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Research Article

Adaptive Mixed Finite Element Methods for Parabolic Optimal Control Problems

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We will investigate the adaptive mixed finite element methods for parabolic optimal control problems. The state and the costate are approximated by the lowest-order Raviart-Thomas mixed finite element spaces, and the control is approximated by piecewise constant elements. We derive a posteriori error estimates of the mixed finite element solutions for optimal control problems. Such a posteriori error estimates can be used to construct more efficient and reliable adaptive mixed finite element method for the optimal control problems. Next we introduce an adaptive algorithm to guide the mesh refinement. A numerical example is given to demonstrate our theoretical results.

1. Introduction

Optimal control problems are very important models in science and engineering numerical simulation. Finite element method of optimal control problems plays an important role in numerical methods for these problems. Let us mention two early papers devoted to linear optimal control problems by Falk [1] and Geveci [2]. Knowles was concerned with standard finite element approximation of parabolic time optimal control problems in [3]. In [4] Gunzburger and Hou investigated the finite element approximation of a class of constrained nonlinear optimal control problems. For quadratic optimal control problem governed by linear parabolic equation, Liu and Yan derived a posteriori error estimates for both the state and the control approximation in [5]. Systematic introductions of the finite element method for optimal control problems can be found in [6–10].

Adaptive finite element approximation was the most important means of boosting the accuracy and efficiency of finite element discretization. The literature in this aspect was huge, see, for example, [11, 12]. Adaptive finite element method was widely used in engineering numerical simulation. There has been extensive studies on adaptive finite element approximation for optimal control problems. In [13], the authors have introduced some basic concept of adaptive finite element discretization for optimal control of partial differential equations. A posteriori error estimators for distributed elliptic optimal control problems were contained in Li et al. [14]. Recently an adaptive finite element method for the estimation of distributed parameter in elliptic equation was discussed by Feng et al. [15]. Note that all the above works aimed at standard finite element method.

In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. When the objective functional contains the gradient of the state variable, mixed finite element methods should be used for discretization of the state equation with which both the scalar variable and its flux variable can be approximated in the same accuracy. In [16–20] we have done some primary works on a priori error estimates and superconvergence for linear optimal control problems by mixed finite element methods. We considered a posteriori error estimates of mixed finite element methods for quadratic and general optimal control problems in [21–23].

In [24], the authors discussed the mixed finite element approximation for general optimal control problems governed by parabolic equation. And then, they derived a posteriori error estimates of mixed finite element solution. In this paper, we study the adaptive mixed finite element methods for the parabolic optimal control problems. We construct the mixed finite element discretization for the original problems and derive a useful posteriori error indicators. Furthermore, we provide an adaptive algorithm to guide the multimesh refinement. Finally, a numerical experiment shows that this algorithm works very well with the adaptive multimesh discretization.

The plan of this paper is as follows. In the next section, we construct the mixed finite element discretization for the parabolic optimal control problems. Then, we derive a posteriori error estimates for the mixed finite element solutions in Section 3. Next, we introduce an adaptive algorithm to guide the mesh refinement in Section 4. Finally, a numerical example is given to demonstrate our theoretical results in Section 5.

2. Mixed Methods of Optimal Control Problems

In this section, we investigate the mixed finite element approximation for parabolic optimal control problems. We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a seminorm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = (\int_0^T \|v\|_{W^{m,p}(\Omega)}^s dt)^{1/s}$ for $s \in [1,\infty)$ and the standard modification for $s = \infty$. The details can be found in [25].

The parabolic optimal control problems that we are interested in are as follows:

$$\min_{u \in K \subset U} \left\{ \int_0^T (g_1(\mathbf{p}) + g_2(y) + j(u))\, dt \right\},$$

$$y_t(x,t) + \operatorname{div} \mathbf{p}(x,t) = f + u(x,t), \quad x \in \Omega,$$

$$\mathbf{p}(x,t) = -A(x)\nabla y(x,t), \quad x \in \Omega,$$
$$y(x,t) = 0, \quad x \in \partial\Omega,\ t \in J, \qquad y(x,0) = y_0(x), \quad x \in \Omega, \tag{2.1}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$ is a convex polygon with the boundary $\partial\Omega$. Let K be a closed convex set in $U = L^2(J;L^2(\Omega))$, $f \in L^2(J;L^2(\Omega))$, $J = [0,T]$, $y_0(x) \in H_0^1(\Omega)$. Furthermore, we assume that the coefficient matrix $A(x) = (a_{i,j}(x))_{2\times 2} \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$ is a symmetric 2×2-matrix and there is a constant $c > 0$ satisfying for any vector $\mathbf{X} \in \mathbb{R}^2$, $\mathbf{X}^t A\mathbf{X} \geq c\|\mathbf{X}\|_{\mathbb{R}^2}^2$. j' is positive, g_1', g_2', and j' are locally Lipschitz on $L^2(\Omega)^2$, W, U, and that there is a $c > 0$ such that $(j'(u) - j'(\tilde{u}), u - \tilde{u}) \geq c\|u - \tilde{u}\|_0$, for all $u, \tilde{u} \in U$.

Now we will describe the mixed finite element discretization of parabolic optimal control problems (2.1). Let

$$\mathbf{V} = H(\operatorname{div};\Omega) = \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^2, \operatorname{div}\mathbf{v} \in L^2(\Omega)\right\}, \qquad W = L^2(\Omega). \tag{2.2}$$

The Hilbert space $\mathbf{V}$ is equipped with the following norm:

$$\|\mathbf{v}\|_{\operatorname{div}} = \|\mathbf{v}\|_{H(\operatorname{div};\Omega)} = \left(\|\mathbf{v}\|_{0,\Omega}^2 + \|\operatorname{div}\mathbf{v}\|_{0,\Omega}^2\right)^{1/2}. \tag{2.3}$$

We recast (2.1) as the following weak form: find $(\mathbf{p}, y, u) \in \mathbf{V} \times W \times K$ such that

$$\min_{u\in K\subset U}\left\{\int_0^T (g_1(\mathbf{p}) + g_2(y) + j(u))\,dt\right\}, \tag{2.4}$$

$$\left(A^{-1}\mathbf{p}, \mathbf{v}\right) - (y, \operatorname{div}\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.5}$$

$$(y_t, w) + (\operatorname{div}\mathbf{p}, w) = (f + u, w), \quad \forall w \in W, \tag{2.6}$$

$$y(x,0) = y_0(x), \quad \forall x \in \Omega. \tag{2.7}$$

Similar to [26], the optimal control problems (2.4)–(2.7) have a unique solution $(\mathbf{p}, y, u)$, and a triplet $(\mathbf{p}, y, u)$ is the solution of (2.4)–(2.7) if and only if there is a costate $(\mathbf{q}, z) \in \mathbf{V} \times W$ such that $(\mathbf{p}, y, \mathbf{q}, z, u)$ satisfies the following optimality conditions:

$$\left(A^{-1}\mathbf{p}, \mathbf{v}\right) - (y, \operatorname{div}\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.8}$$

$$(y_t, w) + (\operatorname{div}\mathbf{p}, w) = (f + u, w), \quad \forall w \in W, \tag{2.9}$$

$$y(x,0) = y_0(x), \quad \forall x \in \Omega, \tag{2.10}$$

$$\left(A^{-1}\mathbf{q}, \mathbf{v}\right) - (z, \operatorname{div}\mathbf{v}) = -\left(g_1'(\mathbf{p}), \mathbf{v}\right), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.11}$$

$$-(z_t, w) + (\operatorname{div}\mathbf{q}, w) = \left(g_2'(y), w\right), \quad \forall w \in W, \tag{2.12}$$

$$z(x,T) = 0, \quad \forall x \in \Omega, \tag{2.13}$$

$$\int_0^T \left(j'(u) + z, \tilde{u} - u\right)_U dt \geq 0, \quad \forall \tilde{u} \in K, \tag{2.14}$$

where $(\cdot,\cdot)_U$ is the inner product of U. In the rest of the paper, we will simply write the product as $(\cdot,\cdot)$ whenever no confusion should be caused.

Let $\mathcal{T}_h$ be regular triangulation of Ω. They are assumed to satisfy the angle condition which means that there is a positive constant C such that, for all $\tau \in \mathcal{T}_h$, $C^{-1}h_\tau^2 \leq |\tau| \leq Ch_\tau^2$, where $|\tau|$ is the area of τ, h_τ is the diameter of τ and $h = \max h_\tau$. In addition C or c denotes a general positive constant independent of h.

Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote the Raviart-Thomas space [27] of the lowest order associated with the triangulation $\mathcal{T}_h$ of Ω. P_k denotes the space of polynomials of total degree at most k. Let $\mathbf{V}(\tau) = \{\mathbf{v} \in P_0^2(\tau) + x \cdot P_0(\tau)\}$, $W(\tau) = P_0(\tau)$. We define

$$\begin{aligned}
\mathbf{V}_h &:= \{\mathbf{v}_h \in \mathbf{V} : \forall \tau \in \mathcal{T}_h, \mathbf{v}_h|_\tau \in \mathbf{V}(\tau)\}, \\
W_h &:= \{w_h \in W : \forall \tau \in \mathcal{T}_h, w_h|_\tau \in W(\tau)\}, \\
K_h &:= \{\tilde{u}_h \in K : \forall \tau \in \mathcal{T}_h, \tilde{u}_h|_\tau = \text{constant}\}.
\end{aligned} \tag{2.15}$$

The mixed finite element discretization of (2.4)–(2.7) is as follows: compute $(\mathbf{p}_h, y_h, u_h) \in \mathbf{V}_h \times W_h \times K_h$ such that

$$\begin{gathered}
\min_{u_h \in K_h} \left\{ \int_0^T \left(g_1(\mathbf{p}_h) + g_2(y_h) + j(u_h)\right) dt \right\}, \\
\left(A^{-1}\mathbf{p}_h, \mathbf{v}_h\right) - \left(y_h, \operatorname{div}\mathbf{v}_h\right) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\
\left(y_{ht}, w_h\right) + \left(\operatorname{div}\mathbf{p}_h, w_h\right) = \left(f + u_h, w_h\right), \quad \forall w_h \in W_h, \\
y_h(x,0) = y_0^h(x), \quad \forall x \in \Omega,
\end{gathered} \tag{2.16}$$

where $y_0^h(x) \in W_h$ is an approximation of y_0. The optimal control problem (2.16) again has a unique solution $(\mathbf{p}_h, y_h, u_h)$, and a triplet $(\mathbf{p}_h, y_h, u_h)$ is the solution of (2.16) if and only if there is a costate $(\mathbf{q}_h, z_h) \in \mathbf{V}_h \times W_h$ such that $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$\begin{gathered}
\left(A^{-1}\mathbf{p}_h, \mathbf{v}_h\right) - \left(y_h, \operatorname{div}\mathbf{v}_h\right) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\
(y_{ht}, w_h) + (\operatorname{div}\mathbf{p}_h, w_h) = \left(f + u_h, w_h\right), \quad \forall w_h \in W_h, \\
y_h(x,0) = y_0(x), \quad \forall x \in \Omega, \\
\left(A^{-1}\mathbf{q}_h, \mathbf{v}_h\right) - (z_h, \operatorname{div}\mathbf{v}_h) = -\left(g_1'(\mathbf{p}_h), \mathbf{v}_h\right), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\
-(z_{ht}, w_h) + (\operatorname{div}\mathbf{q}_h, w_h) = \left(g_2'(y_h), w_h\right), \quad \forall w_h \in W_h, \\
z_h(x,T) = 0, \quad \forall x \in \Omega, \\
\left(j'(u_h) + z_h, \tilde{u}_h - u_h\right) \geq 0, \quad \forall \tilde{u}_h \in K_h.
\end{gathered} \tag{2.17}$$

We now consider the fully discrete approximation for the semidiscrete problem. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$, and $t_i = i\Delta t$, $i \in \mathbb{Z}$. Also, let

$$\psi^i = \psi^i(x) = \psi(x, t_i), \qquad d_t\psi^i = \frac{\psi^i - \psi^{i-1}}{\Delta t}. \tag{2.18}$$

For $i = 1, 2, \ldots, N$, construct the finite element spaces $\mathbf{V}_h^i \in \mathbf{V}$, $W_h^i \in W$ (similar as $\mathbf{V}_h$). Similarly, construct the finite element spaces $K_h^i \in K_h$ with the mesh $\mathcal{T}_h^i$. Let h_τ^i denote the maximum diameter of the element τ^i in $\mathcal{T}_h^i$. Define mesh functions $\tau(\cdot)$ and mesh size functions $h_\tau(\cdot)$ such that $\tau(t)|_{t\in(t_{i-1},t_i]} = \tau^i$, $h_\tau(t)|_{t\in(t_{i-1},t_i]} = h_{\tau_i}$. For ease of exposition, we will denote $\tau(t)$ and $h_\tau(t)$ by τ and h_τ, respectively.

The following fully discrete approximation scheme is to find $(\mathbf{p}_h^i, y_h^i, u_h^i) \in \mathbf{V}_h^i \times W_h^i \times K_h^i$, $i = 1, 2, \ldots, N$, such that

$$\min_{u_h^i \in K_h^i} \left\{ \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left(g_1\left(\mathbf{p}_h^i\right) + g_2\left(y_h^i\right) + j\left(u_h^i\right) \right) \right\}, \tag{2.19}$$

$$\left(A^{-1}\mathbf{p}_h^i, \mathbf{v}_h\right) - \left(y_h^i, \operatorname{div}\mathbf{v}_h\right) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^i, \tag{2.20}$$

$$\left(d_t y_h^i, w_h\right) + \left(\operatorname{div}\mathbf{p}_h^i, w_h\right) = \left(f(x, t_i) + u_h^i, w_h\right), \quad \forall w_h \in W_h^i, \tag{2.21}$$

$$y_h^0(x, 0) = y_0^h(x), \quad \forall x \in \Omega. \tag{2.22}$$

It follows that the optimal control problems (2.19)–(2.22) have a unique solution $(\mathbf{p}_h^i, y_h^i, u_h^i)$, $i = 1, 2, \ldots, N$, and that a triplet $(\mathbf{p}_h^i, y_h^i, u_h^i) \in \mathbf{V}_h^i \times W_h^i \times K_h^i$, $i = 1, 2, \ldots, N$, is the solution of (2.19)–(2.22) if and only if there is a costate $(\mathbf{q}_h^{i-1}, z_h^{i-1}) \in \mathbf{V}_h^i \times W_h^i$ such that $(\mathbf{p}_h^i, y_h^i, \mathbf{q}_h^{i-1}, z_h^{i-1}, u_h^i) \in (\mathbf{V}_h^i \times W_h^i)^2 \times K_h^i$ satisfies the following optimality conditions:

$$\left(A^{-1}\mathbf{p}_h^i, \mathbf{v}_h\right) - \left(y_h^i, \operatorname{div} \mathbf{v}_h\right) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^i, \tag{2.23}$$

$$\left(d_t y_h^i, w_h\right) + \left(\operatorname{div} \mathbf{p}_h^i, w_h\right) = \left(f(x, t_i) + u_h^i, w_h\right), \quad \forall w_h \in W_h^i, \tag{2.24}$$

$$y_h^0(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.25}$$

$$\left(A^{-1}\mathbf{q}_h^{i-1}, \mathbf{v}_h\right) - \left(z_h^{i-1}, \operatorname{div} \mathbf{v}_h\right) = -\left(g_1'\left(\mathbf{p}_h^i\right), \mathbf{v}_h\right), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^i, \tag{2.26}$$

$$-\left(d_t z_h^i, w_h\right) + \left(\operatorname{div} \mathbf{q}_h^{i-1}, w_h\right) = \left(g_2'\left(y_h^i\right), w_h\right), \quad \forall w_h \in W_h^i, \tag{2.27}$$

$$z_h^N(x, T) = 0, \quad \forall x \in \Omega, \tag{2.28}$$

$$\left(j'\left(u_h^i\right) + z_h^{i-1}, \tilde{u}_h - u_h^i\right) \geq 0, \quad \forall \tilde{u}_h \in K_h^i. \tag{2.29}$$

For $i = 1, 2, \ldots, N$, let

$$\begin{aligned}
Y_h|_{(t_{i-1}, t_i]} &= \frac{((t_i - t)y_h^{i-1} + (t - t_{i-1})y_h^i)}{\Delta t}, \\
Z_h|_{(t_{i-1}, t_i]} &= \frac{((t_i - t)z_h^{i-1} + (t - t_{i-1})z_h^i)}{\Delta t}, \\
P_h|_{(t_{i-1}, t_i]} &= \frac{((t_i - t)\mathbf{p}_h^{i-1} + (t - t_{i-1})\mathbf{p}_h^i)}{\Delta t}, \\
Q_h|_{(t_{i-1}, t_i]} &= \frac{((t_i - t)\mathbf{q}_h^{i-1} + (t - t_{i-1})\mathbf{q}_h^i)}{\Delta t}, \\
U_h|_{(t_{i-1}, t_i]} &= u_h^i.
\end{aligned} \tag{2.30}$$

For any function $w \in C(J; L^2(\Omega))$, let

$$\widehat{w}(x, t)|_{t \in (t_{i-1}, t_i]} = w(x, t_i), \qquad \widetilde{w}(x, t)|_{t \in (t_{i-1}, t_i]} = w(x, t_{i-1}). \tag{2.31}$$

Then the optimality conditions (2.23)–(2.29) can be restated as.

$$\left(A^{-1}\widehat{P}_h, \mathbf{v}_h\right) - \left(\widehat{Y}_h, \operatorname{div} \mathbf{v}_h\right) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^i, \tag{2.32}$$

$$(Y_{ht}, w_h) + \left(\operatorname{div} \widehat{P}_h, w_h\right) = \left(\widehat{f} + U_h, w_h\right), \quad \forall w_h \in W_h^i, \tag{2.33}$$

$$Y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.34}$$

$$\left(A^{-1}\widetilde{Q}_h, \mathbf{v}_h\right) - \left(\widetilde{Z}_h, \operatorname{div} \mathbf{v}_h\right) = -\left(g_1'\left(\widehat{P}_h\right), \mathbf{v}_h\right), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^i, \tag{2.35}$$

$$-(Z_{ht}, w_h) + \left(\operatorname{div} \widetilde{Q}_h, w_h\right) = \left(g_2'\left(\widehat{Y}_h\right), w_h\right), \quad \forall w_h \in W_h^i, \tag{2.36}$$

$$Z_h(x, T) = 0, \quad \forall x \in \Omega, \tag{2.37}$$

$$\left(j'(U_h) + \widetilde{Z}_h, \widetilde{u}_h - U_h\right) \geq 0, \quad \forall \widetilde{u}_h \in K_h^i. \tag{2.38}$$

In the rest of the paper, we will use some intermediate variables. For any control function $U_h \in K$, we first define the state solution $(\mathbf{p}(U_h), y(U_h), \mathbf{q}(U_h), z(U_h))$ which satisfies

$$\left(A^{-1}\mathbf{p}(U_h), \mathbf{v}\right) - (y(U_h), \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.39}$$

$$(y_t(U_h), w) + (\operatorname{div} \mathbf{p}(U_h), w) = (f + U_h, w), \quad \forall w \in W, \tag{2.40}$$

$$y(U_h)(x, 0) = y_0(x), \quad \forall x \in \Omega, \tag{2.41}$$

$$\left(A^{-1}\mathbf{q}(U_h), \mathbf{v}\right) - (z(U_h), \operatorname{div} \mathbf{v}) = -(g_1'(\mathbf{p}(U_h)), \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.42}$$

$$-(z_t(U_h), w) + (\operatorname{div} \mathbf{q}(U_h), w) = (g_2'(y(U_h)), w), \quad \forall w \in W, \tag{2.43}$$

$$z(U_h)(x, T) = 0, \quad \forall x \in \Omega. \tag{2.44}$$

3. A Posteriori Error Estimates

In this section we study a posteriori error estimates of the mixed finite element approximation for the parabolic optimal control problems. Fixed given $u \in K$, let $\mathcal{M}_1$, $\mathcal{M}_2$ be the inverse operators of the state equation (2.6), such that $\mathbf{p}(u) = \mathcal{M}_1 u$ and $y(u) = \mathcal{M}_2 u$ are the solutions of the state equations (2.6). Similarly, for given $U_h \in K_h$, $P_h(U_h) = \mathcal{M}_{1h} U_h$, $Y_h(U_h) = \mathcal{M}_{2h} U_h$ are the solutions of the discrete state equation (2.33). Let

$$\begin{aligned} F(u) &= g_1(\mathcal{M}_1 U) + g_2(\mathcal{M}_2 U) + j(u), \\ F_h(U_h) &= g_1(\mathcal{M}_{1h} U_h) + g_2(\mathcal{M}_{2h} U_h) + j(U_h). \end{aligned} \tag{3.1}$$

It can be shown that

$$\begin{aligned}
\left(F'(u), v\right) &= \left(j'(u) + z, v\right),\\
\left(F'(U_h), v\right) &= \left(j'(U_h) + z(U_h), v\right),\\
\left(F_h'(U_h), v\right) &= \left(j'(U_h) + \widetilde{Z}_h, v\right).
\end{aligned} \tag{3.2}$$

It is clear that F and F_h are well defined and continuous on K and K_h^i. Also the functional F_h can be naturally extended on K. Then (2.4) and (2.19) can be represented as

$$\min_{u \in K} \{F(u)\}, \tag{3.3}$$

$$\min_{U_h \in K_h^i} \{F_h(U_h)\}. \tag{3.4}$$

In many application, $F(\cdot)$ is uniform convex near the solution u. The convexity of $F(\cdot)$ is closely related to the second order sufficient conditions of the optimal control problems, which are assumed in many studies on numerical methods of the problem. For instance, in many applications, $u \to g_1(\mathcal{M}_1 U)$ and $u \to g_2(\mathcal{M}_2 U)$ are convex. Then there is a constant $c > 0$, independent of h, such that

$$\int_0^T \left(F'(u) - F'(U_h), u - U_h\right)_U \geq c \|u - U_h\|^2_{L^2(J; L^2(\Omega))}. \tag{3.5}$$

Theorem 3.1. *Let u and U_h be the solutions of* (3.3) *and* (3.4), *respectively. Assume that $K_h^i \subset K$. In addition, assume that $(F_h'(U_h))|_\tau \in H^s(\tau)$, for all $\tau \in \mathcal{T}_h$, $(s = 0, 1)$, and there is a $v_h \in K_h^i$ such that*

$$\left|\left(F_h'(U_h), v_h - u\right)\right| \leq C \sum_{\tau \in \mathcal{T}_h} h_\tau \left\|F_h'(U_h)\right\|_{H^s(\tau)} \|u - U_h\|^s_{L^2(\tau)}. \tag{3.6}$$

Then one has

$$\|u - U_h\|^2_{L^2(J; L^2(\Omega))} \leq C\eta_1^2 + C\left\|z(U_h) - \widetilde{Z}_h\right\|^2_{L^2(J; L^2(\Omega))}, \tag{3.7}$$

where

$$\eta_1^2 = \int_0^T \sum_{\tau \in \mathcal{T}_h} h_\tau^{1+s} \left\|j'(U_h) + \widetilde{Z}_h\right\|^{1+s}_{H^1(\tau)}. \tag{3.8}$$

Proof. It follows from (3.3) and (3.4) that

$$\begin{aligned}&\int_0^T \left(F'(u), u - v\right) \leq 0, \quad \forall v \in K,\\&\int_0^T \left(F_h'(U_h), U_h - v_h\right) \leq 0, \quad \forall v_h \in K_h^i \subset K.\end{aligned} \tag{3.9}$$

Then it follows from assumptions (3.5), (3.6) and Schwartz inequality that

$$\begin{aligned}c\|u - U_h\|^2_{L^2(J;L^2(\Omega))}&\\ &\leq \int_0^T \left(F'(u) - F'(U_h), u - U_h\right)\\ &\leq \int_0^T \left\{\left(F_h'(U_h), v_h - u\right) + \left(F_h'(U_h) - F'(U_h), u - U_h\right)\right\}\\ &\leq C\int_0^T \left\{\sum_{\tau\in\mathcal{T}_h} h_\tau^{1+s}\|F_h'(U_h)\|^{1+s}_{H^s(\tau)} + \|F_h'(U_h) - F'(U_h)\|^2_{L^2(\Omega)}\right\}\\ &\quad + \frac{c}{2}\|u - U_h\|^2_{L^2(J;L^2(\Omega))}.\end{aligned} \tag{3.10}$$

It is not difficult to show

$$F_h'(U_h) = j'(U_h) + \widetilde{Z}_h, \qquad F'(U_h) = j'(U_h) + z(U_h), \tag{3.11}$$

where $z(U_h)$ is defined in (2.39)–(2.44). Thanks to (3.11), it is easy to derive

$$\|F_h'(U_h) - F'(U_h)\|_{L^2(\Omega)} \leq C\left\|\widetilde{Z}_h - z(U_h)\right\|_{L^2(\Omega)}. \tag{3.12}$$

Then by estimates (3.10) and (3.12) we can prove the requested result (3.7). □

In order to estimate the a posteriori error of the mixed finite element approximation solution, we will use the following dual equations:

$$\begin{aligned}-\varphi_t - \operatorname{div}\left(A\nabla\varphi\right) = G, &\quad x \in \Omega,\ t \in (0, T],\\ \varphi|_{\partial\Omega} = 0, &\quad t \in J,\\ \varphi(x, T) = 0, &\quad x \in \Omega,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\psi_t - \operatorname{div}\left(A^*\nabla\psi\right) = G, &\quad x \in \Omega,\ t \in (0, T],\\ \psi|_{\partial\Omega} = 0, &\quad t \in J,\\ \psi(x, 0) = 0, &\quad x \in \Omega.\end{aligned} \tag{3.14}$$

The following well-known stability results are presented in [28].

Lemma 3.2. *Let φ and ψ be the solutions of* (3.13), *and* (3.14), *respectively. Then, for $v = \varphi$ or $v = \psi$,*

$$\begin{aligned} \|v\|_{L^\infty(J;L^2(\Omega))} &\le C\|G\|_{L^2(J;L^2(\Omega))}, \\ \|\nabla v\|_{L^2(J;L^2(\Omega))} &\le C\|G\|_{L^2(J;L^2(\Omega))}, \\ \left\|D^2 v\right\|_{L^2(J;L^2(\Omega))} &\le C\|G\|_{L^2(J;L^2(\Omega))}, \\ \|v_t\|_{L^2(J;L^2(\Omega))} &\le C\|G\|_{L^2(J;L^2(\Omega))}, \end{aligned} \tag{3.15}$$

where $D^2 v = \partial^2 v/\partial x_i \partial x_j$, $1 \le i,\ j \le 2$.

Now, we are able to derive the main result.

Theorem 3.3. *Let $(Y_h, P_h, Z_h, Q_h, U_h)$ and $(y(U_h), \mathbf{p}(U_h), z(U_h), \mathbf{q}(U_h), U_h)$ be the solutions of* (2.32)–(2.38) *and* (2.39)–(2.44). *Then,*

$$\|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))} + \|P_h - \mathbf{p}(U_h)\|^2_{L^2(J;L^2(\Omega))} \le C\eta_2^2, \tag{3.16}$$

where

$$\begin{aligned} \eta_2^2 = {} & \left\|Y_{ht} + \operatorname{div}\widehat{P}_h - \widehat{f} - U_h\right\|^2_{L^2(J;L^2(\Omega))} \\ & + \|(Y_h - y(U_h))(x,0)\|^2_{L^2(\Omega)} + \left\|\widehat{f} - f\right\|^2_{L^2(J;L^2(\Omega))} \\ & + \left\|\left(\widehat{Y}_h - Y_h\right)_t\right\|^2_{L^2(J;L^2(\Omega))} + \left\|\widehat{P}_h - P_h\right\|^2_{L^2(J;H(\operatorname{div};\Omega))}. \end{aligned} \tag{3.17}$$

Proof. Letting φ be the solution of (3.13) with $G = Y_h - y(U_h)$, we infer

$$\begin{aligned} & \|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))} \\ & \quad = \int_0^T (Y_h - y(U_h), F)\,dt \\ & \quad = \int_0^T (Y_h - y(U_h), -\varphi_t - \operatorname{div}(A\nabla\varphi))\,dt \\ & \quad = \int_0^T (((Y_h - y(U_h))_t, \varphi) - (P_h - \mathbf{p}(U_h), \nabla\varphi))\,dt \\ & \qquad + \|(Y_h - y(U_h))(x,0)\|^2_{L^2(\Omega)} \\ & \quad = \int_0^T ((Y_{ht} - y_t(U_h), \varphi) + (\operatorname{div}(P_h - \mathbf{p}(U_h)), \varphi))\,dt \\ & \qquad + \|(Y_h - y(U_h))(x,0)\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.18}$$

Then it follows from (2.39)-(2.40) that

$$\begin{aligned}
&\|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))} \\
&\quad = \int_0^T \Big((Y_{ht}, \varphi) + \big(\operatorname{div} \widehat{P}_h, \varphi\big) - (y_t(U_h), \varphi) - (\operatorname{div} \mathbf{p}(U_h), \varphi) + \big(\operatorname{div}\big(P_h - \widehat{P}_h\big), \varphi\big)\Big) dt \\
&\qquad + \|(Y_h - y(U_h))(x,0)\|^2_{L^2(\Omega)} \\
&\quad = \int_0^T \Big(\big(Y_{ht} + \operatorname{div} \widehat{P}_h - \widehat{f} - U_h, \varphi\big) + \big(\widehat{f} - f, \varphi\big) + \big(\operatorname{div}\big(P_h - \widehat{P}_h\big), \varphi\big)\Big) dt \\
&\qquad + \|(Y_h - y(U_h))(x,0)\|^2_{L^2(\Omega)} \\
&\quad = \int_0^T \Big(\big(Y_{ht} + \operatorname{div} \widehat{P}_h - \widehat{f} - U_h, \varphi\big) + \big(\widehat{f} - f, \varphi\big) + \big(\operatorname{div}\big(P_h - \widehat{P}_h\big), \varphi\big)\Big) dt \\
&\qquad + \|(Y_h - y(U_h))(x,0)\|^2_{L^2(\Omega)} \\
&\quad \le C(\delta) \Big\| Y_{ht} + \operatorname{div} \widehat{P}_h - \widehat{f} - U_h \Big\|^2_{L^2(J;L^2(\Omega))} + C(\delta) \Big\| \widehat{f} - f \Big\|^2_{L^2(J;L^2(\Omega))} \\
&\qquad + C(\delta) \Big\| \widehat{P}_h - P_h \Big\|^2_{L^2(J;H(\operatorname{div};\Omega))} + C(\delta) \Big\| \widehat{Y}_h - Y_h \Big\|^2_{L^2(J;L^2(\Omega))} \\
&\qquad + \|(Y_h - y(U_h))(x,0)\|^2_{L^2(\Omega)} + C\delta \|\varphi\|^2_{L^2(J;L^2(\Omega))},
\end{aligned} \tag{3.19}$$

where and after, δ is an arbitrary positive number, $C(\delta)$ is the constant dependent on δ^{-1}.

Now, we are in the position of estimating the error $\|P_h - \mathbf{p}(U_h)\|^2_{L^2(J;L^2(\Omega))}$. First, we derive from (2.32)-(2.33) and (2.39)-(2.40) the following useful error equations:

$$\Big(A^{-1}\big(\widehat{P}_h - \mathbf{p}(U_h)\big), \mathbf{v}_h\Big) - \Big(\widehat{Y}_h - y(u_h), \operatorname{div} \mathbf{v}_h\Big) = 0, \tag{3.20}$$

$$\Big(\big(\widehat{Y}_h - y(U_h)\big)_t, w_h\Big) + \Big(\operatorname{div}\big(\widehat{P}_h - \mathbf{p}(U_h)\big), w_h\Big) = \Big(\widehat{f} - f, w_h\Big) - \Big(\big(Y_h - \widehat{Y}_h\big)_t, w_h\Big), \tag{3.21}$$

where $\mathbf{v}_h \in \mathbf{V}_h$, $w_h \in W_h$. Choose $v_h = \widehat{P}_h - \mathbf{p}(U_h)$ and $w_h = \widehat{Y}_h - y(U_h)$ as the test functions and add the two relations of (3.20)-(3.21) such that

$$\begin{aligned}
&\Big(A^{-1}\big(\widehat{P}_h - \mathbf{p}(U_h)\big), \widehat{P}_h - \mathbf{p}(U_h)\Big) + \Big(\big(\widehat{Y}_h - y(U_h)\big)_t, \widehat{Y}_h - y(U_h)\Big) \\
&\quad = \Big(\widehat{f} - f, \widehat{Y}_h - y(U_h)\Big) - \Big(\big(Y_h - \widehat{Y}_h\big)_t, \widehat{Y}_h - y(U_h)\Big).
\end{aligned} \tag{3.22}$$

Then, using ϵ-Cauchy inequality, we can find an estimate as follows:

$$c\left\|\widehat{P}_h - \mathbf{p}(U_h)\right\|^2_{L^2(\Omega)} + \left(\left(\widehat{Y}_h - y(U_h)\right)_t, \widehat{Y}_h - y(U_h)\right) \leq C\left\|\widehat{Y}_h - y(U_h)\right\|^2_{L^2(\Omega)} + C\left\|\widehat{f} - f\right\|^2_{L^2(\Omega)} + C\left\|\left(\widehat{Y}_h - Y_h\right)_t\right\|^2_{L^2(\Omega)}. \tag{3.23}$$

Note that

$$\left(\left(\widehat{Y}_h - y(U_h)\right)_t, \widehat{Y}_h - y(U_h)\right) = \frac{1}{2}\frac{\partial}{\partial t}\left\|\widehat{Y}_h - y(U_h)\right\|^2_{L^2(\Omega)}, \tag{3.24}$$

then, using the assumption on A, we verify that

$$c\left\|\widehat{P}_h - \mathbf{p}(U_h)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\left\|\widehat{Y}_h - y(U_h)\right\|^2_{L^2(\Omega)} \leq C\left\|\widehat{Y}_h - y(U_h)\right\|^2_{L^2(\Omega)} + C\left\|\widehat{f} - f\right\|^2_{L^2(\Omega)} + C\left\|\left(\widehat{Y}_h - Y_h\right)_t\right\|^2_{L^2(\Omega)}. \tag{3.25}$$

Integrating (3.25) in time and since $\widehat{Y}_h(0) - y(U_h)(0) = 0$, applying Gronwall's lemma, we can easily obtain the following error estimate:

$$\left\|\widehat{P}_h - \mathbf{p}(U_h)\right\|_{L^2(J;L^2(\Omega))} + \left\|\widehat{Y}_h - y(U_h)\right\|_{L^\infty(J;L^2(\Omega))} \leq C\left\|\widehat{f} - f\right\|_{L^2(J;L^2(\Omega))} + C\left\|\left(\widehat{Y}_h - Y_h\right)_t\right\|_{L^2(J;L^2(\Omega))}. \tag{3.26}$$

Using the triangle inequality and (3.26), we deduce that

$$\|P_h - \mathbf{p}(U_h)\|_{L^2(J;L^2(\Omega))} \leq C\left(\left\|\widehat{f} - f\right\|_{L^2(J;L^2(\Omega))} + \left\|\left(\widehat{Y}_h - Y_h\right)_t\right\|_{L^2(J;L^2(\Omega))} + \left\|\widehat{P}_h - P_h\right\|_{L^2(J;L^2(\Omega))}\right). \tag{3.27}$$

Then letting δ be small enough, it follows from (3.18)–(3.27) that

$$\|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))} + \|P_h - \mathbf{p}(U_h)\|^2_{L^2(J;L^2(\Omega))} \leq C\eta_2^2. \tag{3.28}$$

This proves (3.16). □

Similarly, letting ψ be the solution of (3.14) with $G = Z_h - z(U_h)$, with the aid of (2.26)-(2.27), (2.42)-(2.43), we can conclude the following.

Theorem 3.4. *Let* $(Y_h, P_h, Z_h, Q_h, U_h)$ *and* $(y(U_h), \mathbf{p}(U_h), z(U_h), \mathbf{q}(U_h), U_h)$ *be the solutions of* (2.32)–(2.38) *and* (2.39)–(2.44). *Then,*

$$\|Z_h - z(U_h)\|^2_{L^2(J;L^2(\Omega))} + \|Q_h - \mathbf{q}(U_h)\|^2_{L^2(J;L^2(\Omega))} \le C\left(\eta_2^2 + \eta_3^2\right), \tag{3.29}$$

where

$$\begin{aligned}\eta_3^2 = &\left\|g_2'\left(\widehat{Y}_h\right) + Z_{ht} - \operatorname{div}\left(\widetilde{Q}_h\right)\right\|^2_{L^2(J;L^2(\Omega))}\\ &+ \left\|\widetilde{Z}_h - Z_h\right\|^2_{L^2(J;L^2(\Omega))} + \left\|\left(\widetilde{Z}_h - Z_h\right)_t\right\|^2_{L^2(J;L^2(\Omega))}\\ &+ \left\|Y_h - \widehat{Y}_h\right\|^2_{L^2(J;L^2(\Omega))} + \left\|\widetilde{Q}_h - Q_h\right\|^2_{L^2(J;H(\operatorname{div};\Omega))}.\end{aligned} \tag{3.30}$$

Let $(\mathbf{p}, y, \mathbf{q}, z, u)$ and $(P_h, Y_h, Q_h, Z_h, U_h)$ be the solutions of (2.8)–(2.14) and (2.32)–(2.38), respectively. We decompose the errors as follows:

$$\begin{aligned}\mathbf{p} - P_h &= \mathbf{p} - \mathbf{p}(U_h) + \mathbf{p}(U_h) - P_h := \epsilon_1 + \varepsilon_1,\\ y - Y_h &= y - y(U_h) + y(U_h) - Y_h := r_1 + e_1,\\ \mathbf{q} - Q_h &= \mathbf{q} - \mathbf{q}(U_h) + \mathbf{q}(U_h) - Q_h := \epsilon_2 + \varepsilon_2,\\ z - Z_h &= z - z(U_h) + z(U_h) - Z_h := r_2 + e_2.\end{aligned} \tag{3.31}$$

From (2.8)–(2.13) and (2.39)–(2.44), we derive the error equations:

$$\left(A^{-1}\epsilon_1, \mathbf{v}\right) - (r_1, \operatorname{div}\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{3.32}$$

$$(r_{1t}, w) + (\operatorname{div}\epsilon_1, w) = (u - U_h, w), \quad \forall w \in W, \tag{3.33}$$

$$\left(A^{-1}\epsilon_2, \mathbf{v}\right) - (r_2, \operatorname{div}\mathbf{v}) = -\left(g_1'(\mathbf{p}) - g_1'(\mathbf{p}(U_h))\mathbf{v}\right), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{3.34}$$

$$(r_{2t}, w) + (\operatorname{div}\epsilon_2, w) = \left(g_2'(y) - g_2'(y(U_h)), w\right), \quad \forall w \in W. \tag{3.35}$$

Theorem 3.5. *There is a constant* $C > 0$, *independent of* h, *such that*

$$\|\epsilon_1\|_{L^2(J;L^2(\Omega))} + \|r_1\|_{L^2(J;L^2(\Omega))} \le C\|u - U_h\|_{L^2(J;L^2(\Omega))}, \tag{3.36}$$

$$\|\epsilon_2\|_{L^2(J;L^2(\Omega))} + \|r_2\|_{L^2(J;L^2(\Omega))} \le C\|u - U_h\|_{L^2(J;L^2(\Omega))}. \tag{3.37}$$

Proof. Part I

Choose $\mathbf{v} = \epsilon_1$ and $w = r_1$ as the test functions and add the two relations of (3.32)-(3.33), then we have

$$\left(A^{-1}\epsilon_1, \epsilon_1\right) + (r_{1t}, r_1) = (u - U_h, r_1). \tag{3.38}$$

Then, using the Cauchy inequality, we can find an estimate as follows:

$$\left(A^{-1}\epsilon_1, \epsilon_1\right) + (r_{1t}, r_1) \leq C\left(\|r_1\|^2_{L^2(\Omega)} + \|u - U_h\|^2_{L^2(\Omega)}\right). \tag{3.39}$$

Note that

$$(r_{1t}, r_1) = \frac{1}{2}\frac{\partial}{\partial t}\|r_1\|^2_{L^2(\Omega)}, \tag{3.40}$$

then, using the assumption on A, we can obtain that

$$\|\epsilon_1\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|r_1\|^2_{L^2(\Omega)} \leq C\left(\|r_1\|^2_{L^2(\Omega)} + \|u - U_h\|^2_{L^2(\Omega)}\right). \tag{3.41}$$

Integrating (3.41) in time and since $r_1(0) = 0$, applying the Gronwall's lemma, we can easily obtain the following error estimate

$$\|\epsilon_1\|^2_{L^2(J;L^2(\Omega))} + \|r_1\|^2_{L^2(J;L^2(\Omega))} \leq C\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \tag{3.42}$$

This implies (3.36).

Part II

Similarly, choose $v = \epsilon_2$ and $w = r_2$ as the test functions and add the two relations of (3.34)-(3.35), then we can obtain that

$$\left(A^{-1}\epsilon_2, \epsilon_2\right) + (r_{2t}, r_2) = \left(g_2'(y) - g_2'(y(U_h)), r_2\right) - \left(g_1'(\mathbf{p}) - g_1'(\mathbf{p}(U_h)), \epsilon_2\right). \tag{3.43}$$

Then, using the Cauchy inequality, we can find an estimate as follows:

$$\left(A^{-1}\epsilon_2, \epsilon_2\right) + (r_{2t}, r_2) \leq C\left(\|r_1\|^2_{L^2(\Omega)} + \|r_2\|^2_{L^2(\Omega)} + \|\epsilon_1\|^2_{L^2(\Omega)}\right) + \frac{c}{2}\|\epsilon_2\|^2_{L^2(\Omega)}. \tag{3.44}$$

Note that

$$(r_{2t}, r_2) = \frac{1}{2}\frac{\partial}{\partial t}\|r_2\|^2_{L^2(\Omega)}, \tag{3.45}$$

then, using the assumption on A, we verify that

$$\|\epsilon_2\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|r_2\|^2_{L^2(\Omega)} \le C\Big(\|r_1\|^2_{L^2(\Omega)} + \|r_2\|^2_{L^2(\Omega)} + \|\epsilon_1\|^2_{L^2(\Omega)}\Big). \tag{3.46}$$

Integrating (3.46) in time and since $r_2(T) = 0$, applying the Gronwall's lemma, we can easily obtain the following error estimate

$$\|\epsilon_2\|^2_{L^2(J;L^2(\Omega))} + \|r_2\|^2_{L^2(J;L^2(\Omega))} \le C\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \tag{3.47}$$

Then (3.37) follows from (3.47) and the previous statements immediately. □

Collecting Theorems 3.1–3.5, we can derive the following results.

Theorem 3.6. *Let* $(\mathbf{p}, y, \mathbf{q}, z, u)$ *and* $(P_h, Y_h, Q_h, Z_h, U_h)$ *be the solutions of* (2.8)–(2.14) *and* (2.32)–(2.38)*, respectively. In addition, assume that* $(j'(U_h) + \tilde{Z}_h)|_\tau \in H^s(\tau)$*, for all* $\tau \in \mathcal{T}_h$*,* $(s = 0, 1)$*, and that there is a* $v_h \in K_h$ *such that*

$$\left|\Big(j'(U_h) + \tilde{Z}_h, v_h - u\Big)\right| \le C\sum_{\tau\in\mathcal{T}_h} h_\tau \left\|j'(U_h) + \tilde{Z}_h\right\|_{H^s(\tau)} \|u - U_h\|^s_{L^2(\tau)}. \tag{3.48}$$

Then one has that, for all $t \in (0, T]$*,*

$$\begin{aligned}&\|u - U_h\|^2_{L^2(J;L^2(\Omega))} + \|y - Y_h\|^2_{L^2(J;L^2(\Omega))} + \|\mathbf{p} - P_h\|^2_{L^2(J;L^2(\Omega))}\\ &\quad + \|z - Z_h\|^2_{L^2(J;L^2(\Omega))} + \|\mathbf{q} - Q_h\|^2_{L^2(J;L^2(\Omega))} \le C\sum_{i=1}^{3}\eta_i^2,\end{aligned} \tag{3.49}$$

where η_1*,* η_2*, and* η_3 *are defined in Theorems 3.1–3.4.*

4. An Adaptive Algorithm

In the section, we introduce an adaptive algorithm to guide the mesh refine process. A posteriori error estimates which have been derived in Section 3 are used as an error indicator to guide the mesh refinement in adaptive finite element method.

Now, we discuss the adaptive mesh refinement strategy. The general idea is to refine the mesh such that the error indicator like η is equally distributed over the computational mesh. Assume that an a posteriori error estimator η has the form of $\eta^2 = \sum_{\tau_i}\eta^2_{\tau_i}$, where τ_i is the finite elements. At each iteration, an average quantity of all $\eta^2_{\tau_i}$ is calculated, and each $\eta^2_{\tau_i}$ is then compared with this quantity. The element τ_i is to be refined if $\eta^2_{\tau_i}$ is larger than this quantity. As $\eta^2_{\tau_i}$ represents the total approximation error over τ_i, this strategy makes sure that higher density of nodes is distributed over the area where the error is higher.

Based on this principle, we define an adaptive algorithm of the optimal control problems (2.1) as follows.

Starting from initial triangulations $\mathcal{T}_{h_0}$ of Ω, we construct a sequence of refined triangulation $\mathcal{T}_{h_j}$ as follows. Given $\mathcal{T}_{h_j}$, we compute the solutions $(P_h, Y_h, Q_h, Z_h, U_h)$ of the system (2.32)–(2.38) and their error estimator

$$\begin{aligned}
\eta_\tau^2 = {} & \int_0^T \sum_{\tau\in\mathcal{T}_h} h_\tau^{1+s} \left\| j'(U_h) + \widetilde{Z}_h \right\|_{H^1(\tau)}^{1+s} + \left\| Y_{ht} + \operatorname{div} \widehat{P}_h - \widehat{f} - U_h \right\|_{L^2(J;L^2(\tau))}^2 \\
& + \left\| (Y_h - y(U_h))(x,0) \right\|_{L^2(\tau)}^2 + \left\| \widehat{f} - f \right\|_{L^2(J;L^2(\tau))}^2 \\
& + \left\| \left(\widehat{Y}_h - Y_h\right)_t \right\|_{L^2(J;L^2(\tau))}^2 + \left\| \widehat{P}_h - P_h \right\|_{L^2(J;H(\mathrm{div};\tau))}^2 \\
& + \left\| g_2'\left(\widehat{Y}_h\right) + Z_{ht} - \operatorname{div}\left(\widetilde{Q}_h\right) \right\|_{L^2(J;L^2(\tau))}^2 \\
& + \left\| \widetilde{Z}_h - Z_h \right\|_{L^2(J;L^2(\tau))}^2 + \left\| \left(\widetilde{Z}_h - Z_h\right)_t \right\|_{L^2(J;L^2(\tau))}^2 \\
& + \left\| Y_h - \widehat{Y}_h \right\|_{L^2(J;L^2(\tau))}^2 + \left\| \widetilde{Q}_h - Q_h \right\|_{L^2(J;H(\mathrm{div};\tau))}^2, \\
E_j = {} & \sum_{\tau\in\mathcal{T}_h} \eta_\tau^2.
\end{aligned} \tag{4.1}$$

Then we adopt the following mesh refinement strategy. All the triangles $\tau \in \mathcal{T}_{h_j}$ satisfying $\eta_\tau^2 \geq (\alpha E_j / n)$ are divided into four new triangles in $\mathcal{T}_{h_{j+1}}$ by joining the midpoints of the edges, where n is the numbers of the elements of $\mathcal{T}_{h_j}$, α is a given constant. In order to maintain the new triangulation $\mathcal{T}_{h_{j+1}}$ to be regular and conformal, some additional triangles need to be divided into two or four new triangles depending on whether they have one or more neighbor which have refined. Then we obtain the new mesh $\mathcal{T}_{h_{j+1}}$. The above procedure will continue until $E_j \leq \mathrm{tol}$, where tol is a given tolerance error.

5. Numerical Example

The purpose of this section is to illustrate our theoretical results by numerical example. Our numerical example is the following optimal control problem:

$$\min_{u\in K\subset U} \left\{ \frac{1}{2} \int_0^T \left(\|\mathbf{p} - \mathbf{p}_d\|^2 + \|y - y_d\|^2 + \|u - u_0\|^2 \right) dt \right\}, \tag{5.1}$$

$$y_t + \operatorname{div} \mathbf{p} = f + u, \qquad \mathbf{p} = -\nabla y, \quad x \in \Omega, \; y|_{\partial\Omega} = 0, \; y(x,0) = 0, \tag{5.2}$$

$$-z_t + \operatorname{div} \mathbf{q} = y - y_d, \qquad \mathbf{q} = -(\nabla z + \mathbf{p} - \mathbf{p}_d), \quad x \in \Omega, \; z|_{\partial\Omega} = 0, \; z(x,T) = 0. \tag{5.3}$$

In our example, we choose the domain $\Omega = [0,1] \times [0,1]$. Let Ω be partitioned into $\mathcal{T}_h$ as described Section 2. For the constrained optimization problem,

$$\min_{u \in K} F(u), \tag{5.4}$$

where $F(u)$ is a convex functional on U and $K = \{u \in L^2(\Omega) : u \geq 0 \;\; a.e. \;\text{ in }\; \Omega \times J\}$, the iterative scheme reads $(n = 0, 1, 2, \ldots)$

$$\begin{gathered} b\big(u_{n+(1/2)}, v\big) = b(u_n, v) - \rho_n\big(F'(u_n), v\big), \quad \forall v \in U, \\ u_{n+1} = P_K^b\big(u_{n+(1/2)}\big), \end{gathered} \tag{5.5}$$

where $b(\cdot,\cdot)$ is a symmetric and positive definite bilinear form such that there exist constants c_0 and c_1 satisfying

$$\begin{gathered} |b(u,v)| \leq c_1 \|u\|_U \|v\|_U, \quad \forall u, v \in U, \\ b(u,u) \geq c_0 \|u\|_U^2, \end{gathered} \tag{5.6}$$

and the projection operator $P_K^b U \rightarrow K$ is defined. For given $w \in U$ find $P_K^b w \in K$ such that

$$b\Big(P_K^b w - w, P_K^b w - w\Big) = \min_{u \in K} b(u - w, u - w). \tag{5.7}$$

The bilinear form $b(\cdot,\cdot)$ provides suitable preconditioning for the projection algorithm. An application of (5.5) to the discretized parabolic optimal control problem yields the following algorithm:

$$\begin{gathered} b\big(u_{n+(1/2)}, v_h\big) = b(u_n, v_h) - \rho_n \int_0^T (u_n + z_n, v_h), \quad \forall v_h \in U_h, \\ \int_0^T \big((\mathbf{p}_n, \mathbf{v}_h) - \big(y_n, \operatorname{div} \mathbf{v}_h\big)\big) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \int_0^T \big((y_{nt}, w_h) + (\operatorname{div} \mathbf{p}_n, w_h)\big) + \big(y_n(0), w(0)\big) = \int_0^T \big(f + u_n, w_h\big), \quad \forall w_h \in W_h, \\ \int_0^T ((\mathbf{q}_n, \mathbf{v}_h) - (z_n, \operatorname{div} \mathbf{v}_h)) = -\int_0^T \big(\mathbf{p}_n - \mathbf{p}_d, v_h\big), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \int_0^T (-(z_{nt}, w_h) + (\operatorname{div} \mathbf{q}_n, w_h)) + (z_n(T), w_h(T)) = \int_0^T \big(y_n - y_d, w_h\big), \quad \forall w_h \in W_h, \\ u_{n+1} = P_K^b\big(u_{n+(1/2)}\big), \quad u_{n+(1/2)}, u_n \in U_h. \end{gathered} \tag{5.8}$$

The main computational effort is to solve the four state and costate equations and to compute the projection $P_K^b u_{n+(1/2)}$. In this paper we use a fast algebraic multigrid solver to solve the state and costate equations. Then it is clear that the key to saving computing time is how to compute $P_K^b u_{n+(1/2)}$ efficiently. If one uses the C^0 finite elements to approximate to the control, then one has to solve a global variational inequality, via, for example, semismooth Newton method. The computational load is not trivial. For the piecewise constant elements, $K_h = \{u_h : u_h \geq 0\}$ and $b(u,v) = (u,v)_U$, then

$$P_K^b u_{n+(1/2)}|_\tau = \max\left(0, \operatorname{avg}\left(u_{n+(1/2)}\right)|_\tau\right), \tag{5.9}$$

where $\operatorname{avg}(u_{n+(1/2)})|_\tau$ is the average of $u_{n+(1/2)}$ over τ.

In solving our discretized optimal control problem, we use the preconditioned projection gradient method with $b(u,v) = (u,v)_U$ and a fixed step size $\rho = 0.8$. We now briefly describe the solution algorithm to be used for solving the numerical examples in this section.

5.1. Algorithm

(1) Solve the discretized optimization problem with the projection gradient method on the current meshes, and calculate the error estimators η_i.

(2) Adjust the meshes using the estimators, and update the solution on new meshes, as described.

Now, we present a numerical example to illustrate our theoretical results.

Example 5.1. We choose the state function by

$$y(x_1,x_2) = \sin \pi x_1 \sin \pi x_2 \sin \pi t \tag{5.10}$$

and the function $f(x_1,x_2) = y_t + \operatorname{div} \mathbf{p} - u$ with

$$\begin{aligned} \mathbf{p}(x_1,x_2) &= -(\pi \cos \pi x_1 \sin \pi x_2 \sin \pi t, \pi \sin \pi x_1 \cos \pi x_2 \sin \pi t), \\ \mathbf{q}(x_1,x_2) &= \mathbf{p}_d(x_1,x_2) = \mathbf{p}(x_1,x_2). \end{aligned} \tag{5.11}$$

The costate function can be chosen as

$$z(x_1,x_2) = \sin \pi x_1 \sin \pi x_2 \sin \pi t. \tag{5.12}$$

It follows from (5.2)-(5.3) that

$$y_d(x_1,x_2) = y + z_t - \operatorname{div} \mathbf{q}. \tag{5.13}$$

Table 1: Numerical results on uniform and adaptive meshes.

	On uniform mesh			On adaptive mesh		
	u	y	z	u	y	z
Nodes	8097	8097	8097	1089	1393	1393
Sides	23968	23968	23968	2825	3819	3819
Elements	15872	15872	15872	6348	2423	2423
Dofs	15872	15872	15872	6348	2423	2423
Total L^2 error	6.915e − 03	4.065e − 3	4.018e − 3	6.527e − 03	4.346e − 3	4.323e − 3

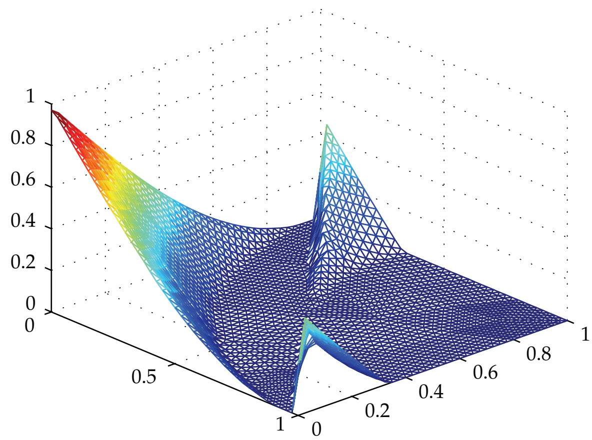

Figure 1: The profile of the control solution u at $t = 0.25$.

We assume that

$$\lambda = \begin{cases} 0.5, & x_1 + x_2 > 1.0, \\ 0.0, & x_1 + x_2 \leq 1.0, \end{cases}$$
$$u_0(x_1, x_2) = 1 - \sin\frac{\pi x_1}{2} - \sin\frac{\pi x_2}{2} + \lambda. \tag{5.14}$$

Thus, the control function is given by

$$u(x_1, x_2) = \max(u_0 - z, 0). \tag{5.15}$$

In this example, the optimal control has a strong discontinuity, introduced by u_0. The exact solution for the control u is plotted in Figure 1. The control function u is discretized by piecewise constant functions, whereas the state $(y, \mathbf{p})$ and the costate $(z, \mathbf{q})$ were approximation by the lowest-order Raviart-Thomas mixed finite elements. In Table 1, numerical results of u, y, and z on uniform and adaptive meshes are presented. It can be founded that the adaptive meshes generated using our error indicators can save substantial computational work, in comparison with the uniform meshes. At the same time, for the discontinuous control variable u, the accuracy has been improved obviously from the uniform meshes to the adaptive meshes in Table 1.

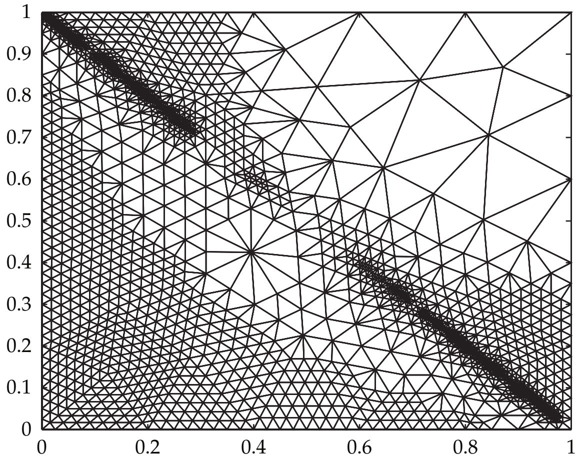

Figure 2: The adaptive meshes of the control solution u at $t = 0.25$.

In Figure 2, the adaptive mesh for u at $t = 0.25$ is shown. In the computing, we use $\eta_1 - \eta_3$ as the error indicators in the adaptive finite element method. It can be founded that the mesh adapts well to be neighborhood of the discontinuity, and a higher density of node points is indeed distributed along them.

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