Research Article

A Note on Stabilization of Discrete Nonlinear Systems

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We deal with the stabilization problem of discrete nonlinear systems. We construct a control Lyapunov function on discrete nonlinear systems. Then, we present a new method to construct a continuous state feedback law.

1. Introduction

For nonlinear dynamical systems, Lyapunov function-based methods play a vital role in stability analysis since the Lyapunov functions have been used to design stabilizing feedback laws that render asymptotically controllable systems ISS to actuator errors and small observation noise [1]. This is an important and challenging problem for the general class of nonlinear systems. Artstein [2] proves that, for smooth control-affine systems evolving on R^n with controls in R^m, there exists a globally stabilizing feedback control law continuously on $R^n \setminus \{0\}$ if the system has a smooth control Lyapunov function. Sontag [3] provided an explicit proof of Artstein's Theorem which uses a "universal formula" for the stabilizing controller. Shahmansoorian et al. [4] presented a new stabilizing control law with respect to a control Lyapunov function. Many researchers dealt with the effects of control Lyapunov functions and obtained considerably meaningful results, just as in [5–12].

The stabilization of discrete systems is studied by means of Lyapunov functions in this paper. Xiushan et al. [13] also studied discrete system, but the control Lyapunov function and the control law they give are not easy to get. We will present an explicit control Lyapunov function and feedback law stabilizing a discrete-time control system. Furthermore, a new way to construct a continuous state feedback law is designed.

2. Main Results

The problem amounts to find a function $u : R^n \rightarrow R^m$ with $u(0) = 0$, such that the closed-loop system

$$x_{k+1} = f(x_k, u(x_k)) \tag{2.1}$$

is asymptotically stable at the origin. Of course we assume that $f(0,0) = 0$, but the origin is not asymptotically stable equilibrium for the unforced system $x_{k+1} = f(x_k, 0)$.

Equation (2.1) describes the behavior of the system in the sense that its evolution depends not only on an initial state, but also on a sequence of input signal. In this paper, we address the following discrete-time nonlinear control systems of the form:

$$x_{k+1} = f(x_k) + u_k g(x_k), \tag{2.2}$$

which is said to be affine-in-control, where f and g are continuous on R^n and $f(0) = 0$, $\{u_k\} \in U \subset R$ (a sequence of input signal). The system (2.2) is globally asymptotically stable at the origin if there exists a map $x \rightarrow u(x)$, such that the system

$$x_{k+1} = f(x_k) + u(x_k)g(x_k) \tag{2.3}$$

is globally asymptotically stable at $0 \in R^n$.

Definition 2.1. A smooth, proper, and positive definite function V mapping R^n into R is said to be a control Lyapunov for the discrete system (2.2) if and only if

$$\inf_{\{u_k\} \subset U} (\Delta V(x_k) = V(x_{k+1}) - V(x_k)) < 0, \tag{2.4}$$

for all $x \in R^n \setminus \{0\}$.

Remark 2.2. Suppose that $V(x) = x^T P x$, then

$$\Delta V(x_k) = (f(x_k) + u_k g(x_k))^T P(f(x_k) + u_k g(x_k)) - x_k^T P x_k \tag{2.5}$$

is a function about x_k and u_k.

Theorem 2.3. *Suppose that* $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, *and let*

$$\begin{aligned} \Omega_0 &= \left\{ x \in R^2 \setminus \{0\} \mid g(x) = 0 \right\}, \\ \Omega_1 &= \left\{ x \in R^2 \setminus \{0\} \mid g(x) \neq 0 \right\}. \end{aligned} \tag{2.6}$$

Assume that there exist constants $A > 0$, $B > 0$, such that $\sup_{x \in R^2}(\|f\|/\|x\|) = A < 1$, and

$$\sup_{x \in R^2} \frac{f_1 g_2 - f_2 g_1}{\left(g_1^2 + g_2^2\right)\left(x_1^2 + x_2^2\right)} = B < +\infty, \tag{2.7}$$

then there exists a positive definite matrix P, such that $V(x) = x^T P x$ is a control Lyapunov function of system (2.2).

Proof. For $(x, u) \in R^2 \times R$, let

$$h_x(u) = u^2 g^T P g + 2u f^T P g + f^T P f - x^T P x. \tag{2.8}$$

Obviously, $\Delta V = V(x_{k+1}) - V(x_k) = h_{x_k}(u_k)$.

(i) If $x_k \in \Omega_0$, ΔV does not depend on u_k, then we just need

$$f^T P f - x^T P x < 0. \tag{2.9}$$

Suppose $\lambda_{\max}, \lambda_{\min} > 0$ are the eigenvalues of P, then

$$f^T P f - x^T P x \le \lambda_{\max} \|f\|^2 - \lambda_{\min} \|x\|^2 \le (A\lambda_{\max} - \lambda_{\min})\|x\|^2 < 0. \tag{2.10}$$

So, it is easy for us to give a suitable positive definite matrix P such that which satisfies the relationship.

(ii) If $x_k \in \Omega_1$, $g^T(x_k)Pg(x_k) > 0$, for the same reason, there exists $u_k \in R$ such that $h_{x_k}(u_k) < 0$. Let

$$u_k = -\frac{f^T(x_k)Pg(x_k)}{g^T(x_k)Pg(x_k)}. \tag{2.11}$$

An easy computation shows that $\Delta V < 0$ is equal to

$$f^T P\left(g f^T - f g^T\right)Pg + g^T P g x^T P x > 0, \tag{2.12}$$

and let $P = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}$, through a simple computation, we can get

$$g^T P g x^T P x > \lambda_{\min}^4 \|g\|^2 \|x\|^2 > (f_1 g_2 - f_2 g_1)^2 \left(p_1 p_2 - p_3^2\right) = f^T P\left(f g^T - g f^T\right)Pg. \tag{2.13}$$

Thus, we obtain

$$\frac{\lambda_{\min}^4}{\lambda_{\min} \cdot \lambda_{\max}} > \frac{(f_1 g_2 - f_2 g_1)^2}{\|g\|^2 \|x\|^2}. \tag{2.14}$$

So, if

$$\sup_{x \in R^2} \frac{f_1 g_2 - f_2 g_1}{\left(g_1^2 + g_2^2\right)\left(x_1^2 + x_2^2\right)} = B < +\infty, \tag{2.15}$$

we can choose suitable positive definite matrix P.

Then, we can choose a suitable positive definite matrix P satisfies (i) and (ii), such that $V(x) = x^T P x$ is a control Lyapunov function of system (2.2). The proof is end. $\square$

Corollary 2.4. *If $V(x)$ is a control Lyapunov function for the discrete-time control system* (2.2), *then the control*

$$u = \begin{cases} -\dfrac{f^T P g}{g^T P g} & x \in \Omega_1 \\[2mm] 0 & x \in \Omega_0 \end{cases} \tag{2.16}$$

globally asymptotically stabilizes the equilibrium $x = 0$ of the system (2.2).

Proposition 2.5. *Suppose that $g(x) \neq 0$ for any $0 \neq x \in R^2$. $V(x)$ is a control Lyapunov function for the discrete-time system* (2.1), *then the control Law* (2.16) *is smooth and globally asymptotically stabilizes the equilibrium $x = 0$ of the system* (2.1).

Theorem 2.6. *If there exists a quadratic control Lyapunov function for the discrete-time system* (2.2), *there exists a continuous stabilizing feedback law $u : R^n \to R$, and $u(0) = 0$.*

Proposition 2.7. *Consider*

$$H_x(u) = g^2(x)u^2 + 2f(x)g(x)u + f^2(x) - x^2 = 0, \tag{2.17}$$

where $x \in R$, $f, g : R \to R$, $f^2(x) - x^2 \leq 0$.

Assume that

$$u_{1,2} = \frac{-f(x) \pm x \, \text{sgn}\left(xg(x)\right)}{g(x)} \tag{2.18}$$

are the roots of (2.17). *If $u_{1,2}$ only have one discontinuity of the second kind $x = x_0$, then one can construct one continuous function $u(x)$, such that $u(0) = 0$ and $H_x(u(x)) < 0$.*

Proof. Since x_0 is the only second category discontinuous point of the roots of u_1, u_2, so for any $\varepsilon > 0$, there must exist $\delta > 0$, such that $(1/|u_1|), (1/|u_2|) < \varepsilon$ for any $x \in (x_0 - \delta, x_0 + \delta)$ thus, we can let

$$u_1^* = \begin{cases} u_1 & x \notin (x_0 - \delta, x_0 + \delta), \\[2mm] u_1(x_0 + \delta) + \dfrac{(u_1(x_0 + \delta) - u_1(x_0 - \delta))(x - x_0 + \delta)}{2\delta} & x \in [x_0 - \delta, x_0 + \delta]. \end{cases} \tag{2.19}$$

Define $u_2^\star$ in the same way and let

$$u(x) = \frac{u_1^\star + u_2^\star}{2}, \tag{2.20}$$

then it is easy for us to verify that $u(x)$ is what we want to get. □

Proposition 2.8. *Consider*

$$H_x(u) = g^T P g u^2 + 2 f^T P g u + f^T P f - x^T P x = 0, \tag{2.21}$$

where $x \in R^n$, $f, g : R^n \to R^n$, $f^T P f - x^T P x \le 0$.

If x_0 is the only second category discontinuous point of the roots

$$u_{1,2} = \frac{-f^T P g \pm \sqrt{f^T P g f^T P g - g^T P g f^T P f + g^T P g x^T P x}}{g^T P g} \tag{2.22}$$

of (2.21), *then one can construct one continuous function $u(x)$, such that $u(0) = 0$ and $H_x(u(x)) < 0$.*

Proof. For all $\varepsilon > 0$, there must exist $\delta > 0$, such that $(1/|u_1|) < \varepsilon$ for any $x \in B_\delta(x_0)$. Let $m = \inf_{x \in \overline{B}_\delta(x_0)} (1/u_1(x))$, $M = \sup_{x \in \overline{B}_\delta(x_0)} (1/u_1(x))$. Obviously, there exists $x_1, x_2 \in \overline{B}_\delta(x_0)$ satisfying $u_1(x_1) = (1/m)$, $u_1(x_2) = (1/M)$, then we can let

$$u_1^\star = \begin{cases} u_1 & x \notin B_\delta(x_0), \\ \dfrac{1}{M} + (x - x_2)\dfrac{(1/M) - (1/m)}{x_2 - x_1} & x \in \overline{B}_\delta(x_0). \end{cases} \tag{2.23}$$

Using the same way, we can get $u_2^\star$, then we can get what we want to get.

By Proposition 2.8, it is easy for us to prove Theorem 2.6. □

3. Example

Consider a discrete-time nonlinear system

$$x_{k+1} = a x_k^2 + (x_k + b) u_k, \tag{3.1}$$

where $0 < a < 1, b > 0$. Let $V(x) = x^2$, then it is easy for us to obtain

$$H_x(u) = (x + b)^2 u^2 + 2ax^2(x + b)u + a^2 x^4 - x^2,$$

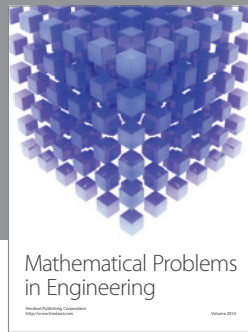$$u_{1,2} = \frac{-ax^2 \pm |x| \operatorname{sgn}(x + b)}{x + b}. \tag{3.2}$$

By Theorem 2.6, we can get that a continuous control law

$$u(x) = \begin{cases} -3ab - 5ax & x \in \left[-\dfrac{3}{2}b, -\dfrac{b}{2}\right] \\[3mm] \dfrac{-ax^2}{x+1} & x \notin \left[-\dfrac{3}{2}b, -\dfrac{b}{2}\right] \end{cases} \tag{3.3}$$

globally asymptotically stabilizes the equilibrium $x = 0$ of the system (2.21).

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