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Research Article

Totally Umbilical Proper Slant and Hemislant Submanifolds of an LP-Cosymplectic Manifold

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In the present note, we study slant and hemislant submanifolds of an LP-cosymplectic manifold which are totally umbilical. We prove that every totally umbilical proper slant submanifold M of an LP-cosymplectic manifold $\overline{M}$ is either totally geodesic or if M is not totally geodesic in $\overline{M}$ then we derive a formula for slant angle of M. Also, we obtain the integrability conditions of the distributions of a hemi-slant submanifold, and then we give a result on its classification.

1. Introduction

A manifold $\overline{M}$ with Lorentzian paracontact metric structure (ϕ, ξ, η, g) satisfying $(\overline{\nabla}_X \phi)Y = 0$ is called an LP-cosymplectic manifold, where $\overline{\nabla}$ is the Levi-Civita connection corresponding to the Lorentzian metric g on $\overline{M}$. The study of slant submanifolds was initiated by Chen [1]. Since then, many research papers have appeared in this field. Slant submanifolds are the natural generalization of both holomorphic and totally real submanifolds. Lotta [2] defined and studied these submanifolds in contact geometry. Later on, Cabrerizo et al. studied slant, semi-slant, and bislant submanifolds in contact geometry [3, 4]. In particular, totally umbilical proper slant submanifold of a Kaehler manifold has also been studied in [5]. Recently, Khan et al. [6] studied these submanifolds in the setting of Lorentzian paracontact manifolds.

The idea of hemi-slant submanifolds was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds [7]. Recently, these submanifolds are studied by Sahin for their warped products [8]. In this paper, we study slant and hemi-slant submanifolds of an LP-cosymplectic manifold. We prove that a totally umbilical proper slant submanifold M is either totally geodesic in $\overline{M}$ or if it is not totally geodesic, then the slant angle $\theta = \tan^{-1}(\sqrt{g(X,Y)/\eta(X)\eta(Y)})$. Also, we define hemi-slant submanifolds of an LP-contact manifold. After we find integrability conditions of the distributions, we investigate a classification of totally umbilical hemi-slant submanifolds of an LP-cosymplectic manifold.

2. Preliminaries

Let $\overline{M}$ be a n-dimensional paracontact manifold with the Lorentzian paracontact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a contravariant vector field, η is a 1-form, and g is a Lorentzian metric with signature $(-,+,+,\dots,+)$ on $\overline{M}$, satisfying [9],

$$\phi^2 = X + \eta(X)\xi, \qquad \eta(\xi) = -1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \operatorname{rank}(\phi) = n-1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi), \tag{2.2}$$

for all $X, Y \in T\overline{M}$.

A Lorentzian paracontact metric structure on $\overline{M}$ is called a *Lorentzian para-cosymplectic structure* if $\overline{\nabla}\phi = 0$, where $\overline{\nabla}$ denotes the Levi-Civita connection with respect to g. The manifold $\overline{M}$ in this case is called a *Lorentzian para-cosymplectic* (in brief, an *LP-cosymplectic*) manifold [10]. From formula $\overline{\nabla}\phi = 0$, it follows that $\overline{\nabla}_X\xi = 0$.

Let M be a submanifold of a Lorentzian almost paracontact manifold $\overline{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g). Let the induced metric on M also be denoted by g, then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.4}$$

for any X, Y in TM and N in $T^{\perp}M$, where TM is the Lie algebra of vector field in M and $T^{\perp}M$ is the set of all vector fields normal to M. $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental form, and A_N is the Weingarten endomorphism associated with N. It is easy to see that

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.5}$$

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.6}$$

where PX is the tangential component and FX is the normal component of ϕX. Similarly for $N \in T^{\perp}M$, we write

$$\phi N = BN + CN, \tag{2.7}$$

where BN is the tangential component and CN is the normal component of ϕN.

The covariant derivatives of the tensor fields ϕ, P, and F are defined as

$$\left(\overline{\nabla}_X\phi\right)Y = \overline{\nabla}_X\phi Y - \phi\overline{\nabla}_X Y, \quad \forall X, Y \in T\overline{M}, \tag{2.8}$$

$$\left(\overline{\nabla}_X P\right)Y = \nabla_X PY - P\nabla_X Y, \quad \forall X, Y \in TM, \tag{2.9}$$

$$\left(\overline{\nabla}_X F\right)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \quad \forall X, Y \in TM. \tag{2.10}$$

Moreover, for an LP-cosymplectic manifold, one has

$$\left(\overline{\nabla}_X P\right)Y = A_{FY}X + Bh(X, Y), \tag{2.11}$$

$$\left(\overline{\nabla}_X F\right)Y = Ch(X, Y) - h(X, PY). \tag{2.12}$$

A submanifold M is said to be *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \tag{2.13}$$

where H is the mean curvature vector. Furthermore, if $h(X, Y) = 0$ for all $X, Y \in TM$, then M is said to be *totally geodesic*, and if $H = 0$, then M is *minimal* in $\overline{M}$.

A submanifold M of a paracontact manifold $\overline{M}$ is said to be a *slant submanifold* if for any $x \in M$ and $X \in T_xM - \langle\xi\rangle$, the angle between ϕX and T_xM is constant. The constant angle $\theta \in [0, \pi/2]$ is then called *slant angle* of M. The tangent bundle TM of M is decomposed as

$$TM = D \oplus \langle\xi\rangle, \tag{2.14}$$

where the orthogonal complementary distribution D of $\langle\xi\rangle$ is known as the *slant distribution* on M. If μ is ϕ-invariant subspace of the normal bundle $T^{\perp}M$, then

$$T^{\perp}M = FTM \oplus \mu. \tag{2.15}$$

Khan et al. [6] proved the following theorem for a slant submanifold M of a Lorentzian paracontact manifold $\overline{M}$ with slant angle θ.

Theorem 2.1. *Let M be a submanifold of an LP-contact manifold $\overline{M}$ such that $\xi \in TM$, then M is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(I + \eta \otimes \xi). \tag{2.16}$$

Furthermore, if θ is slant angle of M, then $\lambda = cos^2\theta$.

Thus, one has the following consequences of formula (2.16)*:*

$$g(PX,PX) = \cos^2\theta\left[g(X,Y) + \eta(X)\eta(Y)\right], \tag{2.17}$$

$$g(FX,FY) = \sin^2\theta\left[g(X,Y) + \eta(X)\eta(Y)\right], \tag{2.18}$$

for any $X, Y \in TM$*.*

3. Totally Umbilical Proper Slant Submanifold

In this section, we consider M as a totally umbilical proper slant submanifold of an LP-cosymplectic manifold $\overline{M}$. Such submanifolds we always consider tangent to the structure vector field ξ.

Theorem 3.1. *A nontrivial totally umbilical proper slant submanifold M of an LP-cosymplectic manifold $\overline{M}$ is either totally geodesic or if it is not totally geodesic in $\overline{M}$, then the slant angle $\theta = \tan^{-1}(\sqrt{g(X,Y)/\eta(X)\eta(Y)})$, for any $X, Y \in TM$.*

Proof. For any $X, Y \in TM$, (2.11) gives

$$\left(\overline{\nabla}_X P\right)Y = A_{FY}X + Bh(X,Y). \tag{3.1}$$

Taking the product with ξ and using (2.9), we obtain

$$g(\nabla_X PY, \xi) = g(A_{FY}X, \xi) + g(Bh(X,Y), \xi). \tag{3.2}$$

Using (2.5) and the fact that M is totally umbilical, the above equation takes the form

$$-g(PY, \nabla_X\xi) = g(H, FY)\eta(X) + g(X,Y)g(BH, \xi). \tag{3.3}$$

Then, from the characteristic equation of LP-cosymplectic manifold, we obtain

$$0 = g(H, FY)\eta(X). \tag{3.4}$$

Thus, from (3.4), it follows that either $H \in \mu$ or M is trivial.

Now, for an LP-cosymplectic manifold, one has, from (2.8),

$$\overline{\nabla}_X\phi Y = \phi\overline{\nabla}_X Y, \tag{3.5}$$

for any $X, Y \in TM$. From (2.3) and (2.6), we obtain

$$\overline{\nabla}_X PY + \overline{\nabla}_X FY = \phi(\nabla_X Y + h(X,Y)). \tag{3.6}$$

Again using (2.3), (2.4), and (2.6), we get

$$\nabla_X PY + h(X, PY) - A_{FY}X + \nabla^{\perp}_X FY = P\nabla_X Y + F\nabla_X Y + \phi h(X, Y). \tag{3.7}$$

As M is totally umbilical, then

$$\nabla_X PY + h(X, PY) - A_{FY}X + \nabla^{\perp}_X FY = P\nabla_X Y + F\nabla_X Y + g(X, Y)\phi H. \tag{3.8}$$

Taking the inner product with ϕH and using the fact that $H \in \mu$, we obtain

$$g(h(X, PY), \phi H) + g\left(\nabla^{\perp}_X FY, \phi H\right) = g(F\nabla_X Y, \phi H) + g(X, Y)g(\phi H, \phi H). \tag{3.9}$$

Then from (2.2) and (2.13), we get

$$g(X, PY)g(H, \phi H) + g\left(\nabla^{\perp}_X FY, \phi H\right) = g(F\nabla_X Y, \phi H) + g(X, Y)\|H\|^2. \tag{3.10}$$

Again, using (2.2) and the fact that $H \in \mu$, then ϕH is also lies in μ; thus, we obtain

$$g\left(\nabla^{\perp}_X FY, \phi H\right) = g(X, Y)\|H\|^2. \tag{3.11}$$

Then, from (2.4), we derive

$$g\left(\overline{\nabla}_X FY, \phi H\right) = g(X, Y)\|H\|^2. \tag{3.12}$$

Now, for any $X \in TM$, one has

$$\left(\overline{\nabla}_X \phi\right)H = \overline{\nabla}_X \phi H - \phi\overline{\nabla}_X H. \tag{3.13}$$

Using the fact that as $\overline{M}$ is an LP-cosymplectic manifold, we obtain

$$\overline{\nabla}_X \phi H = \phi\overline{\nabla}_X H. \tag{3.14}$$

Using (2.4), (2.6), and (2.7), we obtain

$$-A_{\phi H}X + \nabla^{\perp}_X \phi H = -PA_H X - FA_H X + B\nabla^{\perp}_X H + C\nabla^{\perp}_X H. \tag{3.15}$$

Taking the product in (3.15) with FY for any $Y \in TM$ and using the fact $C\nabla^{\perp}_X H \in \mu$, the above equation gives

$$g\left(\nabla^{\perp}_X \phi H, FY\right) = -g(FA_H X, FY). \tag{3.16}$$

Using (2.18), we obtain

$$g\left(\overline{\nabla}_X FY, \phi H\right) = \sin^2\theta\left[g(A_H X, Y) + \eta(A_H X)\eta(Y)\right], \tag{3.17}$$

then, from (2.5) and (2.13), we get

$$g\left(\overline{\nabla}_X FY, \phi H\right) = \sin^2\theta\left[g(X, Y) + \eta(X)\eta(Y)\right]\|H\|^2. \tag{3.18}$$

Thus, from (3.12) and (3.18), we derive

$$\left[\cos^2\theta g(X, Y) - \sin^2\theta\eta(X)\eta(Y)\right]\|H\|^2 = 0. \tag{3.19}$$

Hence, (3.19) gives either $H = 0$ or if $H \neq 0$, then the slant angle of M is $\theta = \tan^{-1}(\sqrt{g(X,Y)/\eta(X)\eta(Y)})$. This proves the theorem completely. □

4. Hemislant Submanifolds

In the following section, we assume that M is a hemi-slant submanifold of an LP-cosymplectic manifold $\overline{M}$ such that the structure vector field ξ tangent to M. First, we define a hemi-slant submanifold, and then we obtain the integrability conditions of the involved distributions D_1 and D_2 in the definition of a hemi-slant submanifold M of an LP-cosymplectic manifold $\overline{M}$.

Definition 4.1. A submanifold M of an LP-contact manifold $\overline{M}$ is said to be a hemi-slant submanifold if there exist two orthogonal complementary distributions D_1 *and* D_2 satisfying

(i) $TM = D_1 \oplus D_2 \oplus \langle\xi\rangle$,

(ii) D_1 is a slant distribution with slant angle $\theta \neq \pi/2$,

(iii) D_2 is totally real that is, $\phi D_2 \subseteq T^{\perp}M$.

If μ is ϕ-invariant subspace of the normal bundle $T^{\perp}M$, then in case of hemi-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = FD_1 \oplus FD_2 \oplus \mu. \tag{4.1}$$

In the following, we obtain the integrability conditions of involved distributions in the definition of hemi-slant submanifold.

Proposition 4.2. *Let M be a hemi-slant submanifold of an LP-cosymplectic manifold $\overline{M}$, then the anti-invariant distribution D_2 is integrable if and only if*

$$A_{FZ}W = A_{FW}Z, \tag{4.2}$$

for any $Z, W \in D_2$.

Proof. For any $Z, W \in D_2$, one has

$$\phi[Z, W] = \phi\overline{\nabla}_Z W - \phi\overline{\nabla}_W Z. \tag{4.3}$$

Using (2.8), we obtain

$$\phi[Z, W] = \overline{\nabla}_Z \phi W - \overline{\nabla}_W \phi Z. \tag{4.4}$$

Then, from (2.4), we derive

$$\phi[Z, W] = -A_{FW}Z + \nabla^{\perp}_Z FW + A_{FZ}W - \nabla^{\perp}_W FZ. \tag{4.5}$$

As D_2 is an anti-invariant distribution, then the tangential part of (4.5) should be identically zero; hence, we obtain the required result. □

Proposition 4.3. *Let M be a hemi-slant submanifold of an LP-cosymplectic manifold $\overline{M}$, then the invariant distribution $D_1 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g\Big(h(X, PY) - h(Y, PX) + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX, FZ\Big) = 0, \tag{4.6}$$

for any $X, Y \in D_1 \oplus \langle\xi\rangle$ and $Z \in D_2$.

Proof. For any $X, Y \in D_1 \oplus \langle\xi\rangle$, one has

$$\phi[X, Y] = \phi\overline{\nabla}_X Y - \phi\overline{\nabla}_Y X. \tag{4.7}$$

Then, from (2.8) and the fact that $\overline{M}$ is LP-cosymplectic, we obtain

$$\phi[X, Y] = \overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X. \tag{4.8}$$

Using (2.6), we get

$$\phi[X, Y] = \overline{\nabla}_X PY + \overline{\nabla}_X FY - \overline{\nabla}_Y PX - \overline{\nabla}_Y FX. \tag{4.9}$$

Thus, from (2.3) and (2.4), we derive

$$\phi[X, Y] = \nabla_X PY + h(X, PY) - A_{FY}X + \nabla^{\perp}_X FY - \nabla_Y PX - h(Y, PX) + A_{FX}Y - \nabla^{\perp}_Y FX. \tag{4.10}$$

Taking the product in (4.10) with FZ, for any $Z \in D_2$, we obtain

$$g(\phi[X, Y], FZ) = g\Big(h(X, PY) + \nabla^{\perp}_X FY - h(Y, PX) - \nabla^{\perp}_Y FX, FZ\Big). \tag{4.11}$$

Thus, the assertion follows from (4.11) after using (2.2) and the fact that ξ is tangential to D_1. □

Now, we consider M as a totally umbilical hemi-slant submanifold of an LP-cosymplectic manifold $\overline{M}$. For any $X, Y \in TM$, one has

$$\overline{\nabla}_X \phi Y = \phi \overline{\nabla}_X Y. \tag{4.12}$$

Using this fact, if we take for any $Z, W \in D_2$, then from (2.3) and (2.4), the above equation takes the form

$$-A_{FW} Z + \nabla^{\perp}_Z FW = \phi(\nabla_Z W + h(Z, W)). \tag{4.13}$$

Thus, on using (2.6) and (2.7), we obtain

$$-A_{FW} Z + \nabla^{\perp}_Z FW = P\nabla_Z W + F\nabla_Z W + Bh(Z, W) + Ch(Z, W). \tag{4.14}$$

Equating the tangential components, we get

$$P\nabla_Z W = -A_{FW} Z - Bh(Z, W). \tag{4.15}$$

Taking the product with $V \in D_2$, we obtain

$$g(P\nabla_Z W, V) = -g(A_{FW} Z, V) - g(Bh(Z, W), V). \tag{4.16}$$

Using (2.2), (2.5), and the fact that $PW = 0$, for any $W \in D_2$, thus, the above equation takes the form

$$0 = g(h(Z, V), FW) + g(Bh(Z, W), V). \tag{4.17}$$

As M is totally umbilical, we derive

$$0 = g(Z, V)g(H, FW) + g(Z, W)g(BH, V). \tag{4.18}$$

Thus, (4.18) has a solution if either $Z = W = V = \xi$, that is, $\dim D_2 = 1$ or $H \in \mu$ or $D_2 = \{0\}$. Hence, we state the following theorem.

Theorem 4.4. *Let M be a totally umbilical hemi-slant submanifold of an LP-cosymplectic manifold $\overline{M}$, then at least one of the following statements is true:*

(i) *the dimension of anti-invariant distribution is one, that is,* $\dim D_2 = 1$*,*

(ii) *the mean curvature vector $H \in \mu$,*

(iii) *M is proper slant submanifold of $\overline{M}$.*

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