Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 575679, 11 pages
doi:10.1155/2011/575679

Research Article

Generalized Jacobi Elliptic Function Solution to a Class of Nonlinear Schrödinger-Type Equations

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Received 17 December 2010; Accepted 10 February 2011

Academic Editor: Cristian Toma

With the help of the generalized Jacobi elliptic function, an improved Jacobi elliptic function method is used to construct exact traveling wave solutions of the nonlinear partial differential equations in a unified way. A class of nonlinear Schrödinger-type equations including the generalized Zakharov system, the Rangwala-Rao equation, and the Chen-Lee-Lin equation are investigated, and the exact solutions are derived with the aid of the homogenous balance principle.

1. Introduction

Nonlinear phenomena appear in a wide variety of scientific fields, such as applied mathematics, physics and engineering problems. However, solving nonlinear partial differential equations (NLPDEs) corresponding to the nonlinear problems is often complicate. Especially, obtaining their explicit solutions is even more difficult. Up to now, a lot of new methods for solving NLPDEs are developed, for example, Bäckland transformation method, inverse scattering method, Darboux transformation method, Hirota's bilinear method, homogeneous balance method, Jacobi elliptic function method, tanh-function method, variational iteration method, the sine-cosine method, F-expansion method, Lucas Riccati method, and so on [1–15]. But, generally speaking, all of the above methods have their own advantages and shortcomings, respectively.

Nowadays, many exact solutions of NLPDEs can be written as a polynomial in several elementary or special functions which satisfy first-order nonlinear ordinary differential equation (NLODE) with a sixth-degree nonlinear term. The aim of this paper, motivated by [13, 15], is to perform a first-order NLODE with sixth-degree nonlinear term which is, in nature, an extension of a type of elliptic equation, into a new algebraic or new auxiliary equation method to seek exact solutions to a class of nonlinear Schrödinger-type equations.

The rest of this paper is organized as follows. In Section 2, we give the description of the generalized improved Jacobi elliptic function method. In Section 3, we apply this method to the generalized Zakharov system, the Rangwala-Rao equation, and the Chen-Lee-Lin equation. Finally, we conclude the paper and give some futures and comments.

2. Description of the Improved Jacobi Elliptic Function Method

The main idea of this method is to take full advantage of the elliptic equation that the generalized Jacobi elliptic functions (GJEFs) satisfy [13, 16–18]. The desired elliptic equation read

$$F'(\xi) = \sqrt{A_0 + A_2F^2(\xi) + A_4F^4(\xi) + A_6F^6(\xi)}, \quad ' \equiv \frac{d}{d\xi}, \tag{2.1}$$

where $\xi \equiv \xi(x,t)$ and A_0, A_2, A_4, A_6 are constants.

Case 1. If $A_0 = 1, A_2 = -(1 + k_1^2 + k_2^2), A_4 = k_1^2 + k_2^2 + k_1^2k_2^2$ and $A_6 = -k_1^2k_2^2$, then (2.1) has a solution $s(\xi, k_1, k_2)$.

Case 2. If $A_0 = 1 - k_1^2 - k_2^2 + k_1^2k_2^2, A_2 = 2k_1^2 + 2k_2^2 - 3k_1^2k_2^2 - 1, A_4 = 3k_1^2k_2^2 - k_1^2 - k_2^2$ and $A_6 = -k_1^2k_2^2$, then (2.1) has a solution $c(\xi, k_1, k_2)$.

Case 3. If $A_0 = k_1^2 - 1 - k_2^2 + k_2^2k_1^{-2}, A_2 = 2k_2^2 + 2 - k_1^2 - 3k_2^2k_1^{-2}, A_4 = 3k_2^2k_1^{-2} - k_2^2 - 1$ and $A_6 = -k_2^2k_1^{-2}$, then (2.1) has a solution $d_1(\xi, k_1, k_2)$.

Case 4. If $A_0 = k_2^2 - 1 - k_1^2 + k_1^2k_2^{-2}, A_2 = 2k_1^2 + 2 - k_2^2 - 3k_1^2k_2^{-2}, A_4 = 3k_1^2k_2^{-2} - k_1^2 - 1$ and $A_6 = -k_1^2k_2^{-2}$, then (2.1) has a solution $d_2(\xi, k_1, k_2)$.

$s(\xi, k_1, k_2)$ is the generalized Jacobi elliptic sine function, ξ is an independent variable, k_1, k_2 $(0 \le k_2 \le k_1 \le 1)$ are two modulus of the GJEFs, $c(\xi, k_1, k_2)$ is the generalized Jacobi elliptic cosine function, $d_1(\xi, k_1, k_2)$ is the generalized Jacobi elliptic function of the third kind, and $d_2(\xi, k_1, k_2)$ is the generalized Jacobi elliptic function of the forth kind [13, 16–18]. The definitions and properties of the GJEFs are given in the appendix.

For a given NLPDEs involving the two independent variables x, t,

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{2.2}$$

where P is in general a polynomial function of its argument and the subscripts denote the partial derivatives, by using the traveling wave transformation, Equation (2.2) possesses the following ansätz:

$$u(x,t) = U(\xi), \quad \xi = k(x - \omega t), \tag{2.3}$$

where k, ω are constants to be determined later. Substituting (2.3) into (2.2) yields an ordinary differential equation (ODE):

$O(u(\xi), u(\xi)_{\xi}, u(\xi)_{\xi\xi}, \ldots) = 0$. Then, $u(\xi)$ is expanded into a polynomial of $F(\xi)$ in the form

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i F^i(\xi). \tag{2.4}$$

The processes take the following steps.

Step 1. Determine n in (2.4) by balancing the linear term(s) of the highest order with the nonlinear term(s) in (2.2).

Step 2. Substituting (2.4) with (2.1) into (2.2), then the left-hand side of (2.2) can be converted into a polynomial in $F(\xi)$. Setting each coefficient of the polynomial to zero yields system of algebraic equations for $a_0, a_1, \ldots, a_n$, k and ω.

Step 3. Solving this system obtained in Step 2, then $a_0, a_1, \ldots, a_n$, k and ω can be expressed by A_0, A_2, A_4, A_6. Substituting these into (2.4), then general form of traveling wave solution of (2.2) can be obtained. In the following section, we apply this method to class of nonlinear Schrödinger-type equations to obtain new quasidoubly periodic solution.

3. Applications

In the following, we use the improved Jacobi elliptic function method to seek exact traveling wave solutions of class of nonlinear Schrödinger-type equations which are of interest in plasma physics, wave propagation in nonlinear optical fibers, Ginzburg-Landau theory of superconductivity, and so forth.

3.1. Generalized Zakharov's System

In the interaction of laser-plasma the system of Zakharov's equation plays an important role. This system has wide interest and attention for many scientists.

Let us consider the generalized Zakharov system [19]

$$\begin{gathered} u_{tt} - c_s^2 u_{xx} = \beta\left(|E|^2\right)_{xx}, \\ iE_t + \alpha E_{xx} - \delta_1 uE + \delta_2 |E|^2 E + \delta_3 |E|^4 E = 0. \end{gathered} \tag{3.1}$$

When $\delta_2 = \delta_3 = 0$, the generalized Zakharov system reduces to the famous Zakharov system which describe the propagation Langmuir waves in plasmas. The real unknown function $u(x,t)$ is the fluctuation in the ion density about its equilibrium value, and the complex unknown function $E(x,t)$ is the slowly varying envelope of highly oscillatory electron field. The parameters α, β, δ_1, δ_2, δ_3, and c_s are real numbers, where c_s is proportional to the ion acoustic speed (or electron sound speed). Here, we seek its traveling wave solution in the forms

$$E(x,t) = H(\xi)e^{i(kx-\omega t)}, \qquad u(x,t) = u(\xi), \quad \xi = x - ct, \tag{3.2}$$

where k, ω, and c are constants and $H(\xi)$ is real function. Therefore, system (3.1) reduces to

$$\left(c^2 - c_s^2\right)u'' = \beta\left(H^2\right)_{\xi\xi}, \tag{3.3}$$

$$\alpha H'' + i(2\alpha k - c)H' + \left(\omega - \alpha k^2\right)H - \delta_1 uH + \delta_2 H^3 + \delta_3 H^5 = 0. \tag{3.4}$$

Integrating (3.3) with respect to ξ and taking the integration constants to zero yield

$$u = \frac{\beta}{c^2 - c_s^2}H^2, \quad c^2 - c_s^2 \neq 0. \tag{3.5}$$

Substituting (3.5) into (3.4) results in

$$H'' + \frac{1}{\alpha}\left[\left(\omega - \alpha k^2\right)H + \left(\delta_2 - \frac{\beta\delta_1}{c^2 - c_s^2}\right)H^3 + \delta_3 H^5\right] = 0, \quad c = 2\alpha k,\ \alpha \neq 0. \tag{3.6}$$

According to Step 3, we assume that (3.6) possesses the solutions in the form

$$H(\xi) = a_0 + a_1 F(\xi). \tag{3.7}$$

Substituting (3.7) with (2.1) into (3.6) and equating each of the coefficients of $F^i(\xi)$, $i = 0, 1, \ldots, 5$ to zero, we obtain system of algebraic equations. To avoid tediousness, we omit the overdetermined algebraic equations. From the output of Maple, we obtain the following solution:

$$a_0 = 0, \qquad c = 2\alpha k, \qquad \omega = \alpha\left(k^2 - A_2\right), \qquad a_1 = \pm\sqrt{\frac{3A_6}{2\delta_3 A_4}\left(\delta_2 - \frac{\beta\delta_1}{4\alpha^2 k^2 - c_s^2}\right)}. \tag{3.8}$$

Now, based on the solutions of (2.1), one can obtain new types of quasiperiodic wave solution of the generalized Zakharov system. We obtain the general formulae of the solution of system (3.1)

$$\begin{aligned} u(x,t) &= \frac{3A_6\beta}{2\delta_3 A_4\left(4\alpha^2k^2 - c_s^2\right)}\left(\delta_2 - \frac{\beta\delta_1}{4\alpha^2 k^2 - c_s^2}\right)F^2(x - 2\alpha k t), \\ E(x,t) &= \pm\sqrt{\frac{3A_6}{2\delta_3 A_4}\left(\delta_2 - \frac{\beta\delta_1}{4\alpha^2 k^2 - c_s^2}\right)}F(x - 2\alpha k t)e^{i(kx - \omega t)}. \end{aligned} \tag{3.9}$$

By selecting the special values of the A_0, A_2, A_4, A_6 and the corresponding function $F(\xi)$, we have the following solutions of the generalized Zakharov system (3.1):

$$\begin{aligned}
u_1(x,t) &= \frac{-3k_1^2k_2^2\beta}{2\delta_3\left(k_1^2+k_2^2+k_1^2k_2^2\right)\left(4\alpha^2k^2-c_s^2\right)}\left(\delta_2-\frac{\beta\delta_1}{4\alpha^2k^2-c_s^2}\right)s^2(x-2\alpha kt,k_1,k_2),\\
E_1(x,t) &= \pm\sqrt{\frac{-3k_1^2k_2^2}{2\delta_3\left(k_1^2+k_2^2+k_1^2k_2^2\right)}\left(\delta_2-\frac{\beta\delta_1}{4\alpha^2k^2-c_s^2}\right)}s(x-2\alpha kt,k_1,k_2)e^{i(kx-\omega t)},\\
u_2(x,t) &= \frac{-3k_1^2k_2^2\beta}{2\delta_3\left(3k_1^2k_2^2-k_1^2-k_2^2\right)\left(4\alpha^2k^2-c_s^2\right)}\left(\delta_2-\frac{\beta\delta_1}{4\alpha^2k^2-c_s^2}\right)c^2(x-2\alpha kt,k_1,k_2),\\
E_2(x,t) &= \pm\sqrt{\frac{-3k_1^2k_2^2}{2\delta_3\left(3k_1^2k_2^2-k_1^2-k_2^2\right)}\left(\delta_2-\frac{\beta\delta_1}{4\alpha^2k^2-c_s^2}\right)}c(x-2\alpha kt,k_1,k_2)e^{i(kx-\omega t)}.
\end{aligned}\tag{3.10}$$

We omitted the reminder solutions for simplicity.

3.2. Rangwala-Rao Equation

The Rangwala-Rao equation [20] is

$$u_{xt}-\beta_1u_{xx}+u+iT\beta_2|u|^2u_x=0,\quad T=\pm1,\tag{3.11}$$

where β_1, β_2 are real constants. Rangwala and Rao introduced Equation (3.11) as the integrability condition when they studied the mixed, derivative, nonlinear Schrödinger equations and looked for the Bäcklund transformation and solitary wave solutions.

Suppose the exact solutions of (3.11) is of the form

$$u(x,t)=e^{-i\omega t}e^{i\varphi(x-ct)}H(x-ct),\tag{3.12}$$

where $\omega,\ c$ are constants determined later and φ, H are undetermined functions with one variable only. Set the relation of φ, H as

$$\varphi'(\xi)=\frac{\omega}{2(c+\beta_1)}+\frac{T\beta_2}{4(c+\beta_1)}H^2(\xi),\qquad '=\frac{d}{d\xi},\ \xi=x-ct.\tag{3.13}$$

Substituting (3.12) with (3.13) into (3.11) simultaneously yields

$$H''-\frac{4(c+\beta_1)-\omega^2}{4(c+\beta_1)^2}H-\frac{T\beta_2\omega}{2(c+\beta_1)^2}H^3+\frac{3T^2\beta_2^2}{16(c+\beta_1)^2}H^5=0.\tag{3.14}$$

According to the homogeneous balance principle, we suppose that the exact solutions of (3.14) take the form

$$H(\xi) = a_0 + a_1 F(\xi). \tag{3.15}$$

Substituting (3.15) with (2.1) into (3.14) and equating each of the coefficients of $F^i(\xi)$, $i = 0, 1, \ldots, 5$ to zero, we obtain system of algebraic equations. Solving this system with the aid of Maple, we obtain the following solution:

$$a_0 = 0, \qquad \omega = 2\sqrt{(c+\beta_1)\left[1-(c+\beta_1)A_2\right]}, \qquad a_1 = \pm\sqrt{\frac{-8A_6\omega}{3TA_4\beta_2}}. \tag{3.16}$$

The general formulae of the solutions of Rangwala-Rao equation

$$u(x,t) = \pm\sqrt{\frac{-8A_6\omega}{3TA_4\beta_2}} F(x-ct) e^{-i\omega t} e^{i\varphi(x-ct)}, \tag{3.17}$$

with $\varphi(\xi) = \omega/6A_4(c+\beta_1)\int[3A_4 - 4A_6F^2(\xi)]d\xi$, $\omega = 2\sqrt{(c+\beta_1)[1-(c+\beta_1)A_2]}$.

By selecting the special values of the A_0, A_2, A_4, A_6 and the corresponding function $F(\xi)$, we have the following intensities of the solutions of the Rangwala-Rao equation.

When $A_0 = 1$, $A_2 = -(1+k_1^2+k_2^2)$, $A_4 = k_1^2+k_2^2+k_1^2k_2^2$ and $A_6 = -k_1^2k_2^2$, we have

$$|u_1|^2 = -\frac{16k_1^2k_2^2\sqrt{(c+\beta_1)\left[1+(1+k_1^2+k_2^2)(c+\beta_1)\right]}}{3T\beta_2(k_1^2+k_2^2+k_1^2k_2^2)} s^2(x-ct,k_1,k_2), \tag{3.18}$$

and when $A_0 = 1-k_1^2-k_2^2+k_1^2k_2^2$, $A_2 = 2k_1^2+2k_2^2-3k_1^2k_2^2-1$, $A_4 = 3k_1^2k_2^2-k_1^2-k_2^2$, and $A_6 = -k_1^2k_2^2$, we have

$$|u_2|^2 = -\frac{16k_1^2k_2^2\sqrt{(c+\beta_1)\left[1-(2k_1^2+2k_2^2-3k_1^2k_2^2)(c+\beta_1)\right]}}{3T\beta_2(3k_1^2k_2^2-k_1^2-k_2^2)} c^2(x-ct,k_1,k_2). \tag{3.19}$$

We omitted the reminder intensities for simplicity.

3.3. *Chen-Lee-Lin Equation*

The Chen-Lee-Lin equation [20] is

$$iu_t + u_{xx} + i\delta|u|^2u_x = 0, \tag{3.20}$$

where δ is a real constant. Similarly as before, we suppose the exact solution of (3.20) is of the form

$$u(x,t) = e^{-i\omega t}e^{i\varphi(x-ct)}H(x-ct). \tag{3.21}$$

Set the relation of φ, H as

$$\varphi'(\xi) = \frac{c}{2} - \frac{\delta}{4}H^2(\xi), \qquad ' = \frac{d}{d\xi}, \ \xi = x - ct. \tag{3.22}$$

Substituting (3.21) with (3.22) into (3.20) simultaneously yields

$$H'' + \left(\omega + \frac{c^2}{4}\right)H - \frac{c\delta}{2}H^3 + \frac{3\delta^2}{16}H^5 = 0. \tag{3.23}$$

According to the homogeneous balance principle, we suppose that the exact solutions of (3.23) take the form

$$H(\xi) = a_0 + a_1 F(\xi). \tag{3.24}$$

Substituting (3.24) with (2.1) into (3.23) and equating each of the coefficients of $F^i(\xi)$, $i = 0, 1, \ldots, 5$ to zero, we obtain system of algebraic equations. Solving this system with the aid of Maple, we obtain the following solution:

$$a_0 = 0, \qquad \omega = A_2 - \frac{c^2}{4}, \qquad a_1 = \pm 2\sqrt{\frac{-cA_6}{\delta A_4}}. \tag{3.25}$$

The general formulae of the solution of Chen-Lee-Lin equation

$$u(x,t) = \pm 2\sqrt{\frac{-cA_6}{\delta A_4}}F(x-ct)e^{-i\omega t}e^{i\varphi(x-ct)}, \tag{3.26}$$

with $\varphi(\xi) = (c/2A_4)\int[A_4 - 2A_6F^2(\xi)]d\xi$ and $\omega = A_2 - c^2/4$. By selecting the special values of the A_0, A_2, A_4, A_6 and the corresponding function $F(\xi)$, we have the following intensities of the solutions of the Chen-Lee-Lin equation.

When $A_0 = 1$, $A_2 = -(1 + k_1^2 + k_2^2)$, $A_4 = k_1^2 + k_2^2 + k_1^2k_2^2$ and $A_6 = -k_1^2k_2^2$, we have

$$|u_1|^2 = -\frac{4ck_1^2k_2^2}{\delta\left(k_1^2 + k_2^2 + k_1^2k_2^2\right)}s^2(x - ct, k_1, k_2), \tag{3.27}$$

and when $A_0 = 1 - k_1^2 - k_2^2 + k_1^2k_2^2$, $A_2 = 2k_1^2 + 2k_2^2 - 3k_1^2k_2^2 - 1$, $A_4 = 3k_1^2k_2^2 - k_1^2 - k_2^2$ and $A_6 = -k_1^2k_2^2$, we have

$$|u_2|^2 = -\frac{4ck_1^2k_2^2}{\delta\left(3k_1^2k_2^2 - k_1^2 - k_2^2\right)}c^2(x - ct, k_1, k_2). \tag{3.28}$$

We omitted the reminder intensities for simplicity.

Besides the solutions obtained above, the ODE Equation (2.1), albeit with different parameters, has been studied in the different context [21–24]. It has been shown that this equation possesses abundant solutions, Including Weierstrass function solutions, kink solutions, periodic solutions, and so forth. To the best of our knowledge, some of our explicit solutions are new.

Notice that the GJEFs are generalization of the Jacobi elliptic, hyperbolic, and trigonometric functions as stated in the appendix. Also, the two modulus parameters k_1 and k_2 describe the degree of the wave energy localization in the obtained solutions.

4. Conclusion

There is no systematic way for solving (2.1). Nevertheless, this ansätz with four arbitrary parameters A_0, A_2, A_4, A_6 is reasonable since its solution can be expressed in terms of functions, such as generalized Jacobi elliptic functions, that appear only in the nonlinear problems. In addition, these functions go back, in some limiting cases, to sn, cn, dn, tanh, sech, sin, and cos functions that describe the double periodic, periodic, solitary, and shock wave propagation. The values of the constants a_i $(i = 0, 1, \ldots, n)$ in (2.4) depend crucially on the nature of differential equations whereas different types of their solutions can be classified in terms of A_0, A_2, A_4, A_6 as shown in Cases 1–4. In this work, we obtain the exact solutions of the generalized Zakharov system, the Rangwala-Rao equation, and the Chen-Lee-Lin equation by using GJEFs. We believe one can apply this method to many other nonlinear partial differential equations in mathematical physics.

Appendix

In this appendix, we review the GJEFs and study some properties of these functions [13, 16–18]. We consider the (pseudo-) hyperelliptic integral

$$y(x, k_1, k_2) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k_1^2 t^2)(1 - k_2^2 t^2)}}. \tag{A.1}$$

We define the generalized Jacobi elliptic sine function as the inverse function $x = s(y, k_1, k_2)$, where y is an independent variable and k_1, k_2 $(0 \leq k_2 \leq k_1 \leq 1)$ are two modulus of the GJEFs. Similarly, $\sqrt{1 - x^2}$, $\sqrt{1 - k_1^2 x^2}$, and $\sqrt{1 - k_2^2 x^2}$ are defined as the generalized Jacobi elliptic cosine function, the generalized Jacobi elliptic function of the third kind, and the generalized Jacobi elliptic function of the forth kind. They are expressed as

$$\sqrt{1 - x^2} = c(y, k_1, k_2), \qquad \sqrt{1 - k_1^2 x^2} = d_1(y, k_1, k_2), \qquad \sqrt{1 - k_2^2 x^2} = d_2(y, k_1, k_2). \tag{A.2}$$

The GJEFs possess the following properties of the triangular functions (we use the abbreviated notations $s(y) \equiv s(y, k_1, k_2)$, $c(y) \equiv (y, k_1, k_2), \ldots$, and so forth):

$$\begin{gathered}
c^2(y) = 1 - s^2(y), \qquad d_1^2(y) = 1 - k_1^2 s^2(y), \\
d_2^2(y) = 1 - k_2^2 s^2(y), \qquad k_1^2 d_2^2(y) - k_2^2 d_1^2(y) = k_1^2 - k_2^2, \\
d_i^2(y) - k_i^2 c^2(y) = 1 - k_i^2, \quad (i = 1, 2).
\end{gathered} \tag{A.3}$$

The first derivatives of these functions are given by

$$\begin{gathered}
s'(y) = c(y) d_1(y) d_2(y), \qquad c'(y) = -s(y) d_1(y) d_2(y), \\
d_1'(y) = -k_1^2 s(y) c(y) d_2(y), \qquad d_2'(y) = -k_2^2 s(y) c(y) d_1(y).
\end{gathered} \tag{A.4}$$

Moreover, in the limiting case $k_2 \to 0$, the GJEF reduced to the usual JEFs

$$\begin{gathered}
s(y, k_1, 0) \longrightarrow \mathrm{sn}(y, k_1), \qquad c(y, k_1, 0) \longrightarrow \mathrm{cn}(y, k_1), \\
d_1(y, k_1, 0), \qquad d_2(y, k_1, 0) \longrightarrow \mathrm{dn}(y, k_1).
\end{gathered} \tag{A.5}$$

When $k_1 \to 1$, $k_2 \to 0$, we have

$$s(y, 1, 0) \longrightarrow \tanh(y), \qquad c(y, 1, 0), \qquad d_1(y, 1, 0), \qquad d_2(y, 1, 0) \longrightarrow \mathrm{sech}(y). \tag{A.6}$$

Also, in the limiting case $k_1 \to 0$, $k_2 \to 0$, we have

$$s(y, 0, 0) \longrightarrow \sin(y), \qquad c(y, 0, 0) \longrightarrow \cos(y), \quad d_1(y, 0, 0), \qquad d_2(y, 0, 0) \longrightarrow 1. \tag{A.7}$$

The GJEFs can be expressed in terms of the standard Jacobi elliptic functions

$$\begin{gathered}
s(y, k_1, k_2) = \frac{\mathrm{sn}(k_2' y, k)}{\sqrt{1 - k_2^2 + k_2^2 \mathrm{sn}^2(k_2' y, k)}}, \qquad c(y, k_1, k_2) = \frac{k_2' \mathrm{cn}(k_2' y, k)}{\sqrt{1 - k_2^2 \mathrm{cn}^2(k_2' y, k)}}, \\
d_1(y, k_1, k_2) = \frac{\sqrt{k_1^2 - k_2^2}\,\mathrm{dn}(k_2' y, k)}{\sqrt{k_1^2 - k_2^2 \mathrm{dn}^2(k_2' y, k)}}, \qquad d_2(y, k_1, k_2) = \frac{\sqrt{k_1^2 - k_2^2}}{\sqrt{k_1^2 - k_2^2 \mathrm{dn}^2(k_2' y, k)}},
\end{gathered} \tag{A.8}$$

with $k_2' = \sqrt{1-k_2^2}$, $k = \sqrt{(k_1^2-k_2^2)/(1-k_2^2)}$, and $0 \leq k_2 \leq k_1 \leq 1$. From the double periodic properties of the Jacobi elliptic functions, one can see that the GJEFs are quasidouble periodic

$$\begin{aligned} s\left(y+\frac{4\mathbf{K}(k)}{k_2'}\right) &= s\left(y+\frac{2i\mathbf{K}(k')}{k_2'}\right) = \pm s(y), \\ c\left(y+\frac{4\mathbf{K}(k)}{k_2'}\right) &= c\left(y+\frac{2\mathbf{K}(k)+2i\mathbf{K}(k')}{k_2'}\right) = \pm c(y), \\ d_1\left(y+\frac{2\mathbf{K}(k)}{k_2'}\right) &= d_1\left(y+\frac{4i\mathbf{K}(k')}{k_2'}\right) = \pm d_1(y), \\ d_2\left(y+\frac{2\mathbf{K}(k)}{k_2'}\right) &= d_2\left(y+\frac{2i\mathbf{K}(k')}{k_2'}\right) = \pm d_2(y), \end{aligned} \tag{A.9}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind and $k' = \sqrt{1-k^2}$ [13, 16–18].

Acknowledgment

This work has been supported by the Deanship of Scientific Research at Qassim University.

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