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Research Article

Generic Lightlike Submanifolds of an Indefinite Cosymplectic Manifold

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Lightlike geometry has its applications in general relativity, particularly in black hole theory. Indeed, it is known that lightlike hypersurfaces are examples of physical models of Killing horizons in general relativity (Galloway, 2007). In this paper, we introduce the definition of generic lightlike submanifolds of an indefinite cosymplectic manifold. We investigate new results on a class of generic lightlike submanifolds M of an indefinite cosymplectic manifold $\overline{M}$.

1. Introduction

In the generalization from Riemannian to semi-Riemannian manifolds, the induced metric may be degenerate (lightlike) therefore there is a natural existence of lightlike submanifolds and for which the local and global geometry is completely different than nondengerate case. In lightlike case, the standard textbook definitions do not make sense- and one fails to use the theory of non-degenerate geometry in the usual way. The primary difference between the lightlike submanifolds and non-degenerate submanifolds is that in the first case, the normal vector bundle intersects with the tangent bundle. Thus, the study of lightlike submanifolds becomes more difficult and different from the study of non-degenerate submanifolds. Moreover, the geometry of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds produce models of different types of horizons (event horizons, Cauchy's horizons, and Kruskal's horizons). The universe can be represented as a four-dimensional submanifold embedded in a $(4+n)$-dimensional spacetime manifold. Lightlike hypersurfaces are also studied in the thoery of electromagnetism [1]. Thus, large number of applications but limited information available, motivated us to do research on this subject matter. Kupeli [2] and Bejancu and Duggal [1] developed the general theory of degenerate (lightlike) submanifolds. They constructed a transversal vector bundle of lightlike submanifold and investigated various properties of these manifolds. The geometry of both lightlike hypersurfaces and half lightlike submanifolds of indefinite cosymplectic manifolds was studied by Jin ([3, 4]). However, a general notion of generic lightlike submanifolds of an indefinite cosymplectic manifold has not been introduced as yet.

The objective of this paper is to study generic r-lightlike submanifolds M of an indefinite cosymplectic manifold $\overline{M}$ subject to the conditions: (1) M is totally umbilical, or (2) $S(TM)$ is totally umbilical in M. In Section 1, we first of all recall some of fundamental formulas in the theory of r-lightlike submanifolds. In Section 2, we newly define generic lightlike submanifolds. After that, we prove some basic theorems which will be used in the rest of this paper. In Section 3, we study generic r-lightlike submanifolds of $\overline{M}$.

2. Lightlike Submanifolds

Let (M, g) be an m-dimensional lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the radical distribution $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r (1 \leq r \leq \min\{m, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $\mathrm{Rad}(TM)$ in TM and $TM^{\perp}$, respectively, called the *screen* and *coscreen distributions* on M, such that

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM), \qquad TM^{\perp} = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right), \tag{2.1}$$

where the symbol $\oplus_{\mathrm{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We use the same notation for any other vector bundle. We use the following range of indices:

$$i, j, k, \ldots \in \{1, \ldots, r\}, \qquad \alpha, \beta, \gamma, \ldots \in \{r+1, \ldots, n\}. \tag{2.2}$$

Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $\Gamma(\mathrm{ltr}(TM)_{|\mathcal{U}})$ consisting of smooth sections of $S(TM)^{\perp}_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\overline{g}\left(N_i, \xi_j\right) = \delta_{ij}, \qquad \overline{g}\left(N_i, N_j\right) = 0, \tag{2.3}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}\ (TM))$. Then we have

$$\begin{aligned} T\overline{M} &= TM \oplus \mathrm{tr}(TM) = \{\mathrm{Rad}(TM) \oplus \mathrm{tr}(TM)\} \oplus_{\mathrm{orth}} S(TM) \\ &= \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\} \oplus_{\mathrm{orth}} S(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right). \end{aligned} \tag{2.4}$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is

(1) *r-lightlike* if $1 \leq r < min\{m, n\}$;

(2) *coisotropic* if $1 \leq r = n < m$;

(3) *isotropic* if $1 \leq r = m < n$;

(4) *totally lightlike* if $1 \leq r = m = n$.

The above three classes (2)–(4) are particular cases of the class (1) as follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$, and $S(TM) = S(TM^{\perp}) = \{0\}$, respectively. The geometry of r-lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, in this paper we consider only r-lightlike submanifolds $M \equiv (M, g, S(TM), S(TM^{\perp}))$, with the following local quasiorthonormal field of frames of $\overline{M}$:

$$\{\xi_1, \dots, \xi_r, N_1, \dots, N_r, F_{r+1}, \dots, F_m, E_{r+1}, \dots, E_n\}, \tag{2.5}$$

where the sets $\{F_{r+1}, \dots, F_m\}$ and $\{E_{r+1}, \dots, E_n\}$ are orthonormal basis of $\Gamma(S(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively.

Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to (2.1). For an r-lightlike submanifold, the local Gauss-Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^{s}(X, Y) E_{\alpha}, \tag{2.6}$$

$$\overline{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X) E_{\alpha}, \tag{2.7}$$

$$\overline{\nabla}_X E_{\alpha} = -A_{E_{\alpha}} X + \sum_{i=1}^{r} \phi_{\alpha i}(X) N_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X) E_{\beta}, \tag{2.8}$$

$$\nabla_X PY = \nabla_X^{*} PY + \sum_{i=1}^{r} h_i^{*}(X, PY) \xi_i, \tag{2.9}$$

$$\nabla_X \xi_i = -A_{\xi_i}^{*} X - \sum_{j=1}^{r} \tau_{ji}(X) \xi_j, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^{*} are induced linear connections on TM and $S(TM)$, respectively, the bilinear forms h_i^{ℓ} and h_{α}^{s} on M are called the *local lightlike* and *screen second fundamental forms* on TM, respectively, h_i^{*} are called the *local radical second fundamental forms* on $S(TM)$. A_{N_i}, $A_{\xi_i}^{*}$, and $A_{E_{\alpha}}$ are linear operators on $\Gamma(TM)$ and τ_{ij}, $\rho_{i\alpha}$, $\phi_{\alpha i}$ and $\sigma_{\alpha\beta}$ are 1-forms on TM. Since $\overline{\nabla}$ is torsion-free, ∇ is also torsion-free and both h_i^{ℓ} and h_{α}^{s} are symmetric. From the fact $h_i^{\ell}(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi_i)$, we know that h_i^{ℓ} are independent of the choice of a screen distribution. We say that

$$h(X, Y) = \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^{s}(X, Y) E_{\alpha} \tag{2.11}$$

is the *second fundamental tensor* of M.

The induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r}\left\{h_i^{\ell}(X,Y)\ \eta_i(Z) + h_i^{\ell}(X,Z)\eta_i(Y)\right\}, \tag{2.12}$$

for all $X, Y \in \Gamma(TM)$, where $\eta_i s$ are the 1-forms such that

$$\eta_i(X) = \overline{g}(X, N_i), \quad \forall X \in \Gamma(TM). \tag{2.13}$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$h_i^{\ell}(X,Y) = g\left(A^*_{\xi_i}X, Y\right) - \sum_{k=1}^{r} h_k^{\ell}(X,\xi_i)\eta_k(Y), \tag{2.14}$$

$$h_i^{\ell}(X,PY) = g\left(A^*_{\xi_i}X, PY\right), \qquad \overline{g}\left(A^*_{\xi_i}X, N_j\right) = 0, \tag{2.15}$$

$$\epsilon_\alpha h_\alpha^s(X,Y) = g(A_{E_\alpha}X, Y) - \sum_{i=1}^{r}\phi_{\alpha i}(X)\eta_i(Y), \tag{2.16}$$

$$\epsilon_\alpha h_\alpha^s(X,PY) = g(A_{E_\alpha}X, PY), \qquad \overline{g}(A_{E_\alpha}X, N_i) = \epsilon_\alpha \rho_{i\alpha}(X), \tag{2.17}$$

$$h_i^*(X,PY) = g(A_{N_i}X, PY), \qquad \eta_j(A_{N_i}X) + \eta_i\left(A_{N_j}X\right) = 0, \tag{2.18}$$

$$\epsilon_\beta \sigma_{\alpha\beta} = -\epsilon_\alpha \sigma_{\beta\alpha}, \quad \forall X, Y \in \Gamma(TM), \tag{2.19}$$

where $\epsilon_\alpha = \overline{g}(E_\alpha, E_\alpha)(=\pm 1)$ is the sign of the vector field E_α. From (2.18), we know that each A_{N_i} is shape operator related to the local second fundamental form h_i^* on $S(TM)$. Replace Y by ξ_j in (2.14), we have

$$h_i^{\ell}(X,\xi_j) + h_j^{\ell}(X,\xi_i) = 0, \tag{2.20}$$

for all $X \in \Gamma(TM)$. It follows

$$h_i^{\ell}(X,\xi_i) = 0, \qquad h_i^{\ell}(\xi_j,\xi_k) = 0. \tag{2.21}$$

Also, replace X by ξ_j in (2.14) and use (2.21), we have

$$h_i^{\ell}(X,\xi_j) = g\left(X, A^*_{\xi_i}\xi_j\right), \qquad A^*_{\xi_i}\xi_j + A^*_{\xi_j}\xi_i = 0, \quad A^*_{\xi_i}\xi_i = 0. \tag{2.22}$$

For an r-lightlike submanifold, replace Y by ξ_i in (2.16), we have

$$h_\alpha^s(X,\xi_i) \ = \ -\epsilon_\alpha \phi_{\alpha i}(X), \quad \forall X \in \Gamma(TM). \tag{2.23}$$

From (2.6), (2.10), and (2.23), for all $X \in \Gamma(TM)$, we have

$$\overline{\nabla}_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j - \sum_{\alpha=r+1}^{n} \epsilon_\alpha \phi_{\alpha i}(X) E_\alpha + \sum_{j=1}^{r} h^\ell_j (X, \xi_i) N_j. \tag{2.24}$$

Definition 2.1. A lightlike submanifold M of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be *irrotational* if $\overline{\nabla}_X \xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(\mathrm{Rad}(TM))$ for all i.

Note 1. For an r-lightlike M, the above definition is equivalent to

$$h^\ell_j (X, \xi_i) = 0, \quad h^s_\alpha (X, \xi_i) = \phi_{\alpha i}(X) = 0, \quad \forall X \in \Gamma(TM). \tag{2.25}$$

Denote by $\overline{R}$ and R the curvature tensors of $\overline{\nabla}$ and ∇, respectively. Using the local Gauss-Weingarten formulas for M, we obtain

$$\begin{aligned}
\overline{R}(X, Y)Z = {} & R(X, Y)Z \\
& + \sum_{i=1}^{r} \left\{ h^\ell_i (X, Z) A_{N_i} Y - h^\ell_i (Y, Z) A_{N_i} X \right\} \\
& + \sum_{\alpha=r_1}^{n} \{ h^s_\alpha (X, Z) A_{E_\alpha} Y - h^s_\alpha (Y, Z) A_{E_\alpha} X \} \\
& + \sum_{i=1}^{r} \Bigg\{ \left(\nabla_X h^\ell_i\right)(Y, Z) - \left(\nabla_Y h^\ell_i\right)(X, Z) \\
& \qquad + \sum_{j=1}^{r} \left[\tau_{ji}(X) h^\ell_j (Y, Z) - \tau_{ji}(Y) h^\ell_j (X, Z) \right] \\
& \qquad + \sum_{\alpha=r+1}^{n} \left[\phi_{\alpha i}(X) h^s_\alpha (Y, Z) - \phi_{\alpha i}(Y) h^s_\alpha (X, Z) \right] \Bigg\} N_i \\
& + \sum_{\alpha=r+1}^{n} \Bigg\{ (\nabla_X h^s_\alpha)(Y, Z) - (\nabla_Y h^s_\alpha)(X, Z) \\
& \qquad + \sum_{i=1}^{r} \left[\rho_{i\alpha}(X) h^\ell_i (Y, Z) - \rho_{i\alpha}(Y) h^s_\alpha (X, Z) \right] \\
& \qquad + \sum_{\beta=r+1}^{n} \left[\sigma_{\beta\alpha}(X) h^s_\beta (Y, Z) - \sigma_{\beta\alpha}(Y) h^s_\beta (X, Z) \right] \Bigg\} E_\alpha,
\end{aligned} \tag{2.26}$$

for all $X, Y, Z \in \Gamma(TM)$. Assume that M is irrotational. Replace Z by ξ_k in (2.26) and use (2.10), (2.15), (2.17), and (2.25), then we have

$$\begin{aligned}\overline{R}(X,Y)\xi_k = R(X,Y)\xi_k &\\ &+\sum_{i=1}^{r}\left\{g\left(A^*_{\xi_i}Y, A^*_{\xi_k}X\right) - g\left(A^*_{\xi_i}X, A^*_{\xi_k}Y\right)\right\}N_i \\ &+\sum_{\alpha=r+1}^{n}\epsilon_\alpha\left\{g\left(A_{E_\alpha}Y, A^*_{\xi_k}X\right) - g\left(A_{E_\alpha}X, A^*_{\xi_k}Y\right)\right\}E_\alpha.\end{aligned} \tag{2.27}$$

Using (2.27) and the fact $R(X,Y)Z \in \Gamma(TM)$ for $X, Y, Z \in \Gamma(TM)$, we get

$$\begin{aligned}\overline{g}\left(\overline{R}(X,Y)Z, \xi_k\right) &= -\overline{g}\left(\overline{R}(X,Y)\xi_k, Z\right) \\ &= -g(R(X,Y)\xi_k, Z) + \sum_{i=1}^{r}\left\{g\left(A^*_{\xi_i}X, A^*_{\xi_k}Y\right) - g\left(A^*_{\xi_i}Y, A^*_{\xi_k}X\right)\right\}\eta_i(Z) \\ &= g(R(X,Y)Z, \xi_k) + \sum_{i=1}^{r}\left\{g\left(A^*_{\xi_i}X, A^*_{\xi_k}Y\right) - g\left(A^*_{\xi_i}Y, A^*_{\xi_k}X\right)\right\}\eta_i(Z) \\ &= \sum_{i=1}^{r}\left\{g\left(A^*_{\xi_i}X, A^*_{\xi_k}Y\right) - g\left(A^*_{\xi_i}Y, A^*_{\xi_k}X\right)\right\}\eta_i(Z), \quad \forall X, Y, Z \in \Gamma(TM).\end{aligned} \tag{2.28}$$

3. Indefinite Cosymplectic Manifolds

An odd dimensional smooth manifold $(\overline{M}, \overline{g})$ is called a contact metric manifold [5, 6] if there exists a $(1,1)$-type tensor field J, a vector field ζ, called the characteristic vector field, and its 1-form θ satisfying

$$\begin{aligned}&J^2X = -X + \theta(X)\zeta, \qquad J\zeta = 0, \qquad \theta \circ J = 0, \qquad \theta(\zeta) = 1, \\ &\overline{g}(\zeta,\zeta) = \epsilon, \qquad \overline{g}(JX, JY) = \overline{g}(X,Y) - \epsilon\theta(X)\theta(Y), \\ &\theta(X) = \epsilon\overline{g}(\zeta, X), \qquad d\theta(X,Y) = \overline{g}(JX, Y), \quad \epsilon = \pm 1,\end{aligned} \tag{3.1}$$

for any vector fields X, Y on $\overline{M}$. Then the set $(J, \theta, \zeta, \overline{g})$ is called a contact metric structure on $\overline{M}$. Note that we may assume that $\epsilon = 1$ without loss of generality [7]. We say that $\overline{M}$ has a normal contact structure [5, 8] if $N_J + d\theta \otimes \zeta = 0$, where N_J is the Nijenhuis tensor field of J. A normal contact metric manifold is called a cosymplectic [9, 10] for which we have

$$\overline{\nabla}_X\theta = 0, \qquad \overline{\nabla}_X J = 0, \tag{3.2}$$

for any vector field X on $\overline{M}$. A cosymplectic manifold $\overline{M} = (\overline{M}, J, \zeta, \theta, \overline{g})$ is called an *indefinite cosymplectic manifold* [3, 4] if $(\overline{M}, \overline{g})$ is a semi-Riemannian manifold of index μ (> 0). For any indefinite cosymplectic manifold, apply $\overline{\nabla}_X$ to $J\zeta = 0$ for any vector field X on $\overline{M}$ and use (3.2), then we have $J(\overline{\nabla}_X\zeta) = 0$. Apply J to this and use (3.1) and $\theta(\overline{\nabla}_X\zeta) = 0$, we get

$$\overline{\nabla}_X\zeta = 0. \tag{3.3}$$

An indefinite cosymplectic manifold $\overline{M}$ is called an *indefinite cosymplectic space form*, denoted by $\overline{M}(c)$, if it has the constant J-sectional curvature c [3, 9, 10]. The curvature tensor $\overline{R}$ of this space form $\overline{M}(c)$ is given by

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{&\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \theta(X)\theta(Z)Y \\ &- \theta(Y)\theta(Z)X + \overline{g}(X,Z)\theta(Y)\zeta - \overline{g}(Y,Z)\theta(X)\zeta \\ &+\overline{g}(JY,Z)JX + \overline{g}(JZ,X)JY - 2\overline{g}(JX,Y)JZ\},\end{aligned} \tag{3.4}$$

for any vector fields X, Y, and Z in $\overline{M}$.

Let M be an m-dimensional r-lightlike submanifold of an $(m+n)$-dimensional indefinite cosymplectic manifold $\overline{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to (2.4). The characteristic vector field ζ of $\overline{M}$ from (2.4) is decomposed by

$$\zeta = P\zeta + \sum_{i=1}^{r} a_i\xi_i + \sum_{i=1}^{r} b_iN_i + \sum_{\alpha=r+1}^{n} e_\alpha E_\alpha, \tag{3.5}$$

where $a_i = \theta(N_i), b_i = \theta(\xi_i)$ and $e_\alpha = \epsilon_\alpha\theta(E_\alpha)$ are smooth functions on $\overline{M}$.

Note 2. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^* = TM/\mathrm{Rad}(TM)$ considered by Kupeli [2]. Thus all screen distributions $S(TM)$ are mutually isomorphic. For this reason, the following definition is well defined.

Definition 3.1 (see [6]). One says that M is *generic lightlike submanifold* of $\overline{M}$ if there exists a screen distribution $S(TM)$ of M such that

$$J\left(S(TM)^{\perp}\right) \subset S(TM). \tag{3.6}$$

Proposition 3.2 (see [3]). *Let M be a lightlike hypersurface of an indefinite cosymplectic manifold $\overline{M}$. Then M is a generic lightlike submanifold of $\overline{M}$.*

Proposition 3.3 (see [4]). *Let M be a 1-lightlike submanifold of codimension 2 of an indefinite cosymplectic manifold $\overline{M}$ such that the coscreen distribution $S(TM^{\perp})$ is spacelike. Then M is a generic lightlike submanifold of $\overline{M}$.*

Theorem 3.4. *Let M be an irrotational generic r-lightlike submanifold of an indefinite cosymplectic space form $\overline{M}(c)$. Then one has $c = 0$.*

Proof. Assume that $b_k \neq 0$ in (3.5). Note that (3.1) implies $\overline{g}(JX, \xi_k) = -\overline{g}(X, J\xi_k)$ for all $X \in \Gamma(TM)$. Then, taking the scalar product with ξ_k to (3.4), and using (2.28), we get

$$\begin{aligned} 4\sum_{i=1}^{r}\Big\{g\Big(A^*_{\xi_i}X, A^*_{\xi_k}Y\Big) - g\Big(A^*_{\xi_i}Y, A^*_{\xi_k}X\Big)\Big\}\eta_i(Z) \\ = c\{b_k g(X,Z)\theta(Y) - b_k g(Y,Z)\theta(X) - \overline{g}(JY,Z)g(X,J\xi_k) \\ -\overline{g}(JZ,X)g(Y,J\xi_k) + 2\overline{g}(JX,Y)g(Z,J\xi_k)\}, \quad \forall X,Y,Z \in \Gamma(TM). \end{aligned} \tag{3.7}$$

Replace Z by $J\xi_k$ and Y by ξ_k in the equation and use (3.1), then we have

$$b_k^2 c g(X, J\xi_k) = 0, \quad \forall X \in \Gamma(TM), \tag{3.8}$$

because $\eta_i(J\xi_k) = 0$ by (3.6). Replacing X by $J\xi_k$ in this equation, we obtain $b_k^4 c = 0$. Since $b_k \neq 0$, we have $c = 0$.

Assume that $b_k = 0$. Then, taking the scalar product with ξ_k to both sides of (3.4) and using (2.28) and (3.1), we obtain

$$\begin{aligned} 4\sum_{i=1}^{r}\Big\{g\Big(A^*_{\xi_i}X, A^*_{\xi_k}Y\Big) - g\Big(A^*_{\xi_i}Y, A^*_{\xi_k}X\Big)\Big\}\eta_i(Z) \\ = c\{-\overline{g}(JY,Z)g(X,J\xi_k) - \overline{g}(JZ,X)g(Y,J\xi_k) \\ +2\overline{g}(JX,Y)g(Z,J\xi_k)\}, \quad \forall X,Y,Z \in \Gamma(TM). \end{aligned} \tag{3.9}$$

Replace Z by JN_k and Y by ξ_k in this equation and use (3.1), then we have

$$cg(X, J\xi_k) = 0, \quad \forall X \in \Gamma(TM), \tag{3.10}$$

because $\eta_i(JN_k) = 0$ by (3.6) and $g(J\xi_k, JN_k) = 1$. Replace X by JN_k in this equation, we get $c = 0$. □

Corollary 3.5. *There exist no irrotational generic r-lightlike submanifolds M of an indefinite cosymplectic space form $\overline{M}(c)$ with $c \neq 0$.*

Proposition 3.6. *Let M be an r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then the characteristic vector field ζ does not belong to* $\mathrm{Rad}(TM)$ *and* $\mathrm{ltr}(TM)$.

Proof. Assume that ζ belongs to $\mathrm{Rad}(TM)$ (or $\mathrm{ltr}(TM)$). Then (3.1) deduces to $\zeta = \sum_{i=1}^{r} a_i\xi_i$ [or $\zeta = \sum_{i=1}^{r} b_i N_i$]. From this, we have

$$1 = \overline{g}(\zeta,\zeta) = \sum_{i,j=1}^{r} a_i a_j \overline{g}(\xi_i, \xi_j) = 0 \quad \left[\text{or } 1 = \overline{g}(\zeta,\zeta) = \sum_{i,j=1}^{r} b_i b_j \overline{g}(N_i, N_j) = 0\right]. \tag{3.11}$$

It is a contradiction. Thus ζ does not belong to $\mathrm{Rad}(TM)$ and $\mathrm{ltr}(TM)$. □

4. Generic Lightlike Submanifolds

If the characteristic vector field ζ is tangent to M, then, by Proposition 3.6, ζ does not belong to $\mathrm{Rad}(TM)$. This enables one to choose a screen distribution $S(TM)$ which contains ζ. This implies that *if ζ is tangent to M, then it belongs to $S(TM)$*. Călin also proved this result in his book [11] which Kang et al. [12] and Duggal and Sahin [5, 8] assumed in their papers. We also assumed this result in this paper. In this case, all of the functions a_i, b_i, and e_α on $\overline{M}$, defined by (3.5), vanish identically.

Theorem 4.1. *Let M be a generic r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then ζ is a parallel vector field on M and $S(TM)$. Furthermore ζ is conjugate to any vector field on M with respect to h and h_i^*. In particular, ζ is an asymptotic vector field on M.*

Proof. Replace Y by ζ to (2.6) and use (3.3) and $\zeta \in \Gamma(TM)$, we get

$$\nabla_X \zeta + \sum_{j=1}^{r} h_j^{\ell}(X,\zeta) N_j + \sum_{\beta=r+1}^{n} h_\beta^s(X,\zeta) E_\beta = 0, \quad \forall X \in \Gamma(TM). \tag{4.1}$$

Taking the scalar product with ξ_i and E_α in this equation by turns, we have

$$\nabla_X \zeta = 0, \qquad h_i^{\ell}(X,\zeta) = 0, \qquad h_\alpha^s(X,\zeta) = 0. \tag{4.2}$$

Thus ζ is parallel on M and conjugate to any vector field on M with respect to h. Replace PY by ζ to (2.9) and use (4.2) and $\zeta \in \Gamma(S(TM))$, we have

$$\nabla_X^* \zeta + \sum_{j=1}^{r} h_j^*(X,\zeta)\xi_j = 0, \quad \forall X \in \Gamma(TM). \tag{4.3}$$

Taking the scalar product with N_i to this equation we have

$$\nabla_X^* \zeta = 0, \qquad h_i^*(X,\zeta) = 0, \quad \forall X \in \Gamma(TM). \tag{4.4}$$

Thus ζ is also parallel on $S(TM)$ and conjugate to any vector field on M with respect to h^*. Thus we have our assertions. □

Definition 4.2. An r-lightlike submanifold M of $\overline{M}$ is said to be *totally umbilical* [13] if there is a smooth vector field $\mathcal{H} \in \Gamma(\mathrm{tr}(TM))$ such that

$$h(X,Y) = \mathcal{H} g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.5}$$

In case $\mathcal{H} = 0$, we say that M is *totally geodesic*.

It is easy to see that M is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions $\mathcal{A}_i$ and $\mathcal{B}_\alpha$ such that

$$h_i^{\ell}(X,Y) = \mathcal{A}_i g(X,Y), \qquad h_\alpha^s(X,Y) = \mathcal{B}_\alpha g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.6}$$

Theorem 4.3. *Let M be a totally umbilical generic r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then M is totally geodesic.*

Proof. From (4.2) and (4.6), we obtain

$$\mathcal{A}_i g(X,\zeta) = 0, \qquad \mathcal{B}_\alpha g(X,\zeta) = 0. \tag{4.7}$$

Replace X by ζ to this equations and use $g(\zeta,\zeta) = 1$, we have $\mathcal{A}_i = 0$ for all i and $\mathcal{B}_\alpha = 0$ for all α. Thus M is totally geodesic. □

Definition 4.4. A screen distribution $S(TM)$ of M is said to be *totally umbilical* [13] in M if, for each locally second fundamental form h_i^*, there exist smooth functions $\mathcal{C}_i$ on any coordinate neighborhood $\mathcal{U}$ in M such that

$$h_i^*(X,PY) = \mathcal{C}_i g(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{4.8}$$

In case $\mathcal{C}_i = 0$ for all i, we say that $S(TM)$ is *totally geodesic* in M.

Due to (2.18) and (4.8), we know that $S(TM)$ is totally umbilical in M if and only if each shape operators A_{N_i} of $S(TM)$ satisfies

$$g(A_{N_i}X,PY) = \mathcal{C}_i g(X,PY), \quad \forall X,Y \in \Gamma(TM), \tag{4.9}$$

for some smooth functions $\mathcal{C}_i$ on $\mathcal{U} \subseteq M$.

Theorem 4.5. *Let M be a generic r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$ such that $S(TM)$ is totally umbilical in M. Then $S(TM)$ is totally geodesic in M.*

Proof. As $S(TM)$ is totally umbilical in M. Replace Y by ζ to (4.8) and use (4.4), we have $\mathcal{C}_i\ g(X,\zeta) = 0$ for all $X \in \Gamma(TM)$. Replace X by ζ to this equation and use the fact $g(\zeta,\zeta) = 1$, we obtain $\mathcal{C}_i = 0$ for all i.

From (3.6), the screen distribution $S(TM)$ splits as follows:

$$S(TM) = \{J(\mathrm{Rad}(TM)) \oplus J(\mathrm{ltr}(TM))\} \oplus_{\mathrm{orth}} J\left(S\left(TM^\perp\right)\right) \oplus_{\mathrm{orth}} H_o, \tag{4.10}$$

where H_o is a non-degenerate almost complex distribution H_o on M with respect to J, that is, $J(H_o) = H_o$. Thus the general decompositions of TM and $T\overline{M}$ in (2.1) and (2.4) reduce, respectively, to

$$TM = H \oplus H', \qquad T\overline{M} = H \oplus H' \oplus \mathrm{tr}(TM), \tag{4.11}$$

where H and H' are $2r$- and r-lightlike distributions on M such that

$$\begin{aligned} H &= \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} J(\mathrm{Rad}(TM)) \oplus_{\mathrm{orth}} H_o, \\ H' &= J(\mathrm{ltr}(TM)) \oplus_{\mathrm{orth}} J\left(S\left(TM^\perp\right)\right). \end{aligned} \tag{4.12}$$

In this case, H is an almost complex distribution of M with respect to J. Consider the local null vector fields U_i and V_i on $S(TM)$ and the local nonnull vector field W_α on $S(TM)$ defined respectively by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_\alpha = -JE_\alpha. \tag{4.13}$$

Denote by S the projection morphism of TM on H with respect to the decomposition (4.11). Then any vector field X on M is expressed as follows:

$$\begin{aligned} X &= SX + \sum_{i=1}^{r} u_i(X)U_i + \sum_{\alpha=r+1}^{m} w_\alpha(X)W_\alpha, \\ JX &= FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{\alpha=r+1}^{m} w_\alpha(X)E_\alpha, \end{aligned} \tag{4.14}$$

where u_i, v_i, and w_α are 1-forms locally defined on M by

$$u_i(X) = g(X, V_i), \qquad v_i(X) = g(X, U_i), \qquad w_i(X) = \epsilon_\alpha g(X, E_\alpha), \tag{4.15}$$

and F is a tensor field of (1,1)-type globally defined on M by

$$FX = JSX, \quad \forall X \in \Gamma(TM). \tag{4.16}$$

Apply J to (2.6), (2.7), (2.8), and (2.24) and use (4.13) and (4.14), we have

$$h_j^\ell(X, U_i) = h_i^*(X, V_j), \qquad h_i^*(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, U_i), \tag{4.17}$$

$$h_j^\ell(X, V_i) = h_i^\ell(X, V_j), \qquad h_i^\ell(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, V_i), \tag{4.18}$$

$$h_i^\ell(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, V_i), \qquad \epsilon_\beta h_\beta^s(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, W_\beta), \tag{4.19}$$

$$\nabla_X U_i = F(A_{N_i}X) + \sum_{j=1}^{r} \tau_{ij}(X)U_j + \sum_{\alpha=r+1}^{r} \rho_{i\alpha}(X)W_\alpha, \tag{4.20}$$

$$\nabla_X V_i = F\left(A^*_{\xi_i}X\right) - \sum_{j=1}^{r} \tau_{ji}(X)V_j - \sum_{\alpha=r+1}^{r} \epsilon_\alpha \phi_{\alpha i}(X)W_\alpha + \sum_{j=1}^{r} h_j^\ell(X, \xi_i)U_j, \tag{4.21}$$

$$\nabla_X W_\alpha = F(A_{E_\alpha}X) + \sum_{i=1}^{r} \phi_{\alpha i}(X)U_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X)W_\beta, \tag{4.22}$$

$$(\nabla_X F)(Y) = \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{\alpha=r+1}^{n} w_\alpha(Y)A_{E_\alpha}X - \sum_{i=1}^{r} h_i^\ell(X, Y)U_i - \sum_{\alpha=r+1}^{n} h_\alpha^s(X, Y)W_\alpha. \tag{4.23}$$

□

Theorem 4.6. *Let M be a generic r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then H is integrable if and only if*

$$h(X, FY) = h(FX, Y), \quad \forall X, Y \in \Gamma(H). \tag{4.24}$$

Moreover, if M is totally umbilical, then H is a parallel distribution on M.

Proof. Take $X, Y \in \Gamma(H)$. Then we have $FY = JY \in \Gamma(H)$. Apply $\overline{\nabla}_X$ to $FY = JY$ and use (2.6), (3.2), (4.13), and (4.14), we have

$$h_i^\ell(X, FY) = g(\nabla_X Y, V_i), \qquad h_\alpha^s(X, FY) = \epsilon_\alpha g(\nabla_X Y, W_\alpha), \tag{4.25}$$

$$(\nabla_X F)(Y) = -\sum_{i=1}^{r} h_i^\ell(X, Y)U_i - \sum_{\alpha=r+1}^{m} h_\alpha^s(X, Y)W_\alpha. \tag{4.26}$$

By directed calculations from two equations of (4.25), we have

$$h(X, FY) - h(FX, Y) = \sum_{i=1}^{r} g([X, Y], V_i)N_i + \sum_{\alpha=r+1}^{m} \epsilon_\alpha g([X, Y], W_\alpha)E_\alpha. \tag{4.27}$$

If H is an integrable distribution on M, then we have $[X, Y] \in \Gamma(H)$ for any $X,\ Y \in \Gamma(H)$. This implies $g([X, Y], V_i) = g([X, Y], W_\alpha) = 0$ for all i and α. Therefore we obtain $h(X, FY) = h(FX, Y)$ for all $X,\ Y \in \Gamma(H)$. Conversely if $h(X, FY) = h(FX, Y)$ for all $X,\ Y \in \Gamma(H)$, then we have $g([X, Y], V_i) = g([X, Y], W_\alpha) = 0$ for all i and α. This implies $[X, Y] \in \Gamma(H)$ for all $X,\ Y \in \Gamma(H)$. Thus H is an integrable distribution of M.

If M is totally umbilical, from Theorem 4.3 and (4.25), we have

$$g(\nabla_X Y, V_i) = g(\nabla_X Y, W_\alpha) = 0, \quad \forall i, \alpha. \tag{4.28}$$

This implies $\nabla_X Y \in \Gamma(H)$ for all $X,\ Y \in \Gamma(H)$. Thus H is a parallel distribution on M. □

Theorem 4.7. *Let M be a generic r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then F is parallel on H with respect to ∇ if and only if H is a parallel distribution on$M$.*

Proof. Assume that F is parallel on H with respect to ∇. For any $X,\ Y \in \Gamma(H)$, we have $(\nabla_X F)Y = 0$. Taking the scalar product with V_k and W_β to (4.26) with $(\nabla_X F)Y = 0$, we have $h_k^\ell(X, Y) = 0$ and $h_\beta^s(X, Y) = 0$ for all $X,\ Y \in \Gamma(H)$ and for each k and β, respectively. From (4.25), we have $g(\nabla_X Y, V_i) = 0$ and $g(\nabla_X Y, W_\alpha) = 0$. This implies $\nabla_X Y \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$. Thus H is a parallel distribution on M.

Conversely, if H is parallel on M, from (4.25) we have

$$h_i^\ell(X, FY) = 0, \qquad h_\alpha^s(X, FY) = 0, \quad \forall X, Y \in \Gamma(H). \tag{4.29}$$

For any $Y \in \Gamma(H)$, we show that $F^2Y = J^2Y = -Y + \theta(Y)\zeta$. Replace Y by FY to the equations and use (4.2), we have $h_i^\ell(X, Y) = 0$ and $h_\alpha^s(X, Y) = 0$ for any $X,\ Y \in \Gamma(H)$. Thus F is parallel on H with respect to ∇ by (4.25). □

Theorem 4.8. *Let M be a generic r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. If F is parallel on TM with respect to ∇, then H is a parallel distribution on M and M is locally a product manifold $M^r \times M^{n-r} \times M^{m-n}$, where M^r, M^{n-r}, and M^{m-n} are leafs of $J(\mathrm{ltr}(TM))$, $J(S(TM^{\perp}))$ and H, respectively.*

Proof. Assume that F is parallel on TM with respect to ∇. Then F is parallel on H with respect to ∇. By Theorem 4.7, H is a parallel distribution on M. Apply the operator F to (4.23) with $(\nabla_X F)Y = 0$, we have

$$\sum_{i=1}^{r} u_i(Y)F(A_{N_i}X) + \sum_{\alpha=r+1}^{n} e_\alpha(Y)F(A_{E_\alpha}X) = 0, \quad \forall X,\ Y \in \Gamma(TM), \tag{4.30}$$

due to $FU_i = FW_\alpha = 0$ for all i and α. Replace Y by U_k and W_β to this equation by turns and use (4.15), we have $F(A_{N_i}X) = 0$ and $F(A_{E_\alpha}X) = 0$. Taking the scalar product with W_β and N_k to (4.23) with $(\overline{\nabla}_X F)Y = 0$ by turns, we have

$$h_\alpha^s(X,Y) = \sum_{i=1}^{r} u_i(Y)w_\alpha(A_{N_i}X) + \sum_{\beta=r+1}^{m} w_\beta(Y)w_\alpha\left(A_{E_\beta}X\right), \tag{4.31}$$

$$\sum_{i=1}^{r} u_i(Y)g(A_{N_i}X, N_k) + \sum_{\alpha=r+1}^{m} w_\alpha(Y)g(A_{E_\alpha}X, N_k) = 0, \tag{4.32}$$

for all $X, Y \in \Gamma(TM)$. Replace Y by ξ_j to (4.31), we get $\phi_{\alpha j}(X) = 0$ due to (2.23). Also replace Y by W_β to (4.32), we have $\rho_{k\beta}(X) = 0$ due to (2.17). From this results, (4.11) and (4.14), we get

$$\nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X)U_j, \qquad \nabla_X W_\alpha = \sum_{\beta=r+1}^{m} \sigma_{\alpha\beta}(X)W_\beta, \quad \forall X \in \Gamma(TM). \tag{4.33}$$

Thus $J(\mathrm{ltr}(TM))$ and $J(S(TM^{\perp}))$ are also parallel distributions on M. By the decomposition theorem of de Rham [14], we show that $M = M^r \times M^{n-r} \times M^{m-n}$, where M^r, M^{n-r}, and M^{m-n} are some leafs of $J(\mathrm{ltr}(TM))$, $J(S(TM^{\perp}))$ and H, respectively. □

Theorem 4.9. *Let M be a generic r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. One has the following assertions.*

(i) *If each V_i is parallel with respect to ∇, then $\tau_{ij} = \phi_{\alpha i} = h^\ell(X, \xi_i) = 0$. In this case M is irrotational. Moreover, one has*

$$A_{\xi_i}^{*}X = \sum_{j=1}^{r} u_j\left(A_{\xi_i}^{*}X\right)U_j + \sum_{\alpha=r+1}^{m} w_\alpha\left(A_{\xi_i}^{*}X\right)W_\alpha, \quad \forall X \in \Gamma(TM). \tag{4.34}$$

(ii) *If each U_i is parallel with respect to ∇, then $\tau_{ij} = \rho_{i\alpha} = 0$ and*

$$A_{N_i}X = \sum_{j=1}^{r} u_j(A_{N_i}X)U_j + \sum_{\alpha=r+1}^{m} w_\alpha(A_{N_i}X)W_\alpha, \quad \forall X \in \Gamma(TM). \tag{4.35}$$

(iii) *If each W_α is parallel with respect to ∇, then $\phi_{\alpha i} = 0$ and*

$$A_{E_\alpha}X = \sum_{i=1}^{r} u_i(A_{E_\alpha}X)U_i + \sum_{\beta=r+1}^{m} w_\beta(A_{E_\alpha}X)W_\beta, \quad \forall X \in \Gamma(TM). \tag{4.36}$$

Moreover, if all of V_i, U_i, and W_α are parallel on TM with respect to ∇, then $S(TM)$ is totally geodesic in M and $\tau_{ij} = \phi_{\alpha i} = \rho_{i\alpha} = 0$ on $\Gamma(TM)$. In this case, each null transversal vector fields N_i of M is a constant on M.

Proof. If V_i is parallel with respect to ∇, then, taking the scalar product with U_k, W_β, and V_k to (4.21) by turns, we have $\tau_{ki}(X) = 0$, $\phi_{\beta i}(X) = 0$ and $h_k^\ell(X, \xi_i) = 0$, respectively. Thus M is irrotational. We have $F(A^*_{\xi_i}X) = 0$ for all $X \in \Gamma(TM)$. From this result and (4.14), we obtain

$$J\left(A^*_{\xi_i}X\right) = \sum_{j=1}^{r} u_j\left(A^*_{\xi_i}X\right)N_j + \sum_{\alpha=r+1}^{m} w_\alpha\left(A^*_{\xi_i}X\right)E_\alpha. \tag{4.37}$$

Apply J to this equation and use $\theta(A^*_\xi X_i) = 0$, we obtain (i). In a similar way, by using (4.13), (4.14), (4.20), and (4.22), we have (ii) and (iii).

Assume that all of V, U and W are parallel on TM with respect to ∇. Substituting the equation of (i) into (4.17)-1, we have

$$u_j(A_{N_i}X) = v_i\left(A^*_{\xi_j}X\right) = g\left(A^*_{\xi_j}X, U_i\right) = 0, \quad \forall X \in \Gamma(TM). \tag{4.38}$$

Also, substituting the equation of (iii) into (4.17)-2, we have

$$w_\alpha(A_{N_i}X) = \epsilon_\alpha v_i(A_{E_\alpha}X) = g(A_{E_\alpha}X, U_i) = 0, \quad \forall X \in \Gamma(TM). \tag{4.39}$$

From the last two equations and the equation of (ii), we see that $A_{N_i}X = 0$ for all $X \in \Gamma(TM)$. From this and (2.18) we see that $S(TM)$ is totally geodesic in M and all 1-forms τ_{ij}, $\phi_{\alpha i}$, and $\rho_{i\alpha}$, defined by (2.7) and (2.8), vanish identically. Using the results and (2.7), we show that N is a constant on M. □

Theorem 4.10. *Let M be a totally umbilical generic r-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$ such that $S(TM)$ is totally umbilical. Then M is locally a product manifold $M^r \times M^s \times M^t$, where M^r, M^s, and M^t are some leafs of* $\mathrm{Rad}(TM)$, $H_o^\perp = \mathrm{Span}\{V_i, U_i, W_\alpha\}$ *and H_o, respectively, and $s = n + r, t = m - n - 2r$.*

Proof. By Theorem 4.6, H is a parallel distribution M. Thus, for all $X, Y \in \Gamma(H_o)$, we have $\nabla_X Y \in \Gamma(H)$. From (2.9) and (4.26), we have

$$\begin{aligned} h_i^*(X, FY) &= g(\nabla_X FY, N_i) = g((\nabla_X F)Y + F(\nabla_X Y), N_i) \\ &= g(F(\nabla_X Y), N_i) = -g(\nabla_X Y, JN_i) = g(\nabla_X Y, U_i), \end{aligned} \tag{4.40}$$

due to $FY \in \Gamma(H_o)$. If $S(TM)$ is totally umbilical in M, then we have $h_i^* = 0$ due to Theorem 4.5. By (2.9) and (4.40), we get

$$g(\nabla_X Y, N_i) = 0, \qquad g(\nabla_X Y, U_i) = 0, \quad \forall X \in \Gamma(TM), \ \forall Y \in \Gamma(H_o). \tag{4.41}$$

These results and (4.25) imply $\nabla_X Y \in \Gamma(H_o)$ for all $X, Y \in \Gamma(H_o)$. Thus H_o is a parallel distribution on $S(TM)$ and $TM = H_o \oplus_{\mathrm{orth}} H_o^{\perp}$, where $H_o^{\perp} = \mathrm{Span}\{\xi_i, V_i, U_i, W_\alpha\}$. By Theorems 4.3 and 4.5, we have $h_i^{\ell} = h_\alpha^s = A_{N_i} = \phi_{\alpha i} = 0$ and $A_{E_\alpha} X = \sum_{i=1}^{r} \rho_{i\alpha}(X)\xi_i$. Thus (2.10) and (4.20)~(4.22) deduce, respectively, to

$$\begin{aligned}
\nabla_X U_i &= \sum_{j=1}^{r} \tau_{ij}(X) U_j + \sum_{\alpha=r+1}^{m} \rho_{i\alpha}(X) W_\alpha, \\
\nabla_X V_i &= -\sum_{j=1}^{r} \tau_{ji}(X) V_j + \sum_{j=1}^{r} h_j^{\ell}(X, \xi_i) U_j, \\
\nabla_X W_\alpha &= -\sum_{i=1}^{r} \rho_{i\alpha}(X) V_j + \sum_{\beta=r+1}^{m} \sigma_{\alpha\beta}(X) W_\beta, \\
\nabla_X \xi &= -\sum_{j=1}^{r} \tau_{ji}(X) \xi_j, \quad \forall X \in \Gamma\left(H_o^{\perp}\right).
\end{aligned} \tag{4.42}$$

Thus $H_o^{\perp}$ is also a parallel distribution on M. Thus we have $M = M^r \times M^s \times M^t$, where M^r, M^s, and M^t are some leafs of $\mathrm{Rad}(TM)$, $H_o^{\perp} = \mathrm{Span}\{V_i, U_i, W_\alpha\}$ and H_o, respectively, and $s = n + r$, $t = m - n - 2r$. □

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