of convergence if the condition $\Delta t = ch$ of

$$\begin{aligned} \|u - u_h\|_{0,\alpha} &\sim O\left(h^{2-\alpha}\right), \\ \|u - u_h\|_{\infty,0} &\sim O\left(h^{2-\alpha}\right), \end{aligned} \tag{4.1}$$

provided that the initial value $\varphi(x)$ is smooth enough.

Example 4.1. The following equation

$$\begin{gathered} \frac{\partial u}{\partial t} = D\left(u^2\, {}_0D_x^{0.5} u(x,t)\right) - 2x(x-1)\left(\frac{2x^{1.5}}{\Gamma(2.5)} - \frac{x^{0.5}}{\Gamma(1.5)}\right)e^{-2t}Du - u(x,t) \\ -u^2 e^{-t}\left(\frac{2x^{0.5}}{\Gamma(1.5)} - \frac{x^{-0.5}}{\Gamma(0.5)}\right), \quad 0 \le x \le 1,\ 0 \le t \le 1, \\ u(x,0) = x(x-1), \quad 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1, \end{gathered} \tag{4.2}$$

has a unique solution $u(x,t) = e^{-t}x(x-1)$.

If we select $\Delta t = ch$ and note that the initial value u^0 is smooth enough, then we have

$$\begin{aligned} \|u - u_h\|_{0,0.75} &\sim O\left(h^{1.25}\right), \\ \|u - u_h\|_{\infty,0} &\sim O\left(h^{1.25}\right). \end{aligned} \tag{4.3}$$

Table 1: Numerical error result for Example 4.1.

h	$\|u-u_h\|_{\infty,0}$	cvge. rate	$\|u-u_h\|_{0,0.75}$	cvge. rate
1/5	2.2216E−003	—	1.0213E−003	—
1/10	1.3551E−003	0.7132	6.0779E−004	0.74875
1/20	5.5865E−004	1.2784	2.3188E−004	1.3901
1/40	3.0515E−004	0.8724	1.0545E−004	1.1367
1/80	1.2423E−004	1.2964	3.9883E−005	1.4027
1/160	5.1033E−005	1.2835	2.1310E−005	0.9042

Table 1 includes numerical calculations over a regular partition of $[0,1]$. We can observe the experimental rates of convergence agree with the theoretical rates for the numerical solution.

Example 4.2. The function $u(x,t) = \cos(t)x^2(2-x)^2$ solves the equation in the following form:

$$\begin{gathered}\frac{\partial u}{\partial t} = {}_0D_x^{1.7}\,u(x,t) + b(u)Du - u[4(1-x)+\tan t] + f(x,t), \quad x \in (0,2), t \in [0,1),\\ u(x,0) = x^2(2-x)^2, \quad 0 \le x \le 2,\\ u(0,t)=0, \quad u(2,t)=0, \quad 0 \le t \le 1,\end{gathered} \tag{4.4}$$

where

$$b(u) = \frac{\sqrt{u}}{\sqrt{\cos t}},$$

$$f(x,t) = \frac{\cos t}{\cos(0.85\pi)}\left[\frac{24\left(x^{2.3}+(2-x)^{2.3}\right)}{\Gamma(3.3)} - \frac{24\left(x^{1.3}+(2-x)^{1.3}\right)}{\Gamma(2.3)} - \frac{8\left(x^{0.3}+(2-x)^{0.3}\right)}{\Gamma(1.3)}\right]. \tag{4.5}$$

If we select $\Delta t = ch$, then

$$\begin{gathered}\|u-u_h\|_{0,0.85} \sim O\left(h^{1.15}\right),\\ \|u-u_h\|_{\infty,0} \sim O\left(h^{1.15}\right).\end{gathered} \tag{4.6}$$

Table 2 shows the error results at different size of space grid. We can observe that the experimental rates of convergence still support the theoretical rates.

Table 2: Numerical error result for Example 4.2.

h	$\|u-u_h\|_{\infty,0}$	cvge. rate	$\|u-u_h\|_{0,0.85}$	cvge. rate
1/5	1.3010E−001	—	3.2223E−002	—
1/10	4.6402E−002	1.4878	1.4133E−002	1.1890
1/20	1.6843E−002	1.4620	6.2946E−003	1.1669
1/40	6.6019E−003	1.6843	2.8571E−003	1.1395
1/80	2.7979E−003	1.2386	1.3137E−003	1.1209
1/160	1.2665E−003	1.1434	6.0848E−004	1.1103

Table 3: Numerical error result for Example 4.3.

h	$\|u-u_h\|_{\infty,0}$	cvge. rate	$\|u-u_h\|_{0,0.85}$	cvge. rate
1/5	8.3052E−002	—	2.8009E−002	—
1/10	3.6038E−002	1.2045	1.0086E−002	1.4735
1/20	1.3839E−002	1.3807	3.2327E−003	1.6414
1/40	5.0631E−003	1.4507	1.1789E−003	1.4554
1/80	1.8920E−003	1.4201	5.9555E−004	0.9851
1/160	9.9899E−004	0.9214	3.0034E−004	0.9876

Example 4.3. Consider the following space-fractional differential equation with the nonhomogeneous boundary conditions,

$$\begin{gathered}\frac{\partial u}{\partial t} = {}_0D_x^{1.7}u(x,t) - \frac{3}{x}\int_0^x u\,dx - \frac{2x^{0.3}e^{-t}}{\Gamma(1.3)}, \quad 0 \le x \le 1,\ 0 \le t \le 1, \\ u(x,0) = x^2, \quad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = e^{-t}, \quad 0 \le t \le 1,\end{gathered} \tag{4.7}$$

whose exact solution is $u(x,t) = e^{-t}x^2$.

We still choose $\Delta t = ch$, then get the convergence rates

$$\begin{gathered}\|u-u_h\|_{0,0.85} \sim O\left(h^{1.15}\right), \\ \|u-u_h\|_{\infty,0} \sim O\left(h^{1.15}\right).\end{gathered} \tag{4.8}$$

The numerical results are presented in Table 3 which are in line with the theoretical analysis.

5. Conclusion

In this paper, we propose a fully discrete Galerkin finite element method to solve a type of fractional advection-diffusion equation numerically. In the temporal direction we use the modified Crank-Nicolson method, and in the spatial direction we use the finite element method. The error analysis is derived on the basis of fractional derivative space. The numerical results agree with the theoretical error estimates, demonstrating that our algorithm is feasible.

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