Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 462507, 11 pages
doi:10.1155/2011/462507

Research Article

A Sextuple Product Identity with Applications

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Received 21 February 2011; Revised 20 March 2011; Accepted 24 March 2011

Academic Editor: Ming Li

We get a new proof of a sextuple product identity depending on the Laurent expansion of an analytic function in an annulus. Many identities, including an identity for $(q;q)_\infty^4$, are obtained from this sextuple product identity.

1. Introduction

For convenience, we let $|q| < 1$ throughout the paper. We employ the standard notation

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad (a,b,\ldots,c;q)_\infty = (a;q)_\infty(b;q)_\infty\cdots(c;q)_\infty. \tag{1.1}$$

Series product has been an interesting topic. The Jacobi triple product is one of the most famous series-product identity. We announce it in the following (see, e.g, [1, page 35, Entry 19] or [2, Equation (2.1)]):

$$\left(q,z,\frac{q}{z};q\right)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{(1/2)n(n-1)} z^n, \quad z \neq 0. \tag{1.2}$$

It is well known that an analytic function has a unique Laurent expansion in an annulus. Bailey [3] used this property to prove the quintuple product identity. By this approach, Cooper [4, 5] and Kongsiriwong and Liu [2] proved many types of the Macdonald identities and some other series-product identities. In this paper, we use this method to deal with a sextuple product identity.

In Section 2, we present the sextuple product identity ((2.1) below) and its proof. Our identity is equivalent to [2, Equation (8.16)] by Kongsiriwong and Liu, which is the simplification of [2, Equation (6.13)]. Kongsiriwong and Liu got [2, Equation (8.16)] from a more general identity. In this section, we give it a direct proof.

In Section 3, we get many identities from this sextuple product identity.

To simplify notation, we often write $\sum_n$ for $\sum_{n=-\infty}^{\infty}$ in the following when no confusion occurs.

2. A New Proof of the Sextuple Product Identity

The starting point of our investigation in this section is the identity in the following theorem.

Theorem 2.1. *For any complex number z with $z \neq 0$, one has*

$$\begin{aligned}\left(q, z, \frac{q}{z}; q\right)_\infty \left(q^3, z^3, \frac{q^3}{z^3}; q^3\right)_\infty &= \left(q^{12}, -q^6, -q^6; q^{12}\right)_\infty \sum_n q^{2n^2-2n} z^{4n} \\ &\quad + 2\left(q^{12}, -q^{12}, -q^{12}; q^{12}\right)_\infty \sum_n q^{2n^2+1} z^{4n+2} \\ &\quad - \left(q^3, -q^3, -q^3; q^3\right)_\infty \sum_n q^{(1/2)(n^2-n)} z^{2n+1}.\end{aligned} \tag{2.1}$$

Before the proof of Theorem 2.1, we need some preparations. The two identities in the following lemma are from [6]. We write them in this version.

Lemma 2.2. *One has*

$$\begin{aligned}&\left(q^8, q^3, q^5; q^8\right)_\infty \left(q^{24}, q^9, q^{15}; q^{24}\right)_\infty + q^2 \left(q^8, q, q^7; q^8\right)_\infty \left(q^{24}, q^3, q^{21}; q^{24}\right)_\infty \\ &\quad = \left(q^2, -q^2, -q^2; q^2\right)_\infty \left(q^6, q^3, q^3; q^6\right)_\infty,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\left(q^8, q, q^7; q^8\right)_\infty \left(q^{24}, q^9, q^{15}; q^{24}\right)_\infty - q \left(q^8, q^3, q^5; q^8\right)_\infty \left(q^{24}, q^3, q^{21}; q^{24}\right)_\infty \\ &\quad = \left(q^2, q, q; q^2\right)_\infty \left(q^6, -q^6, -q^6; q^6\right)_\infty.\end{aligned} \tag{2.3}$$

Proof. For (2.2), see [6, Equation (3.18)]. Equation (2.3) is from [6, Equation (3.21)]. Its proof is similar to that of [6, Equation (3.18)]. □

The lemma above is used to prove the following two identities.

Lemma 2.3. *One has*

$$(q,-q,-q;q)_\infty\left(q^3,-q^3,-q^3;q^3\right)_\infty + (q,iq,-iq;q)_\infty\left(q^3,-iq^3,iq^3;q^3\right)_\infty = 2\left(q^4,-q^4,-q^4;q^4\right)_\infty\left(q^{12},-q^6,-q^6;q^{12}\right)_\infty, \tag{2.4}$$

$$(q,-q,-q;q)_\infty\left(q^3,-q^3,-q^3;q^3\right)_\infty - (q,iq,-iq;q)_\infty\left(q^3,-iq^3,iq^3;q^3\right)_\infty = 2q\left(q^4,-q^2,-q^2;q^4\right)_\infty\left(q^{12},-q^{12},-q^{12};q^{12}\right)_\infty. \tag{2.5}$$

Proof. By (1.2), we have

$$\begin{aligned}(q,-q,-q;q)_\infty\left(q^3,-q^3,-q^3;q^3\right)_\infty &= \frac{1}{4}(q,-1,-q;q)_\infty\left(q^3,-1,-q^3;q^3\right)_\infty\\ &= \frac{1}{4}\sum_m q^{(1/2)(m^2-m)}\sum_n q^{(3/2)(n^2-n)} = \sum_m q^{2m^2+m}\sum_n q^{6n^2+3n}\\ &= \sum_m q^{8m^2+2m}\sum_n q^{24n^2+6n} + q^4\sum_m q^{8m^2+6m}\sum_n q^{24n^2+18n}\\ &\quad + q^3\sum_m q^{8m^2+2m}\sum_n q^{24n^2+18n} + q\sum_m q^{8m^2+6m}\sum_n q^{24n^2+6n},\end{aligned} \tag{2.6}$$

$$\begin{aligned}(q,iq,-iq;q)_\infty\left(q^3,-iq^3,iq^3;q^3\right)_\infty &= \frac{1}{2}(q,i,-iq;q)_\infty\left(q^3,-i,iq^3;q^3\right)_\infty\\ &= \frac{1}{2}\sum_m(-1)^m q^{(1/2)(m^2-m)}i^m\sum_n(-1)^n q^{(3/2)(n^2-n)}i^{3n}\\ &= \sum_m(-1)^m q^{2m^2+m}\sum_n(-1)^n q^{6n^2+3n}\\ &= \sum_m q^{8m^2+2m}\sum_n q^{24n^2+6n} + q^4\sum_m q^{8m^2+6m}\sum_n q^{24n^2+18n}\\ &\quad - q^3\sum_m q^{8m^2+2m}\sum_n q^{24n^2+18n} - q\sum_m q^{8m^2+6m}\sum_n q^{24n^2+6n}.\end{aligned} \tag{2.7}$$

Adding (2.6) and (2.7), we have

$$\begin{aligned}&(q,-q,-q;q)_\infty\left(q^3,-q^3,-q^3;q^3\right)_\infty + (q,iq,-iq;q)_\infty\left(q^3,-iq^3,iq^3;q^3\right)_\infty\\ &\quad= 2\sum_m q^{8m^2+2m}\sum_n q^{24n^2+6n} + 2q^4\sum_m q^{8m^2+6m}\sum_n q^{24n^2+18n}\\ &\quad= 2\left(q^{16},-q^6,-q^{10};q^{16}\right)_\infty\left(q^{48},-q^{18},-q^{30};q^{48}\right)_\infty\\ &\qquad+ 2q^4\left(q^{16},-q^2,-q^{14};q^{16}\right)_\infty\left(q^{48},-q^6,-q^{42};q^{48}\right)_\infty.\end{aligned} \tag{2.8}$$

By (2.2), we have (2.4).

Subtracting (2.7) from (2.6), we obtain

$$\begin{aligned}
&(q,-q,-q;q)_\infty\left(q^3,-q^3,-q^3;q^3\right)_\infty-(q,iq,-iq;q)_\infty\left(q^3,-iq^3,iq^3;q^3\right)_\infty\\
&=2q^3\sum_m q^{8m^2+2m}\sum_n q^{24n^2+18n}+2q\sum_m q^{8m^2+6m}\sum_n q^{24n^2+6n}\\
&=2q^3\left(q^{16},-q^6,-q^{10};q^{16}\right)_\infty\left(q^{48},-q^6,-q^{42};q^{48}\right)_\infty\\
&\quad+2q\left(q^{16},-q^2,-q^{14};q^{16}\right)_\infty\left(q^{48},-q^{18},-q^{32};q^{48}\right)_\infty.
\end{aligned}\tag{2.9}$$

Replacing q in (2.3) by $-q^2$ and, then, applying the resulting identity to the above equation, we get (2.5). This completes the proof. □

Proof of Theorem 2.1. Set

$$f(z,q)=\left(q,z,\frac{q}{z};q\right)_\infty\left(q^3,z^3,\frac{q^3}{z^3};q^3\right)_\infty.\tag{2.10}$$

Then f is an analytic function of z in the annulus $0<|z|<\infty$. Put

$$f(z,q)=\sum_n a_n(q)z^n,\quad 0<|z|<\infty.\tag{2.11}$$

By (2.10), we can easily verify

$$f(z,q)=z^4f(zq,q).\tag{2.12}$$

Combining (2.11) and (2.12) gives

$$\sum_m a_m(q)z^m=\sum_m q^{m-4}a_{m-4}(q)z^m.\tag{2.13}$$

Equate the coefficients of z^m on both sides to get

$$a_m(q)=q^{m-4}a_{m-4}(q).\tag{2.14}$$

Using the above relation, we obtain

$$\begin{aligned}
a_{4m-1}(q)&=q^{2m^2-3m}a_{-1}(q), &\quad a_{4m}(q)&=q^{2m^2-2m}a_0(q),\\
a_{4m+1}(q)&=q^{2m^2-m}a_1(q), &\quad a_{4m+2}(q)&=q^{2m^2}a_2(q).
\end{aligned}\tag{2.15}$$

Substituting the above four identities into (2.11), we have

$$f(z,q) = a_{-1}(q)\sum_m q^{2m^2-3m}z^{4m-1} + a_0(q)\sum_m q^{2m^2-2m}z^{4m} + a_1(q)\sum_m q^{2m^2-m}z^{4m+1} + a_2(q)\sum_m q^{2m^2}z^{4m+2}. \tag{2.16}$$

By (2.10), we also have

$$f(z,q) = f\left(\frac{q}{z},q\right). \tag{2.17}$$

This gives

$$\sum_m a_m(q)z^m = \sum_m q^{-m}a_{-m}(q)z^m. \tag{2.18}$$

Then we have

$$a_m(q) = q^{-m}a_{-m}(q). \tag{2.19}$$

Set $m = 1$ to get

$$a_1(q) = q^{-1}a_{-1}(q). \tag{2.20}$$

By this relation, (2.16) reduces to

$$f(z,q) = a_0(q)\sum_m q^{2m^2-2m}z^{4m} + a_1(q)\sum_m q^{(1/2)(m^2-m)}z^{2m+1} + a_2(q)\sum_m q^{2m^2}z^{4m+2}. \tag{2.21}$$

Now, it remains to determine $a_0(q)$, $a_1(q)$, and $a_2(q)$.

Putting $z = 1$ in (2.21) gives

$$0 = a_0(q)\sum_m q^{2m^2-2m} + a_1(q)\sum_m q^{(1/2)(m^2-m)} + a_2(q)\sum_m q^{2m^2}. \tag{2.22}$$

Set $z = -1$ in (2.21) to get

$$4(q,-q,-q;q)_\infty\left(q^3,-q^3,-q^3;q^3\right)_\infty = a_0(q)\sum_m q^{2m^2-2m} - a_1(q)\sum_m q^{(1/2)(m^2-m)} + a_2(q)\sum_m q^{2m^2}. \tag{2.23}$$

Taking $z = i$ in (2.21) and noting that $\sum_m (-1)^m q^{(1/2)(m^2-m)} = 0$, we have

$$(q, i, -iq; q)_\infty \left(q^3, -i, iq^3; q^3\right)_\infty = a_0(q) \sum_m q^{2m^2-2m} - a_2(q) \sum_m q^{2m^2}. \tag{2.24}$$

Subtracting (2.23) from (2.22) and noting that $\sum_m q^{(1/2)(m^2-m)} = 2(q, -q, -q; q)_\infty$, we obtain

$$a_1(q) = -\left(q^3, -q^3, -q^3; q^3\right)_\infty. \tag{2.25}$$

Add (2.22) and (2.23) to get

$$2(q, -q, -q; q)_\infty \left(q^3, -q^3, -q^3; q^3\right)_\infty = a_0(q) \sum_m q^{2m^2-2m} + a_2(q) \sum_m q^{2m^2}. \tag{2.26}$$

Adding (2.24) and (2.26) and, then, using (1.2) in the resulting equation, we obtain

$$\begin{aligned} &(q, -q, -q; q)_\infty \left(q^3, -q^3, -q^3; q^3\right)_\infty + (q, iq, -iq; q)_\infty \left(q^3, -iq^3, iq^3; q^3\right)_\infty \\ &\quad = 2a_0(q) \left(q^4, -q^4, -q^4; q^4\right)_\infty. \end{aligned} \tag{2.27}$$

By (2.4), we have

$$a_0(q) = \left(q^{12}, -q^6, -q^6; q^{12}\right)_\infty. \tag{2.28}$$

Similarly, subtracting (2.24) from (2.26) and, then using (1.2), we have

$$\begin{aligned} &(q, -q, -q; q)_\infty \left(q^3, -q^3, -q^3; q^3\right)_\infty - (q, iq, -iq; q)_\infty \left(q^3, -iq^3, iq^3; q^3\right)_\infty \\ &\quad = a_2(q) \left(q^4, -q^2, -q^2; q^4\right)_\infty. \end{aligned} \tag{2.29}$$

Applying (2.5) to this equation gives

$$a_2(q) = 2q\left(q^{12}, -q^{12}, -q^{12}; q^{12}\right)_\infty, \tag{2.30}$$

which completes the proof. □

3. Some Applications

In this section, we deduce many modular identities from Theorem 2.1.

Corollary 3.1. *One has*

$$\begin{aligned}3(q;q)_\infty^3\left(q^3;q^3\right)_\infty^3 &= \left(q^{12},-q^6,-q^6;q^{12}\right)_\infty\sum_n 2n(4n-1)q^{2n^2-2n}\\&\quad+2\left(q^{12},-q^{12},-q^{12};q^{12}\right)_\infty\sum_n(2n+1)(4n+1)q^{2n^2+1}\\&\quad-\left(q^3,-q^3,-q^3;q^3\right)_\infty\sum_n n(2n+1)q^{1/2(n^2-n)}.\end{aligned}\tag{3.1}$$

Proof. Dividing both sides of (2.1) by $(1-z)^2$, letting $z\to 1$, and then using L'Hospital's rule twice on the right-hand side gives (3.1). ☐

Corollary 3.2. *One has*

$$\begin{aligned}&\left(q^{24},-q^{12},-q^{12};q^{24}\right)_\infty\left(q^8,-q^4,-q^4;q^8\right)_\infty+4q^4\left(q^{24},-q^{24},-q^{24};q^{24}\right)_\infty\left(q^8,-q^8,-q^8;q^8\right)_\infty\\&\quad+2q\left(q^6,-q^6,-q^6;q^6\right)_\infty\left(q^2,-q^2,-q^2;q^2\right)_\infty\\&\quad=\left(q^2,-q,-q;q^2\right)_\infty\left(q^6,-q^3,-q^3;q^6\right)_\infty,\end{aligned}\tag{3.2}$$

$$\begin{aligned}&\left(q^{36},-q^{18},-q^{18};q^{36}\right)_\infty\left(q^{12},-q^4,-q^8;q^{12}\right)_\infty+2q^5\left(q^{36},-q^{36},-q^{36};q^{36}\right)_\infty\left(q^{12},-q^2,-q^{10};q^{12}\right)_\infty\\&\quad-q\left(q^9,-q^9,-q^9;q^9\right)_\infty\left(q^3,-q,-q^2;q^3\right)_\infty\\&\quad=(q;q)_\infty\left(q^3;q^3\right)_\infty,\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\left(q^{12},-q^6,-q^6;q^{12}\right)_\infty\left(q^4,-q^4,-q^4;q^4\right)_\infty-q\left(q^{12},-q^{12},-q^{12};q^{12}\right)_\infty\left(q^4,-q^2,-q^2;q^4\right)_\infty\\&\quad=\frac{\left(q^2;q^2\right)_\infty}{\left(-q;q^2\right)_\infty}\frac{\left(q^6;q^6\right)_\infty}{\left(-q^3;q^6\right)_\infty},\end{aligned}\tag{3.4}$$

$$
\begin{aligned}
&\left(q^{60},-q^{30},-q^{30};q^{60}\right)_\infty\left(q^{20},-q^{8},-q^{12};q^{20}\right)_\infty+2q^{9}\left(q^{60},-q^{60},-q^{60};q^{60}\right)_\infty\left(q^{20},-q^{2},-q^{18};q^{20}\right)_\infty\\
&\qquad-q^{2}\left(q^{15},-q^{15},-q^{15};q^{15}\right)_\infty\left(q^{5},-q,-q^{4};q^{5}\right)_\infty\\
&\quad=\frac{(q;q)_\infty}{(q,q^4;q^5)_\infty}\frac{(q^5;q^5)_\infty}{(q^3,q^{12};q^{15})_\infty},
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&\left(q^{60},-q^{30},-q^{30};q^{60}\right)_\infty\left(q^{20},-q^{4},-q^{16};q^{20}\right)_\infty+2q^{7}\left(q^{60},-q^{60},-q^{60};q^{60}\right)_\infty\left(q^{20},-q^{6},-q^{14};q^{20}\right)_\infty\\
&\qquad-q\left(q^{15},-q^{15},-q^{15};q^{15}\right)_\infty\left(q^{5},-q^{2},-q^{3};q^{5}\right)_\infty\\
&\quad=\frac{(q;q)_\infty}{(q^2,q^3;q^5)_\infty}\frac{(q^5;q^5)_\infty}{(q^6,q^9;q^{15})_\infty}.
\end{aligned}
\tag{3.6}
$$

Proof. Replace q in (2.1) by q^2 and, then, z by $-q$. Using (1.2) in the resulting identity gives (3.2).

Replace q in (2.1) by q^3 and, then, z by q. Using (1.2) in the resulting identity gives (3.3).

Replace q in (2.1) by q^4 and, then, z by q. Using (1.2) and the fact that $(q^4,-q,-q^3;q^4)_\infty=(q,-q,-q;q)_\infty$ in the resulting identity, we obtain

$$
\begin{aligned}
&\left(\left(q^{48},-q^{24},-q^{24};q^{48}\right)_\infty+2q^{6}\left(q^{48},-q^{48},-q^{48};q^{48}\right)_\infty\right)\left(q^{4},-q^{4},-q^{4};q^{4}\right)_\infty\\
&\quad-q\left(q^{12},-q^{12},-q^{12};q^{12}\right)_\infty\left(q^{4},-q^{2},-q^{2};q^{4}\right)_\infty=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\frac{(q^6;q^6)_\infty}{(-q^3;q^6)_\infty}.
\end{aligned}
\tag{3.7}
$$

By (1.2), we have

$$
\begin{aligned}
\left(q^{12},-q^{6},-q^{6};q^{12}\right)_\infty&=\sum_n q^{6n^2}=\sum_n q^{6(2n)^2}+\sum_n q^{6(2n+1)^2}\\
&=\left(q^{48},-q^{24},-q^{24};q^{48}\right)_\infty+2q^{6}\left(q^{48},-q^{48},-q^{48};q^{48}\right)_\infty.
\end{aligned}
\tag{3.8}
$$

Combining (3.7) and (3.8) gives (3.4).

Replace q in (2.1) by q^5 and, then, z by q^2. Using (1.2) in the resulting identity gives (3.5).

Replace q in (2.1) by q^5 and, then, z by q. Using (1.2) in the resulting identity gives (3.6). □

Obviously, using the same method above, we can get more identities from (2.1).

Now, we deduce a formula for $(q;q)_\infty^4$ from (2.1).

Corollary 3.3. *One has*

$$(q;q)_\infty^4 = 2\sum_m q^{2m^2}\sum_n 2nq^{6n^2+2n} + 2q\sum_m q^{2m^2+2m}\sum_n (2n-1)q^{6n^2-4n} + \sum_m q^{2m^2+m}\sum_n (2n+1)q^{(1/2)(3n^2+n)}. \quad (3.9)$$

Proof. Denote the left-hand side of (2.1) by $f(z)$ and the right-hand side of (2.1) by $g(z)$. Let z_0 be a zero point of $f(z)$. Because (2.1) holds in $0 < |z| < \infty$, z_0 is also a zero point of $g(z)$. If $az_0 = 1$, we have

$$\lim_{z\to z_0}\frac{f(z)}{1-az} = \lim_{z\to z_0}\frac{g(z)}{1-az}. \quad (3.10)$$

Setting $z_0 = a = 1$ in (3.10) and by L'Hospital's rule on the right-hand side, we have

$$\begin{aligned} 0 &= \left(q^3,-q^3,-q^3;q^3\right)_\infty\sum_n (2n+1)q^{(1/2)(n^2-n)} \\ &\quad - 2\left(q^{12},-q^{12},-q^{12};q^{12}\right)_\infty\sum_n (4n+2)q^{2n^2+1} \\ &\quad - \left(q^{12},-q^6,-q^6;q^{12}\right)_\infty\sum_n 4nq^{2n^2-2n}. \end{aligned} \quad (3.11)$$

Let $\omega = e^{(2/3)\pi i}$. Putting $z_0 = \omega$ and $a = \omega^2$ in (3.10) and noting $\omega^{3n} = 1$ for any integer n, we have

$$\begin{aligned} 3(1-\omega)\left(q^3;q^3\right)_\infty^4 &= \left(q^3,-q^3,-q^3;q^3\right)_\infty\sum_n (2n+1)q^{(1/2)(n^2-n)}\omega^{2(n-1)} \\ &\quad - 2\left(q^{12},-q^{12},-q^{12};q^{12}\right)_\infty\sum_n (4n+2)q^{2n^2+1}\omega^{n-1} \\ &\quad - \left(q^{12},-q^6,-q^6;q^{12}\right)_\infty\sum_n 4nq^{2n^2-2n}\omega^n. \end{aligned} \quad (3.12)$$

Taking $z_0 = \omega^2$ and $a = \omega$ in (3.10), we obtain

$$\begin{aligned} 3\left(1-\omega^2\right)\left(q^3;q^3\right)_\infty^4 &= \left(q^3,-q^3,-q^3;q^3\right)_\infty\sum_n (2n+1)q^{(1/2)(n^2-n)}\omega^{n-1} \\ &\quad - 2\left(q^{12},-q^{12},-q^{12};q^{12}\right)_\infty\sum_n (4n+2)q^{2n^2+1}\omega^{2(n-1)} \\ &\quad - \left(q^{12},-q^6,-q^6;q^{12}\right)_\infty\sum_n 4nq^{2n^2-2n}\omega^{2n}. \end{aligned} \quad (3.13)$$

Adding the above three identities together gives

$$\begin{aligned}9\left(q^3;q^3\right)_\infty^4 &= \left(q^3,-q^3,-q^3;q^3\right)_\infty \sum_n (2n+1)q^{(1/2)(n^2-n)}\left(1+\omega^{n-1}+\omega^{2(n-1)}\right)\\ &\quad -2\left(q^{12},-q^{12},-q^{12};q^{12}\right)_\infty \sum_n (4n+2)q^{2n^2+1}\left(1+\omega^{n-1}+\omega^{2(n-1)}\right)\\ &\quad -\left(q^{12},-q^{6},-q^{6};q^{12}\right)_\infty \sum_n 4nq^{2n^2-2n}\left(1+\omega^{n}+\omega^{2n}\right).\end{aligned} \tag{3.14}$$

Using the fact

$$1+\omega^n+\omega^{2n} = \begin{cases} 3, & n \equiv 0 \pmod 3,\\ 0, & n \not\equiv 0 \pmod 3\end{cases} \tag{3.15}$$

in the above identity and, then, replacing q^3 by q, we get

$$\begin{aligned}(q;q)_\infty^4 &= (q,-q,-q;q)_\infty \sum_n (2n+1)q^{(1/2)(3n^2+n)}\\ &\quad -4q\left(q^4,-q^4,-q^4;q^4\right)_\infty \sum_n (2n+1)q^{6n^2+4n}\\ &\quad -2\left(q^4,-q^2,-q^2;q^4\right)_\infty \sum_n 2nq^{6n^2-2n}.\end{aligned} \tag{3.16}$$

Replacing n in the last two sums on the right-hand side of the above identity by $-n$ and, then, applying (1.2) to the resulting equation, we get Corollary 3.3. □

4. Conclusion

Besides the Jacobi triple product (1.2), well-known series-product identities are known as the quintuple product identity, the Winquist identity, and so forth. The formula (2.1) is also such an identity. Recently, we also obtain some other identities of this kind, including the simplifications of the formulae [2, Equations (6.12) and (6.14)], with a different method. These identities are widely used in number theory, combinatorics, and many other fields. literature on this topic abounds. In (2.1), if we replace z by e^{2iz}, then the right-hand side of (2.1) turns into fourier series. For recent papers on the applications of fourier analysis, we refer the readers to [7–9].

Acknowledgment

This research is supported by the Shanghai Natural Science Foundation (Grant no. 10ZR1409100), the National Science Foundation of China (Grant no. 10771093), and the Natural Science Foundation of Education Department of Henan Province of China (Grant no. 2007110025).

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