Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 509724, 19 pages
doi:10.1155/2011/509724

Research Article

Analytical Behavior Prediction for Skewed Thick Plates on Elastic Foundation

Pang-jo Chun and Yun Mook Lim

Department of Civil and Environmental Engineering, Yonsei University, 134 Shinchon-Dong Seodaemun-Gu, Seoul 120-749, Republic of Korea

Correspondence should be addressed to Yun Mook Lim, yunmook@yonsei.ac.kr

Received 11 May 2011; Accepted 1 August 2011

Academic Editor: Paulo Batista Gonçalves

This paper presents analytical solutions for the problem of skewed thick plates under transverse load on a Winkler foundation, which has not been reported in the literature. The thick plate solution is obtained by using a framework of an oblique coordinate system. First, the governing differential equation in that system is derived, and the solution is obtained using deflection and rotation as derivatives of the potential function developed here. This method is applicable for arbitrary loading conditions, boundary conditions, and materials. The solution technique is applied to two illustrative application examples, and the results are compared with numerical solutions. The two approaches yielded results in good agreement.

1. Introduction

Plates resting on an elastic foundation are important structural elements that are used in a wide range of applications including foundation structures and concrete pavements of roads. In the past few decades, there have been efforts to analytically investigate the behavior of plates resting on an elastic foundation. Most of the work has been performed on rectangular plates and circular plates. For example, Timoshenko and Woinowsky-Krieger [1] provided an analytical solution for rectangular thin plates, and Celep [2] presented the behavior of circular thin plates on an elastic foundation.

Due to its mathematical complexity, there are very few studies which have considered skewed plates on an elastic foundation. Kennedy [3] was the first to develop an analytical solution as polynomial series for skewed isotropic thin plates with four edges clamped on an elastic foundation. Ng and Das [4] and Chell et al. [5] developed an analytical solution as trigonometric series for skewed sandwich thin plates on an elastic foundation. Though these studies are informative, they have limitations in terms of loading and

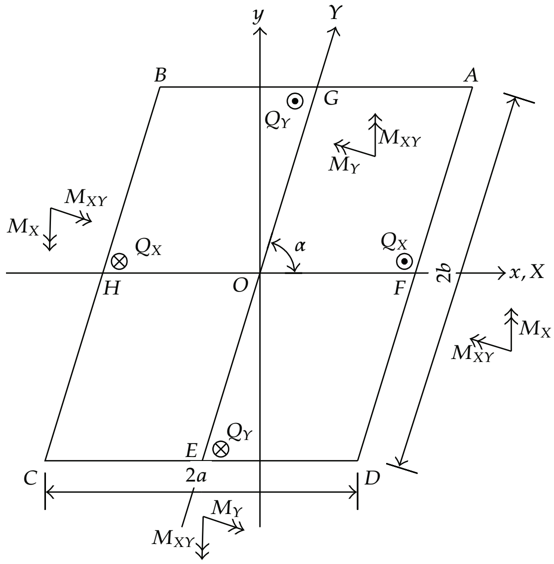

Figure 1: A skewed plate in an oblique coordinate system.

boundary conditions. All of the solutions developed by the above researchers can handle only uniformly distributed load, and a designated boundary condition. The solutions by Kennedy [3] and Ng and Das [4] are limited to clamped plates, and that of Chell et al. [5] is limited to simply supported plates.

In addition, there is another issue; these methods are applicable only to thin plates of which the thickness-to-side ratio is less than 1/20 because Kirchhoff plate theory has been employed to develop the governing equation in these studies. Though the Kirchhoff theory is widely used in plate analysis, it suffers from under predicting deflections when thick plates are analyzed because it neglects the effect of the transverse shear deformation [6]. To resolve this issue, the Mindlin theory [7, 8], which relaxes the perpendicular restriction for transverse normals, was developed so that the effect of transverse shear deformation could be considered. Several studies which have employed the Mindlin theory for analysis are worth mentioning. For example, Kobayashi and Sonoda [9] developed an analytical solution for rectangular thick plates with two opposite edges, simply supported. In their research, Levy-type trigonometric series solutions were derived. Liew et al. [10] and Liu [11] solved the rectangular thick plate problem by the differential quadrature method and offered a number of numerical results under several boundary conditions. Though there are some studies dealing with rectangular thick plates, there has been no research analyzing skewed thick plates.

In the present manuscript, analytical solutions for skewed thick plates on an elastic foundation with arbitrary boundary and loading conditions are newly developed based on the Mindlin theory. The Winkler foundation, which is the simplest model to represent the elastic foundation, is employed here. The present method is new in the following two aspects. First of all, the present method has developed analytical solutions based on the Mindlin theory for skewed thick plates, and such solutions have never previously been developed in the literature. Second, the present method allows arbitrary loading conditions, boundary conditions, and materials.

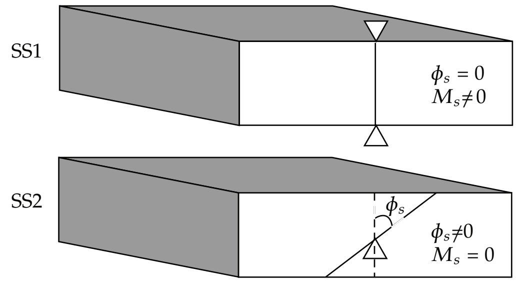

Figure 2: Comparison between SS1 and SS2.

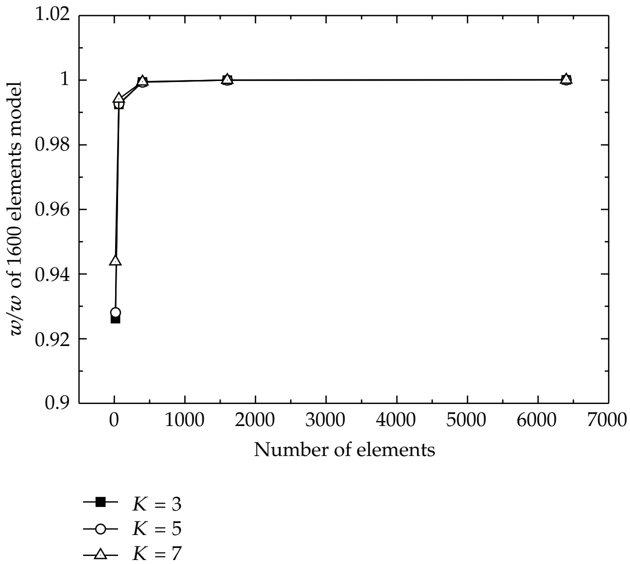

Figure 3: Convergence study of the center deflection of a clamped isotropic 30-degree skewed thick plate under uniform loading.

We first show the derivation of a governing equation in a framework of an oblique coordinate system, and then it is solved using trigonometric series. The governing equation in this system is derived and the solution is obtained using the deflection and rotation as derivatives of a potential function. The solution technique is applied to an illustrative application example with different skew angles and modulus of foundations, and the results are compared with the commercial finite element package ANSYS [12]. The present approach is in reasonable agreement with the FEM solution.

2. Governing Equation under the Oblique Coordinate System

When a plate's boundary profile is a parallelogram, the oblique Cartesian coordinate system can be advantageous. We first present the concept of the oblique coordinate system and then show the governing differential equation of skewed thick plates on a Winkler foundation based on the Mindlin theory. The authors have developed an analytical solution for skewed thick plates without an elastic foundation in [13], and the formulas developed in this chapter have utilized the relationships developed in the research. Figure 1 shows the oblique coordinate system spanned by the X and Y axes, along with the rectangular coordinate system x and y, with angle XOY denoted as skew angle α. Parallelogram ABCD in Figure 1 represents the skewed plate of interest, and the edge lengths CD and AD are 2a and 2b, respectively.

The relationship between the rectangular and oblique coordinate systems can be written as follows [14, 15]:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & -\cot\alpha \\ 0 & \csc\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{pmatrix} \dfrac{\partial}{\partial X} \\ \\ \dfrac{\partial}{\partial Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix}, \tag{2.1}$$
$$\begin{pmatrix} \phi_X \\ \phi_Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix},$$

where ϕ_X and ϕ_Y are, respectively, the rotations normal to the X and Y axes. The corresponding relationships for stress, strain, moment, and shear force under the two coordinate systems are also available in [14]. They are readily derivable according to the principles of statics and continuum mechanics. Their corresponding quantities in the oblique system are derived in this research effort. The resulting flexural stiffness matrix under the oblique coordinate system $[D_o]$ is related to its counterpart in the rectangular system $[D_r]$ as follows [16]:

$$[D_O] = \begin{pmatrix} \sin\alpha & \cos\alpha\cot\alpha & -2\cos\alpha \\ 0 & \csc\alpha & 0 \\ 0 & -\cot\alpha & 1 \end{pmatrix} [D_r] \begin{pmatrix} 1 & 0 & 0 \\ \cos^2\alpha & \sin^2\alpha & \sin\alpha\cos\alpha \\ 2\cos\alpha & 0 & \sin\alpha \end{pmatrix}^{-1}. \tag{2.2}$$

The flexural stiffness matrices relate the moments to the curvatures in the respective coordinate systems. For example, $[D_r]$ in the rectangular coordinate system for isotropic material is [1]:

$$[D_r] = \frac{Et^3}{12} \begin{pmatrix} \dfrac{1}{1-\nu^2} & \dfrac{\nu}{1-\nu^2} & 0 \\ \\ \dfrac{\nu}{1-\nu^2} & \dfrac{1}{1-\nu^2} & 0 \\ \\ 0 & 0 & \dfrac{1}{2(1+\nu)} \end{pmatrix}, \tag{2.3}$$

where E is Young's modulus, ν is Poisson's ratio, and t is the thickness of the plate. Note that (2.2) is also applicable for other, more complex situations, such as orthotropic or anisotropic materials. By using the flexural stiffness matrices, the moment-strain relationship of the rectangular and oblique coordinate systems can be described as in the following (2.4) and (2.5), respectively [6]:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D_r] \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\[2ex] \dfrac{\partial \phi_y}{\partial y} \\[2ex] \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix}, \tag{2.4}$$

$$\begin{Bmatrix} M_X \\ M_Y \\ M_{XY} \end{Bmatrix} = [D_O] \begin{Bmatrix} \dfrac{\partial \phi_X}{\partial X} \\[2ex] \dfrac{\partial \phi_Y}{\partial Y} \\[2ex] \dfrac{\partial \phi_X}{\partial Y} + \dfrac{\partial \phi_Y}{\partial X} \end{Bmatrix}. \tag{2.5}$$

In this derivation, the Mindlin theory is applied, which assumes that strains are linearly distributed along the thickness direction in the plate's cross-section.

Similarly, the extensional stiffness matrix in the oblique coordinate system $[A_O]$ is related to that of the rectangular system $[A_r]$ as follows [13]:

$$[A_O] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ 0 & 1 \end{pmatrix} [A_r] \begin{pmatrix} 1 & 0 \\ -\tan\alpha & \sec\alpha \end{pmatrix}. \tag{2.6}$$

The extensional stiffness matrix relates the shear forces to the shear strains. For example, $[A_r]$ for isotropic material is

$$[A_r] = \frac{Et}{2(1+\nu)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.7}$$

By using the extensional stiffness matrices, the relationships between the shear force and deflection, rotation angle of the rectangular and oblique coordinate system can be described as in the following (2.8) and (2.9), respectively, [6]:

$$\begin{pmatrix} Q_x \\ Q_y \end{pmatrix} = K_s[A_r] \begin{pmatrix} \dfrac{\partial w}{\partial x} + \phi_x \\ \\ \dfrac{\partial w}{\partial y} + \phi_y \end{pmatrix}, \tag{2.8}$$

$$\begin{pmatrix} Q_X \\ Q_Y \end{pmatrix} = K_s[A_O] \begin{pmatrix} \dfrac{\partial w}{\partial X} + \phi_X \\ \\ \dfrac{\partial w}{\partial Y} + \phi_Y \end{pmatrix}, \tag{2.9}$$

where w is the transverse deformation perpendicular to the plane of the plate, and K_s is the shear correction factor to account for nonuniform transverse shear distribution. Hereafter, the components in $[D_o]$ and $[A_o]$ are referred to using their respective elements D_{11} to D_{33} and A_{44} to A_{55} as follows:

$$[D_O] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix}, \qquad [A_O] = \begin{bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{bmatrix}, \tag{2.10}$$

where the diagonal components of $[D_O]$ relate the moments to the curvatures in the same directions. The off-diagonal terms relate the same moments to the curvatures in other directions due to the Poisson's effect and coordinate system obliquity. Similarly, the diagonal components of $[A_o]$ relate the shear forces to the shear strains in the same directions, and off-diagonal terms to the shear strains in other directions due to obliquity.

The following (2.11) to (2.13) are equilibrium conditions of the skewed plates shown in Figure 1.

Equilibrium of force in the z direction (perpendicular to the plane of the plate)

$$\frac{\partial Q_X}{\partial X} + \frac{\partial Q_Y}{\partial Y} = -Q + kw. \tag{2.11}$$

Equilibrium of moments along the X-axis

$$\frac{\partial M_Y}{\partial Y} + \frac{\partial M_{XY}}{\partial X} = Q_Y. \tag{2.12}$$

Equilibrium of moments along the Y-axis

$$\frac{\partial M_X}{\partial X} + \frac{\partial M_{XY}}{\partial Y} = Q_X, \tag{2.13}$$

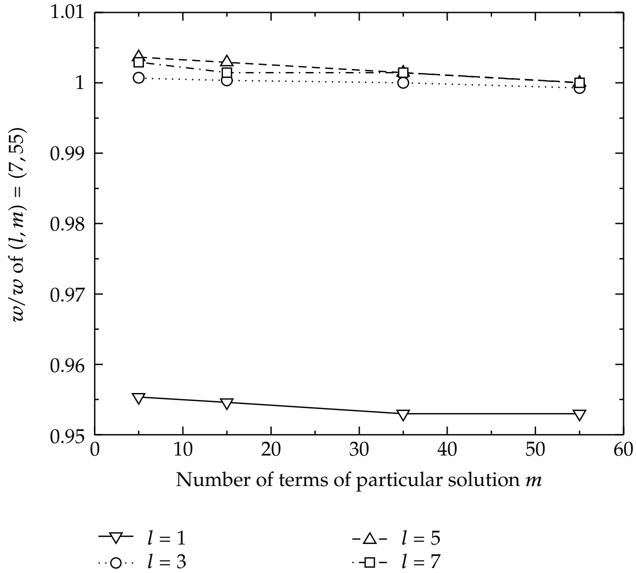

Figure 4: Truncation effect for convergence for center deflection of clamped isotropic 30-degree skewed thick plate under uniform loading.

where Q in (2.11) is the load normal to the upper surface of the plate, and k is the modulus of the foundation. By substituting (2.5), (2.9), and (2.10) into (2.11) to (2.13), the following (2.14) to (2.16) are obtained in the oblique system:

$$\begin{aligned} &K_s A_{45}\left(\frac{\partial^2 w}{\partial X \partial Y} + \frac{\partial \phi_X}{\partial Y}\right) + K_s A_{45}\left(\frac{\partial^2 w}{\partial X \partial Y} + \frac{\partial \phi_Y}{\partial X}\right) + K_s A_{55}\left(\frac{\partial^2 w}{\partial X^2} + \frac{\partial \phi_X}{\partial X}\right) \\ &\quad + K_s A_{44}\left(\frac{\partial^2 w}{\partial Y^2} + \frac{\partial \phi_Y}{\partial Y}\right) = -Q + kw, \end{aligned} \tag{2.14}$$

$$\begin{aligned} &D_{11}\frac{\partial^2 \phi_X}{\partial X^2} + D_{12}\frac{\partial^2 \phi_Y}{\partial X \partial Y} + D_{13}\left(2\frac{\partial^2 \phi_X}{\partial X \partial Y} + \frac{\partial^2 \phi_Y}{\partial X^2}\right) + D_{23}\frac{\partial^2 \phi_Y}{\partial Y^2} + D_{33}\left(\frac{\partial^2 \phi_X}{\partial Y^2} + \frac{\partial^2 \phi_Y}{\partial X \partial Y}\right) \\ &\quad = K_s A_{45}\left(\frac{\partial w}{\partial Y} + \phi_Y\right) + K_s A_{55}\left(\frac{\partial w}{\partial X} + \phi_X\right), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &D_{12}\frac{\partial^2 \phi_X}{\partial X \partial Y} + D_{13}\frac{\partial^2 \phi_X}{\partial X^2} + D_{22}\frac{\partial^2 \phi_Y}{\partial Y^2} + D_{23}\left(\frac{\partial^2 \phi_X}{\partial Y^2} + 2\frac{\partial^2 \phi_Y}{\partial X \partial Y}\right) + D_{33}\left(\frac{\partial^2 \phi_X}{\partial X \partial Y} + \frac{\partial^2 \phi_Y}{\partial X^2}\right) \\ &\quad = K_s A_{44}\left(\frac{\partial w}{\partial Y} + \phi_Y\right) + K_s A_{45}\left(\frac{\partial w}{\partial X} + \phi_X\right). \end{aligned} \tag{2.16}$$

To make the solution process simpler, a new function ψ is introduced below to represent the condition of the skewed thick plate. We assume that w consists of terms up to the 4th derivative, and ϕ_X and ϕ_Y up to the 3rd derivative of ψ, with respect to the spatial variables X and Y. The following relations in (2.17) to (2.19) are obtained to satisfy (2.15) and (2.16).

$$\begin{aligned} w = {} & \left(D_{13}^2 - D_{11}D_{33}\right)\frac{\partial^4\psi}{\partial X^4} + 2(D_{12}D_{13} - D_{11}D_{23})\frac{\partial^4\psi}{\partial X^3\partial Y} \\ & + \left(D_{12}^2 - D_{11}D_{22} - 2D_{13}D_{23} + 2D_{12}D_{33}\right)\frac{\partial^4\psi}{\partial X^2\partial Y^2} \\ & + 2(-D_{13}D_{22} + D_{12}D_{23})\frac{\partial^4\psi}{\partial X\partial Y^3} + \left(D_{23}^2 - D_{22}D_{33}\right)\frac{\partial^4\psi}{\partial Y^4} \\ & + \{A_{44}D_{11} - 2A_{45}D_{13} + A_{55}D_{33}\}K_s\frac{\partial^2\psi}{\partial X^2} \\ & + 2\{A_{44}D_{13} + A_{55}D_{23} - A_{45}(D_{12} + D_{33})\}K_s\frac{\partial^2\psi}{\partial X\partial Y} \\ & + \{A_{55}D_{22} - 2A_{45}D_{23} + A_{44}D_{33}\}K_s\frac{\partial^2\psi}{\partial Y^2} + \left(A_{45}^2 - A_{44}A_{55}\right)K_s^2\psi, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \phi_X = {} & (A_{45}D_{13} - A_{55}D_{33})K_s\frac{\partial^3\psi}{\partial X^3} + \{A_{44}D_{13} - 2A_{55}D_{23} + A_{45}D_{12}\}K_s\frac{\partial^3\psi}{\partial X^2\partial Y} \\ & + \{-A_{55}D_{22} - A_{45}D_{23} + A_{44}(D_{12} + D_{33})\}K_s\frac{\partial^3\psi}{\partial X\partial Y^2} + (-A_{45}D_{22} + A_{44}D_{23})K_s\frac{\partial^3\psi}{\partial Y^3} \\ & + \left(-A_{45}^2 + A_{44}A_{55}\right)K_s^2\frac{\partial\psi}{\partial X}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \phi_Y = {} & (-A_{45}D_{11} + A_{55}D_{13})K_s\frac{\partial^3\psi}{\partial X^3} + \{-A_{44}D_{11} - A_{45}D_{13} + A_{55}(D_{12} + D_{33})\}K_s\frac{\partial^3\psi}{\partial X^2\partial Y} \\ & + \{-2A_{44}D_{13} + A_{55}D_{23} + A_{45}D_{12}\}K_s\frac{\partial^3\psi}{\partial X\partial Y^2} + (A_{45}D_{23} - A_{44}D_{33})K_s\frac{\partial^3\psi}{\partial Y^3} \\ & + \left(-A_{45}^2 + A_{44}A_{55}\right)K_s^2\frac{\partial\psi}{\partial Y}. \end{aligned} \tag{2.19}$$

w has only even derivatives and ϕ_X and ϕ_Y have only odd derivatives though it is assumed that w, ϕ_X, and ϕ_Y may have both even and odd derivatives. This is because the coefficients of odd derivatives of w and the coefficients of even derivatives of ϕ_X and ϕ_Y should be 0 to satisfy (2.14) to (2.16). By substituting these relations into (2.14), the governing equation of the Mindlin skewed thick plate is then formulated as a 6th-order partial differential equation as follows:

$$L(\psi) = -Q, \tag{2.20}$$

where L is a linear differential operator in the oblique coordinate system

$$
\begin{aligned}
L() = {} & A_{55}\left(D_{13}^2 - D_{11}D_{33}\right)K_s\frac{\partial^6\psi}{\partial X^6} + 2\left\{A_{55}(D_{12}D_{13} - D_{11}D_{23}) + A_{45}\left(D_{13}^2 - D_{11}D_{33}\right)\right\} \\
& \times K_s\frac{\partial^6\psi}{\partial X^5\partial Y} + \left\{A_{44}\left(D_{13}^2 - D_{11}D_{33}\right) + 4A_{45}(D_{12}D_{13} - D_{11}D_{23})\right. \\
& \qquad\qquad \left. + A_{55}\left(D_{12}^2 - D_{11}D_{22} - 2D_{13}D_{23} + 2D_{12}D_{33}\right)\right\}K_s\frac{\partial^6\psi}{\partial X^4\partial Y^2} \\
& + \left\{2A_{44}(D_{12}D_{13} - D_{11}D_{23}) + 2A_{55}(-D_{13}D_{22} + D_{12}D_{23})\right. \\
& \quad \left. + 2A_{45}\left(D_{12}^2 - 2D_{13}D_{23} + 2D_{12}D_{33} - D_{11}D_{22}\right)\right\}K_s\frac{\partial^6\psi}{\partial X^3\partial Y^3} \\
& + \left\{A_{44}\left(D_{12}^2 - D_{11}D_{22} - 2D_{13}D_{23} + 2D_{12}D_{33}\right) + A_{55}\left(D_{23}^2 - D_{22}D_{33}\right)\right. \\
& \quad \left. + 4A_{45}(D_{12}D_{23} - D_{13}D_{22})\right\}K_s\frac{\partial^6\psi}{\partial X^2\partial Y^4} \\
& + 2\left\{A_{45}\left(D_{23}^2 - D_{22}D_{33}\right) + A_{44}(D_{12}D_{23} - D_{13}D_{22})\right\}K_s\frac{\partial^6\psi}{\partial X\partial Y^5} \\
& + A_{44}\left(D_{23}^2 - D_{22}D_{33}\right)K_s\frac{\partial^6\psi}{\partial Y^6} + D_{11}\left(A_{44}A_{55} - A_{45}^2\right)K_s^2\frac{\partial^4\psi}{\partial X^4} \\
& + 4D_{13}\left(A_{44}A_{55} - A_{45}^2\right)K_s^2\frac{\partial^4\psi}{\partial X^3\partial Y} + 2(D_{12} + 2D_{33})\left(A_{44}A_{55} - A_{45}^2\right)K_s^2\frac{\partial^4\psi}{\partial X^2\partial Y^2} \\
& + 4D_{23}\left(A_{44}A_{55} - A_{45}^2\right)K_s^2\frac{\partial^4\psi}{\partial X\partial Y^3} \\
& + D_{22}\left(A_{44}A_{55} - A_{45}^2\right)K_s^2\frac{\partial^4\psi}{\partial Y^4} - k\sin\alpha \\
& \times \left\{\left(A_{45}^2 - A_{44}A_{55}\right)K_s^2\psi + (A_{55}D_{22} - 2A_{45}D_{23} + A_{44}D_{33})K_s\frac{\partial^2\psi}{\partial Y^2}\right. \\
& \quad + \left(D_{23}^2 - D_{22}D_{33}\right)\frac{\partial^4\psi}{\partial Y^4} + 2(D_{12}D_{23} - D_{13}D_{22})\frac{\partial^4\psi}{\partial X\partial Y^3} \\
& \quad + 2(A_{44}D_{13} + A_{55}D_{23} - A_{45}(D_{12} + D_{33}))K_s\frac{\partial^2\psi}{\partial X\partial Y} \\
& \quad + (A_{44}D_{11} - 2A_{45}D_{13} + A_{55}D_{33})K_s\frac{\partial^2\psi}{\partial X^2} + \left(D_{13}^2 - D_{11}D_{33}\right)\frac{\partial^4\psi}{\partial X^4} \\
& \quad + \left(D_{12}^2 - D_{11}D_{22} - 2D_{13}D_{23} + 2D_{12}D_{33}\right)\frac{\partial^4\psi}{\partial X^2\partial Y^2} \\
& \quad \left. + 2(D_{12}D_{13} - D_{11}D_{23})\frac{\partial^4\psi}{\partial X^3\partial Y}\right\}.
\end{aligned}
\tag{2.21}
$$

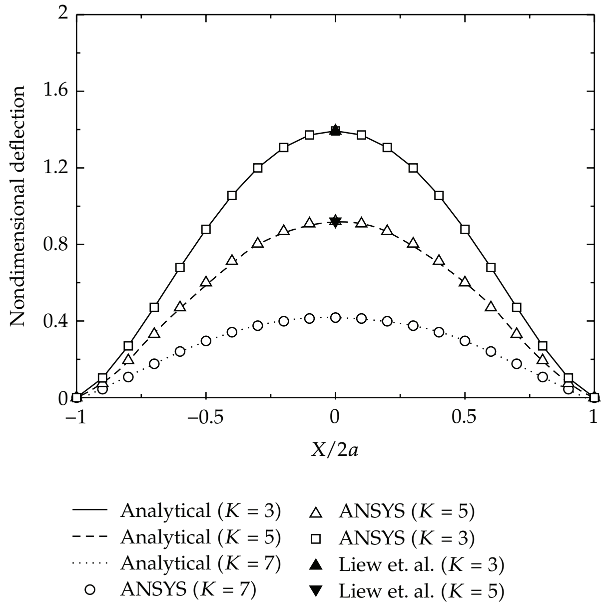

Figure 5: Analytical and numerical solutions by ANSYS and from the literature [10] of the deflection of a clamped isotropic skewed thick plate under uniformly distributed load. (Skew angle = 90°).

3. Analytical Solution in Series Form

In this section, a general solution to the governing differential equation (2.11) is developed as the sum of a fundamental (homogeneous) and a particular (nonhomogeneous) solution, detailed separately, as follows.

3.1. Homogeneous Solution

The homogeneous solution ψ_h is the solution to (2.20) for $Q = 0$, obtained as a trigonometric series in (3.1) below

$$\begin{aligned}\psi_h = \sum_{h=1}^{\infty}\big(&A_h C_{1X1} + iB_h C_{1X2} + C_h C_{2X1} + iD_h C_{2X2} + E_h C_{3X1} + iF_h C_{3X2} + G_h S_{1X1} + iH_h S_{1X2}\\ &+ I_h S_{2X1} + iJ_h S_{2X2} + K_h S_{3X1} + iL_h S_{3X2} + M_h C_{1Y1} + iN_h C_{1Y2} + O_h C_{2Y1} + iP_h C_{2Y2}\\ &+Q_h C_{3Y1} + iR_h C_{3Y2} + S_h S_{1Y1} + iT_h S_{1Y2} + U_h S_{2Y1} + iV_h S_{2Y2} + W_h S_{3Y1} + iX_h S_{3Y2}\big),\end{aligned} \tag{3.1}$$

where $i = \sqrt{-1}$ is the imaginary unit, and C_{eXf}, C_{eYf}, S_{eXf}, and S_{eYf} trigonometric functions are as follows:

$$C_{eXf} = \cos\frac{\pi h(X + \lambda_{eY}Y)}{2a} + (-1)^{f+1}\cos\frac{\pi h\left(X + \overline{\lambda_{eY}}Y\right)}{2a},$$

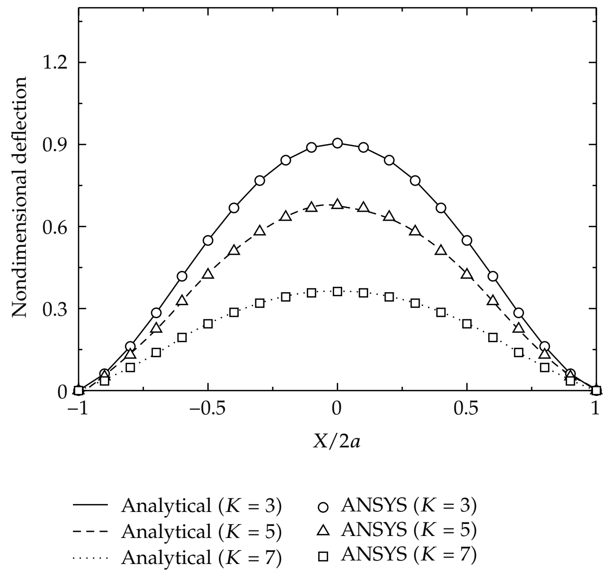

Figure 6: Analytical and numerical solutions by ANSYS of the deflection of a clamped isotropic skewed thick plate under uniformly distributed load. (Skew angle = 60°).

$$
\begin{aligned}
S_{eXf} &= \sin\frac{\pi h(X+\lambda_{eY}Y)}{2a} + (-1)^{f+1}\sin\frac{\pi h\left(X+\overline{\lambda_{eY}}Y\right)}{2a},\\
C_{eYf} &= \cos\frac{\pi h(\lambda_{eX}X+Y)}{2b} + (-1)^{f+1}\cos\frac{\pi h\left(\overline{\lambda_{eX}}X+Y\right)}{2b},\\
S_{eYf} &= \sin\frac{\pi h(\lambda_{eX}X+Y)}{2b} + (-1)^{f+1}\sin\frac{\pi h\left(\overline{\lambda_{eX}}X+Y\right)}{2b},\\
&\qquad\qquad (e=1,2,3,\quad f=1,2),
\end{aligned}
\tag{3.2}
$$

where the bar on λ denotes the conjugate of λ. λ_{1X}, λ_{2X}, λ_{3X}, λ_{1Y}, and λ_{2Y}, and λ_{3Y} are the eigenvalues to be obtained by satisfying $L(\psi_h) = 0$. For example, λ_{eX} is derived by solving the following equation:

$$
L\left(\cos\frac{\pi h(\lambda_{eX}X+Y)}{2a} + \sin\frac{\pi h(\lambda_{eX}X+Y)}{2a}\right) = 0.
\tag{3.3}
$$

The trigonometric function ψ_h in (3.1) has $24l$ unknowns $A_h, B_h, C_h, \ldots$, and $X_h (h = 1,2,3,\ldots,l)$, with l being the number of the trigonometric terms needed for convergence. In addition, as for the coefficients of real parts ($A_h, C_h, \ldots$, and W_h), $(l+1)$ terms are considered so that 12 more unknowns ($A_{l+1}, C_{l+1}, \ldots$, and W_{l+1}) can be determined. Therefore, the homogeneous solution ψ_h has $24l+12$ unknowns, and these will be determined according to the boundary conditions discussed below.

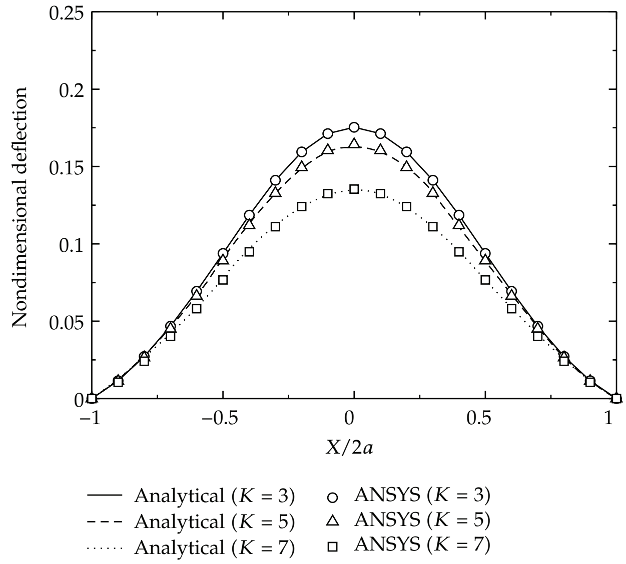

Figure 7: Analytical and numerical solutions by ANSYS of the deflection of a clamped isotropic skewed thick plate under uniformly distributed load. (Skew angle = 30°).

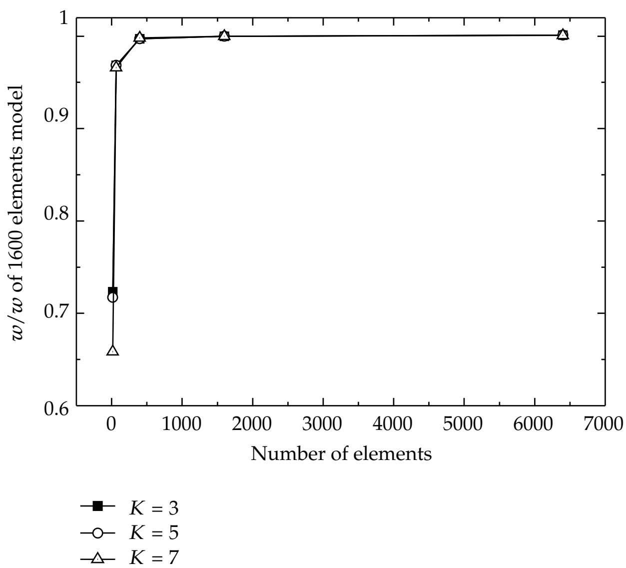

Figure 8: Convergence study of the center deflection of an orthotropic 30-degree skewed thick plate under concentrated loading.

3.2. Particular Solution

For a particular solution in the series form, the transverse load Q in (2.20) is expanded to a trigonometric series as follows:

$$Q(X,Y) = \sum_{j=1,2\ldots}^{\infty}\sum_{k=1,2\ldots}^{\infty}\frac{\cos\alpha}{ab}\int_{-b}^{b}\int_{-a}^{a}Q(\xi,\eta)\sin\frac{j\pi(\xi+a)}{2a}\sin\frac{k\pi(\eta+b)}{2b}d\xi\, d\eta\sin\frac{j\pi(X+a)}{2a}\sin\frac{k\pi(Y+b)}{2b}. \tag{3.4}$$

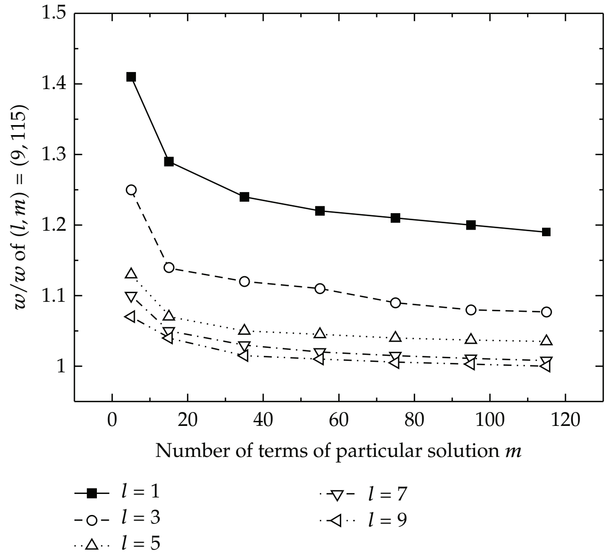

Figure 9: Truncation effect for convergence for maximum deflection of an orthotropic 30-degree skewed thick plate under concentrated loading.

Equation (3.4) can express any transverse load, such as a uniformly distributed load, a concentrated load, a line load, or a patch load. Therefore, the particular solution ψ_p for (2.20) can be written in a series form as

$$\psi_p = \sum_{j=1,2,\ldots}^{m} \sum_{k=1,2,\ldots}^{m} K_{jk} \cos \frac{j\pi(X+a)}{2a} \cos \frac{k\pi(Y+b)}{2b} + L_{jk} \sin \frac{j\pi(X+a)}{2a} \sin \frac{k\pi(Y+b)}{2b}, \quad (3.5)$$

where K_{jk} and L_{jk} are to be determined to satisfy (2.20) and (3.4), and m is the number of the trigonometric terms needed for convergence. The general solution for ψ is derived as the sum of the homogeneous solution and the particular solution as

$$\psi = \psi_h + \psi_p. \quad (3.6)$$

Since no unknowns exist in the particular solution, the total number of unknowns in the general solution is still $24l + 12$, as in the homogeneous solution.

4. Determination of Unknown Constants for Series Solution

Using the Mindlin theory, the boundary conditions for various edges are given below for determining the unknown constants in the homogeneous solution. The normal and tangential directions to the edge are denoted here using the subscripts n and s, respectively. The moments on the edges are noted using these subscripts in a way that is consistent with

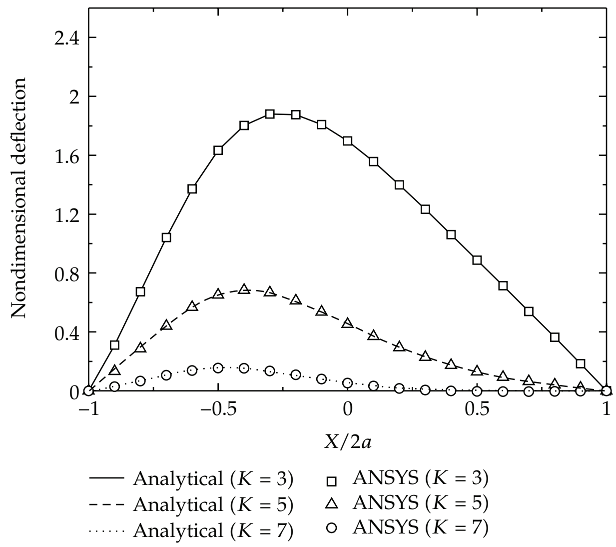

Figure 10: Analytical and numerical solutions by ANSYS of the deflection of an orthotropic skewed thick plate under concentrated load. (Skew angle = 90°).

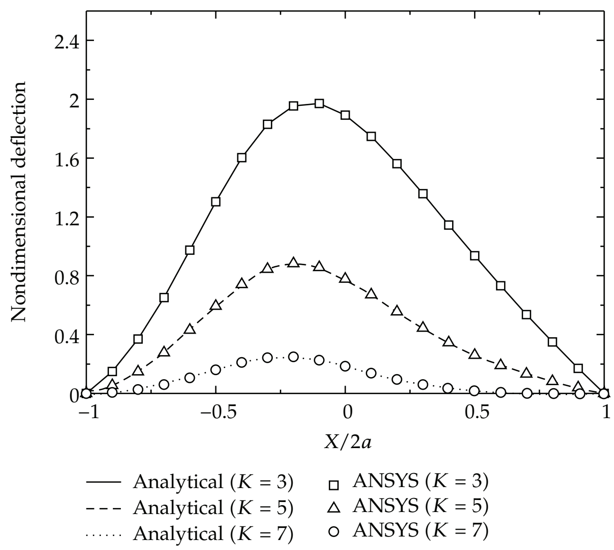

Figure 11: Analytical and numerical solutions by ANSYS of the deflection of an orthotropic skewed thick plate under concentrated load. (Skew angle = 60°).

the directions of the stresses thereby induced. Namely, M_n is for the moment causing normal stresses and M_s is the torsional moment inducing shear stresses.

$$(1)\ \text{Clamped (C)}:\ w = 0, \quad \phi_n = 0, \quad \phi_s = 0. \tag{4.1}$$

$$(2)\ \text{Soft simply supported (SS1)}:\ w = 0, \quad M_n = 0, \quad \phi_s = 0. \tag{4.2}$$

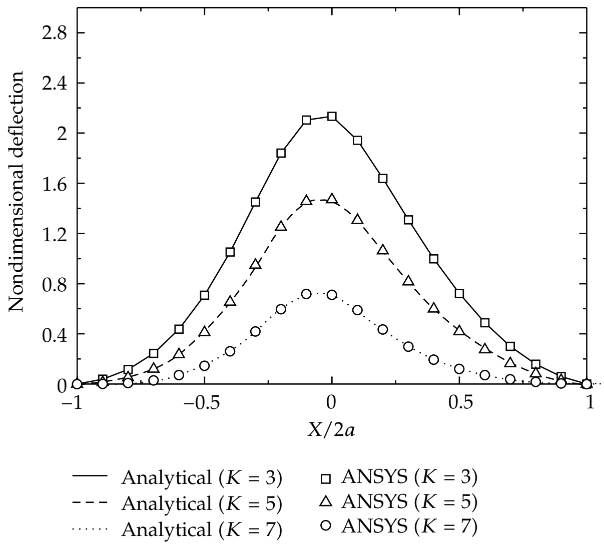

Figure 12: Analytical and numerical solutions by ANSYS of the deflection of an orthotropic skewed thick plate under concentrated load. (Skew angle = 30°).

$$\text{(3) Hard simply supported (SS2)}: \ w = 0, \quad M_n = 0, \quad M_s = 0. \tag{4.3}$$

$$\text{(4) Free (F)}: \ M_n = 0, \quad M_s = 0, \quad Q_n = 0. \tag{4.4}$$

Note that the Kirchhoff theory treats SS1 and SS2 in (4.2) and (4.3) as having the same boundary condition. The difference between them is explained graphically in Figure 2. The boundary condition of SS1 restricts the tangential rotation by supporting two points in the cross-section, thereby generating a nonzero torsional moment. In contrast, the boundary condition of SS2 supports the plate only at one point in the cross-section, allowing a tangential rotation and generating no twisting moment. The boundary conditions in (4.1) to (4.4) can be unified as follows:

$$\Gamma_d(X,Y) = 0 \begin{cases} d = 1,2,3 & (\text{edge CD in Figure 1}) \\ d = 4,5,6 & (\text{edge AB in Figure 1}) \\ d = 7,8,9 & (\text{edge BC in Figure 1}) \\ d = 10,11,12 & (\text{edge AD in Figure 1}), \end{cases} \tag{4.5}$$

where $\Gamma_1(X,Y)$ to $\Gamma_{12}(X,Y)$ represent the left hand side of (4.1) to (4.4). $\Gamma_1(X,Y)$ to $\Gamma_{12}(X,Y)$ are expanded as a Fourier series as follows for the solution method pursued in this paper:

$$\Gamma_d(X,Y) = \frac{a_{0d}}{2} + \sum_{c=1}^{\infty}\left(a_{cd}\cos\left(\frac{c\pi X}{a}\right) + b_{cd}\sin\left(\frac{c\pi X}{a}\right)\right),$$

$$(d = 1,2,\ldots,6) \ (\text{for the edge of } Y = b,-b),$$

Table 1: Deflection at the center of clamped skewed thick plates under uniform loading. The numbers in parentheses show the difference between these results and the analytical solution developed in this research.

Skew angle		$K = 3$	$K = 5$	$K = 7$
90°	Present study	1.39462	0.92291	0.42018
	ANSYS	1.39304 (0.11%)	0.92125 (0.18%)	0.41751 (0.64%)
	Mindlin et al. [7]	1.393 (0.12%)	0.921 (0.20%)	N/A
60°	Present study	0.90921	0.67795	0.36392
	ANSYS	0.90472 (0.49%)	0.67779 (0.02%)	0.36196 (0.54%)
30°	Present study	0.17597	0.16528	0.13701
	ANSYS	0.17541 (0.32%)	0.16417 (0.67%)	0.13537 (0.12%)

Table 2: Deflection at the center of orthotropic skewed thick plates under concentrated loading. The numbers in parentheses show the difference between these results and the analytical solution developed in this research.

Skew angle		$K = 3$	$K = 5$	$K = 7$
90°	Present study	1.71091	0.45867	0.05467
	ANSYS	1.69742 (0.79%)	0.45247 (1.35%)	0.05393 (1.35%)
60°	Present study	1.90910	0.78330	0.18668
	ANSYS	1.8925 (0.87%)	0.77793 (0.69%)	0.18562 (0.57%)
30°	Present study	2.16008	1.48826	0.72032
	ANSYS	2.13448 (1.19%)	1.46995 (1.23%)	0.71173 (1.19%)

$$\Gamma_d(X,Y) = \frac{a_{0d}}{2} + \sum_{c=1}^{\infty}\left(a_{cd}\cos\left(\frac{c\pi Y}{b}\right) + b_{cd}\sin\left(\frac{c\pi Y}{b}\right)\right),$$
$$(d = 7, 8, \ldots, 12) \ \left(\text{for the edge of } X = a, -a\right), \tag{4.6}$$

where coefficients a_{0d}, a_{cd}, and b_{cd} are Fourier coefficients for the boundary condition $\Gamma_d(X,Y)$. Because of the need for truncation, l terms are kept for each of the 12 boundary conditions so that a total of $12(2l+1)$ equations are established for the same number of Fourier coefficients that have the $24l + 12$ unknowns included to be solved for.

5. Application Examples

In this section, two application examples are presented using the developed analytical solution for skewed thick plates. They are also compared with solutions published in [10] and the FEM analysis result obtained using the commercial package ANSYS 11. In the analysis by ANSYS, 2D 8-node quadrilateral plate elements (SHELL99) appropriate for thick plate analysis are used for the skewed plates with various skewed angles. The numbers of nodes and elements are 4961 and 1600, respectively. The convergence figure of transverse displacements along with the number of nodes is provided for both of the application examples. In addition, the effect on convergence of the number of terms l and m in the fundamental and particular solutions is studied. In the following examples, the shear correction factor K_s is taken to be 5/6 since this factor is commonly used in plate analyses.

5.1. Isotropic Clamped Skewed Thick Plate under Uniformly Distributed Load

Isotropic skewed thick plates are analyzed here. As an external force, uniformly distributed load q_0 is applied, and, as skewed angles, $\alpha = 30°, 60°$, and $90°$ are employed. The geometrical properties used in this research are $a = b$, and $t = 0.2a$. The clamped boundary condition is used for all edges. The modulus of foundation k is expressed in a dimensionless form as $K = (16a^4 k/D)^{1/4}$. D is the bending stiffness and is expressed as $D = Et^3/12(1 - \nu^2)$ for the isotropic plate. $K = 3, 5$, and 7 are employed here.

As a first step, convergence of the finite element model is checked. Figure 3 shows the convergence figure of transverse displacements w at the center of the plates when α is $30°$. It can be concluded that the finite element model converges adequately when the number of elements = 1600. Similar results are confirmed for other α.

Then, the numbers of terms in the series solution l and m are determined. The deflection w at the center of the plates for $(l, m) = (5, 55)$ and $(7, 35)$ differs by less than 0.1% from that of $(l, m) = (7, 55)$ as in Figure 4 when $\alpha = 30°$ and $K = 7$. It can be concluded that the solution is already convergent while truncated at $(l, m) = (7, 55)$, and similar results are observed for other α and K. Therefore, $l = 7$ and $m = 55$ are employed in this example.

Figures 5, 6, and 7 give a comparison between the present method and FEM analysis using ANSYS for the deflection w for $\alpha = 30°, 60°$, and $90°$ along line HF defined in Figure 1 and on the top of the plate. Note that the deflection is expressed in a dimensionless form as $1000wD/16q_0a^4$. For the plate of $\alpha = 90°$, results in the literature [10] are also shown in the figure. In addition, Table 1 also shows the comparison results between the present solution, FEM, and literature at the center of the plate. The results show that our analytical and numerical solutions agree with each other very well.

5.2. Orthotropic Skewed Thick Plate under Concentrated Load

Orthotropic thick skewed plates are analyzed in this example, with the following material and geometrical properties: $E_y = 0.5E_x$, $G_{xy} = 0.3E_x$, $G_{xz} = 0.1E_x$, $G_{yz} = 0.08E_x$, $\nu_{xy} = 0.2$, $a = b$, $t = 0.3a$, where E_x and E_y are Young's modulus along the x and y directions, ν_{xy} is the major Poisson's ratio, and G_{xy}, G_{xz}, and G_{yz} are the shear modulus in the xy, xz, and yz planes. These values determine $[D_r]$, $[D_o]$, $[A_r]$, and $[A_o]$ in (2.2) and (2.6). The external transverse force is a concentrated force of Q_0 applied at $(X, Y) = (-a/2,\ b/2)$. Plates with skew angle $\alpha = 30°, 60°$, and $90°$ and the modulus of foundation of $K = 3, 5$, and 7 are analyzed here, where K is expressed in a dimensionless form as $K = (16a^4k/D_{11})^{1/4}$, and D_{11} is the bending stiffness which is expressed as $D_{11} = E_x t^3/12(1 - \nu_{xy}\nu_{yx})$. To show the applicability of this method for the arbitrary boundary condition, the following complex boundary condition is applied in this example: AB in Figure 1 is soft, simply supported condition (SS1), BC is clamped (C), CD is free (F), and DA is a hard, simply supported condition (SS2).

As with the previous example, the convergence of the finite element model should be checked. Figure 8 shows the convergence study of transverse displacements w at the center of the plates when α is $30°$. It can be concluded that the finite element model converges adequately when the number of elements = 1600. Similar results are confirmed for other α.

In addition, the numbers of terms including l and m in (3.1) and (3.5) need to be determined. Figure 9 shows the deflection w at the point where a concentrated load is applied for skew angle $\alpha = 30^\circ$ and $K = 7$, as one of the cases considered, for various l and m values. It is seen that the deflection at $(l, m) = (9, 115)$ is well converged. Similar results are observed for other cases. Therefore $(l, m) = (9, 115)$ is employed here and is also used as the reference for comparison.

For this example, because no previous work in the literature has been found reporting a similar experience, only FEM analysis results by ANSYS are employed for comparison with our analytical solution results. Table 2 shows comparison results of the deflection at the center of the plate $(X, Y) = (0,\ 0)$. The deflection is expressed in a dimensionless form as $100wD_{11}/4Q_0a^2$.

Figures 10, 11, and 12 indicate the results for deflection along line HF in Figure 1. The results indicate that the response behavior for this case is much more complex than the examples above, due to nonsymmetric loading and boundary conditions. Due to the oblique coordinate system, the load at $(X, Y) = (-a/2,\ b/2)$ has different relative relations to the interested responses on $Y = 0$, along with different skew angles. For example, this causes the peak responses of the deflection to move towards the center of the plate with the skew angle decreasing from 90 to 30 degrees, and it makes the deflection of $\alpha = 30^\circ$ larger than that of $\alpha = 90^\circ$.

6. Conclusions

The governing differential equation of skewed thick plates on an elastic foundation in an oblique coordinate system is formulated for the first time in this paper. This work allows derivation of an analytical solution for any boundary condition, loading condition, and material, also for the first time reported. All response quantities including shear forces, moments, stresses, strains, deflections, and rotation angles can be readily derived from the proposed potential function ψ. The two illustrative examples show that the analytical solutions are in good agreement with those reported in literature and numerical solutions by FEM. Further validation of the analytical solution developed here will be provided by an experimental test.

It is also worth noting that the approach to the governing differential equation and its analytical solution developed in this study can be used for further studies including, but not limited to, continuous plate analysis and dynamic analysis, for which only numerical solutions exist. These studies are in progress now and may contribute additional understanding of the behavior of a structure on an elastic foundation.

References

[1] S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, Engineering Societies Monographs, McGraw-Hill, New York, NY, USA, 2nd edition, 1959.

[2] Z. Celep, "Rectangular plates resting on tensionless elastic foundation," *Journal of Engineering Mechanics*, vol. 114, no. 12, pp. 2083–2092, 1988.

[3] J. B. Kennedy, "Influence of elastic support on the bending of parallelogram plates," *Applied Scientific Research*, vol. 18, no. 1, pp. 68–80, 1968.

[4] S. F. Ng and B. Das, "Finite deflection of skew sandwich plates on elastic foundations by the Galerkin method," *Journal of Structural Mechanics*, vol. 14, no. 3, pp. 355–377, 1986.

[5] G. C. Chell, S. Mondal, and G. Bairagi, "Large deflection analysis of rhombic sandwich plates placed on elastic foundation," *Indian Journal of Engineering & Materials Sciences*, vol. 15, no. 1, pp. 7–13, 2008.

[6] J. N. Reddy, *Theory and Analysis of Elastic Plates and Shells*, CRC Press, New York, NY, USA, 2007.

[7] R. D. Mindlin, "Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates," *Journal of Applied Mechanics*, vol. 18, no. 1, pp. 31–38, 1951.

[8] E. Reissner, "The effect of transverse shear deformation on the bending of elastic plates," *Journal of Applied Mechanics*, vol. 12, pp. A69–A77, 1945.

[9] H. Kobayashi and K. Sonoda, "Rectangular Mindlin plates on elastic foundations," *International Journal of Mechanical Sciences*, vol. 31, no. 9, pp. 679–692, 1989.

[10] K. M. Liew, J. B. Han, Z. M. Xiao, and H. Du, "Differential quadrature method for Mindlin plates on Winkler foundations," *International Journal of Mechanical Sciences*, vol. 38, no. 4, pp. 405–421, 1996.

[11] F. L. Liu, "Rectangular thick plates on winkler foundation: differential quadrature element solution," *International Journal of Solids and Structures*, vol. 37, no. 12, pp. 1743–1763, 2000.

[12] ANSYS Release 11.0 Documentation, ANSYS Inc., 2007.

[13] P. Chun, *Skewed bridge behaviors: experimental, analytical, and numerical analysis*, Ph.D. thesis, Wayne State University, Wayne, NJ, USA, 2010.

[14] L. S. D. Morley, *Skew Plates and Structures*, Pergamon Press, New York, NY, USA, 1963.

[15] K. M. Liew and J. B. Han, "Bending analysis of simply supported shear deformable skew plates," *Journal of Engineering Mechanics*, vol. 123, no. 3, pp. 214–221, 1997.

[16] P. J. Chun, G. Fu, and Y. M. Lim, "Analytical solutions for skewed thick plates subjected to transverse loading," *Structural Engineering and Mechanics*, vol. 38, no. 5, pp. 549–571, 2011.