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Research Article

Hybrid Algorithms for Minimization Problems over the Solutions of Generalized Mixed Equilibrium and Variational Inclusion Problems

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We introduce a new general hybrid iterative algorithm for finding a common element of the set of solution of fixed point for a nonexpansive mapping, the set of solution of generalized mixed equilibrium problem, and the set of solution of the variational inclusion for a β-inverse-strongly monotone mapping in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above three sets under some mild conditions. Our results improve and extend the corresponding results of Marino and Xu (2006), Yao and Liou (2010), Tan and Chang (2011), and other authors.

1. Introduction

In the theory of variational inequalities, variational inclusions, and equilibrium problems, the development of an efficient and implementable iterative algorithm is interesting and important. The important generalization of variational inequalities called variational inclusions, have been extensively studied and generalized in different directions to study a wide class of problems arising in mechanics, optimization, nonlinear programming, economics, finance, and applied sciences.

Equilibrium theory represents an important area of mathematical sciences such as optimization, operations research, game theory, complementarity problems, financial mathematics, and mechanics. Equilibrium problems include variational inequalities, optimization problems, Nash equilibria problems, saddle point problems, fixed point problems, and complementarity problems as special cases; for example, see the references herein. Let C be a closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let F be a bifunction of $C \times C$ into $\mathcal{R}$, where $\mathcal{R}$ is the set of real numbers, $\Phi : C \to H$ be a mapping and $\varphi : C \to \mathcal{R}$ be a real-valued function. The *generalized mixed equilibrium problem* for finding $x \in C$ such that

$$F(x,y) + \langle \Phi x, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{GMEP}(F,\varphi,\Phi)$, that is

$$\mathrm{GMEP}(F,\varphi,\Phi) = \{x \in C : F(x,y) + \langle \Phi x, y - x\rangle + \varphi(y) - \varphi(x) \geq 0,\ \forall y \in C\}. \tag{1.2}$$

If $\Phi \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced into the *equilibrium problem* [1] for finding $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{EP}(F)$. This problem contains fixed point problems, includes as special cases numerous problems in physics, optimization, and economics. Some methods have been proposed to solve the equilibrium problem, please consult [2–4].

If $F \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced into the *Hartmann-Stampacchia variational inequality* [5] for finding $x \in C$ such that

$$\langle \Phi x, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $\mathrm{VI}(C,\Phi)$. The variational inequality has been extensively studied in the literature [6].

If $F \equiv 0$ and $\Phi \equiv 0$, the problem (1.1) is reduced into the *minimize problem* for finding $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $\mathrm{Argmin}(\varphi)$.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\theta(x) = \frac{1}{2}\langle Ax, x\rangle - \langle x, y\rangle, \quad \forall x \in F(S), \tag{1.6}$$

where A is a linear bounded operator, $F(S)$ is the fixed point set of a nonexpansive mapping S, and y is a given point in H [7].

Recall, a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \tag{1.7}$$

for all $x, y \in C$. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty [8]. We denote weak convergence and strongly convergence by notations $\rightharpoonup$ and $\to$, respectively. A mapping A of C into H is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \tag{1.8}$$

for all $x, y \in C$. A mapping A of C into H is called α- *inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \tag{1.9}$$

for all $x, y \in C$. It is obvious that any α-inverse-strongly monotone mappings A is monotone and Lipschitz continuous mapping. A linear bounded operator A is *strongly positive* if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \tag{1.10}$$

for all $x \in H$. A self-mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \tag{1.11}$$

for all $x, y \in C$. We use Π_C to denote the collection of all contraction on C. Note that each $f \in \Pi_C$ has a unique fixed point in C.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. The *variational inclusion problem* is to find $x \in H$ such that

$$\theta \in B(x) + M(x), \tag{1.12}$$

where θ is the zero vector in H. The set of solutions of problem (1.12) is denoted by $I(B, M)$. The variational inclusion has been extensively studied in the literature. See, for example, [9–12] and the reference therein.

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$.

Let B be an inverse-strongly monotone mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$, and define

$$Mv = \begin{cases} Bv + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{1.13}$$

Then M is a maximal monotone and $\theta \in Mv$ if and only if $v \in \mathrm{VI}(C, B)$ [13].

Let $M : H \to 2^H$ be a set-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \quad x \in H \tag{1.14}$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is the identity mapping. It is worth mentioning that the resolvent operator is nonexpansive, 1-inverse-strongly monotone, and that a solution of problem (1.12) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$, see [14], that is, $I(B, M) = F(J_{M,\lambda}(I - \lambda B)),\ \forall\lambda > 0$.

In 2000, Moudafi [15] introduced the viscosity approximation method for nonexpansive mapping and proved that if H is a real Hilbert space, the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in C$ chosen arbitrarily,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, \quad n \geq 0, \tag{1.15}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies certain conditions, converges strongly to a fixed point of S (say $\overline{x} \in C$) which is the unique solution of the following variational inequality:

$$\langle (I - f)\overline{x}, x - \overline{x} \rangle \geq 0, \quad \forall x \in F(S). \tag{1.16}$$

In 2006, Marino and Xu [7] introduced a general iterative method for nonexpansive mapping. They defined the sequence $\{x_n\}$ generated by the algorithm $x_0 \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n, \quad n \geq 0, \tag{1.17}$$

where $\{\alpha_n\} \subset (0, 1)$ and A is a strongly positive linear bounded operator. They prove that if $C = H$ and the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.17) converges strongly to a fixed point of S (say $\overline{x} \in H$) which is the unique solution of the following variational inequality:

$$\langle (A - \gamma f)\overline{x}, x - \overline{x} \rangle \geq 0, \quad \forall x \in F(S), \tag{1.18}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S) \cap \mathrm{EP}(F)} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.19}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2010, Yao and Liou [16] introduced the following composite iterative scheme in a real Hilbert space: $x_0 \in C$

$$x_{n+1} = \mu_n P_C\left[\alpha_n f(x_n) + (1 - \alpha_n)Sx_n\right] + (1 - \mu_n)T_r(x_n - rAx_n), \tag{1.20}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\mu_n\} \subset [0,1)$. Furthermore, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to the same point $z \in F(S) \cap \mathrm{EP}(F)$, where P_C is the projection of H onto C.

In 2011, Tan and Chang [11] introduced the following iterative process for $\{T_n : C \to C\}$ be a sequence of nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\big(SP_C\big((1-t_n)J_{M,\lambda}(I-\lambda A)T_\mu(I-\mu B)\big)x_n\big), \quad \forall n \geq 0, \tag{1.21}$$

where $\{\alpha_n\} \subset (0,1)$, $\lambda \in (0,2\alpha]$ and $\mu \in (0,2\beta]$. Then, the sequence $\{x_n\}$ defined by (1.21) converges strongly to a common element of the set of fixed points of nonexpansive mapping, the set of solution of the variational inequality and the generalized equilibrium problem.

In this paper, we modify the iterative methods (1.17), (1.20), and (1.21) by purposing the following new general viscosity iterative method: $x_0 \in C$,

$$x_{n+1} = \xi_n P_C\big[\alpha_n \gamma f(x_n) + (I-\alpha_n A)SJ_{M,\lambda}(x_n - \lambda Bx_n)\big] + (1-\xi_n)T_r(x_n - r\Phi x_n), \tag{1.22}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $r \in (0,2\sigma)$, and $\lambda \in (0,2\beta)$ satisfy some appropriate conditions. Consequently, we show that under some control conditions the sequence $\{x_n\}$ strongly converge to a common element of the set of fixed points of nonexpansive mapping, the solution of the generalized mixed equilibrium problem, and the set of solution of the variational inclusion in a real Hilbert space.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|. \tag{2.1}$$

The following characterizes the projection P_C. We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.1. *The function $u \in C$ is a solution of the variational inequality* (1.4) *if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda\Phi u)$ for all $\lambda > 0$.*

Lemma 2.2. *For a given $z \in H$, $u \in C$, $u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0$, $\forall v \in C$.*

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{2.2}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0. \tag{2.3}$$

Lemma 2.3 (see [17]). *Let $M: H \to 2^H$ be a maximal monotone mapping and let $B: H \to H$ be a monotone and Lipschitz continuous mapping. Then the mapping $L = M + B: H \to 2^H$ is a maximal monotone mapping.*

Lemma 2.4 (see [18]). *Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$, hold for each $y \in H$ with $y \neq x$.*

Lemma 2.5 (see [19]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n, \quad \forall n \geq 0, \tag{2.4}$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\}$ is a sequence in $\mathcal{R}$ such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$.

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

Lemma 2.6 (see [20]). *Let C be a closed convex subset of a real Hilbert space H and let $T: C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, \quad x_n - Tx_n \longrightarrow 0 \tag{2.5}$$

implies $x = Tx$.

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $F: C \times C \to \mathcal{R}$, the nonlinear mapping $\Phi: C \to H$ is continuous monotone and $\varphi: C \to \mathcal{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;

(A3) for each fixed $y \in C$, $x \mapsto F(x,y)$ is weakly upper semicontinuous;

(A4) for each fixed $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous;

(B1) for each $x \in C$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0, \tag{2.6}$$

(B2) C is a bounded set.

Lemma 2.7 (see [21]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F: C \times C \to \mathcal{R}$ be a bifunction mapping satisfies (A1)–(A4) and let $\varphi: C \to \mathcal{R}$ is convex and lower semicontinuous such that $C \cap \operatorname{dom}\varphi \neq \emptyset$. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, then there exists $u \in C$ such that*

$$F(u,y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle. \tag{2.7}$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{ u \in C : F(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x \rangle \geq 0,\ \forall y \in C \right\}, \tag{2.8}$$

for all $x \in H$. Then, the following hold:

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$;*

(iii) $F(T_r) = \mathrm{MEP}(F, \varphi)$*;*

(iv) $\mathrm{MEP}(F, \varphi)$ *is closed and convex.*

Lemma 2.8 (see[7]). *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

3. Strong Convergence Theorems

In this section, we show a strong convergence theorem which solves the problem of finding a common element of $F(S)$, $\mathrm{GMEP}(F, \varphi, \Phi)$, and $I(B, M)$ of inverse-strongly monotone mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, C be a closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathcal{R}$ satisfying (A1)–(A4) and $B, \Phi : C \to H$ be β, σ-inverse-strongly monotone mappings, $\varphi : C \to \mathcal{R}$ is convex and lower semicontinuous function, $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$, $M : H \to 2^H$ be a maximal monotone mapping and A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$, assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let S be a nonexpansive mapping of H into itself and assume that either (B1) or (B2) holds such that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F, \varphi, \Phi) \cap I(B, M) \neq \emptyset. \tag{3.1}$$

Suppose $\{x_n\}$ is a sequences generated by the following algorithm $x_0 \in C$ arbitrarily:

$$x_{n+1} = \xi_n P_C\left[\alpha_n \gamma f(x_n) + (I - \alpha_n A) S J_{M,\lambda}(x_n - \lambda B x_n)\right] + (1 - \xi_n) T_r(x_n - r\Phi x_n), \tag{3.2}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0, 2\beta)$ such that $0 < a \leq \lambda \leq b < 2\beta$ and $r \in (0, 2\sigma)$ with $0 < c \leq d \leq 1 - \sigma$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

(C2) $0 < \liminf_{n\to\infty}\xi_n < \limsup_{n\to\infty}\xi_n < 1$ *and* $\lim_{n\to\infty}((\xi_{n+1} - \xi_n)/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ converges strongly to $q \in \Theta$, where $q = P_\Theta(\gamma f + I - A)(q)$ which solves the following variational inequality:

$$\langle (\gamma f - A)q, p - q \rangle \leq 0, \quad \forall p \in \Theta \tag{3.3}$$

which is the optimality condition for the minimization problem

$$\min_{q\in\Theta}\frac{1}{2}\langle Aq,q\rangle - h(q), \tag{3.4}$$

where h is a potential function for γf (i.e., $h'(q)=\gamma f(q)$ for $q\in H$).

Proof. Because of condition (C1), we may assume without loss of generality, then $\alpha_n \in (0,\|A\|^{-1})$ for all n. By Lemma 2.8, we have $\|I-\alpha_n A\| \le 1-\alpha_n\overline{\gamma}$. Next, we will assume that $\|I-A\| \le \|1-\overline{\gamma}\|$.

Step 1. We will show $\{x_n\},\{u_n\}$ are bounded.

Since B,Φ are β, σ-inverse-strongly monotone mappings, we have

$$\begin{aligned}\|(I-\lambda B)x-(I-\lambda B)y\|^2 &= \|(x-y)-\lambda(Bx-By)\|^2\\ &= \|x-y\|^2 - 2\lambda\langle x-y,Bx-By\rangle + \lambda^2\|Bx-By\|^2\\ &\le \|x-y\|^2 + \lambda(\lambda-2\beta)\|Bx-By\|^2\\ &\le \|x-y\|^2.\end{aligned} \tag{3.5}$$

In similar way, we can obtain

$$\|(I-r\Phi)x-(I-r\Phi)y\|^2 \le \|x-y\|^2. \tag{3.6}$$

It is clear that if $0<\lambda<2\beta$, $0<r<2\sigma$, then $I-\lambda B, I-r\Phi$ are all nonexpansive.

Put $y_n = J_{M,\lambda}(x_n-\lambda Bx_n)$, $n\ge 0$. It follows that

$$\begin{aligned}\|y_n-q\| &= \|J_{M,\lambda}(x_n-\lambda Bx_n)-J_{M,\lambda}(q-\lambda Bq)\|\\ &\le \|x_n-q\|.\end{aligned} \tag{3.7}$$

By Lemma 2.7, we have $u_n = T_r(x_n-r\Phi x_n)$ for all $n\ge 0$. Then, we have

$$\begin{aligned}\|u_n-q\|^2 &= \|T_r(x_n-r\Phi x_n)-T_r(q-r\Phi q)\|^2\\ &\le \|(x_n-r\Phi x_n)-(q-r\Phi q)\|^2\\ &\le \|x_n-q\|^2 + r(r-2\sigma)\|\Phi x_n-\Phi q\|^2\\ &\le \|x_n-q\|^2.\end{aligned} \tag{3.8}$$

Put $z_n = P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)Sy_n]$ for all $n \geq 0$. From (3.2), we deduce that

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\xi_n(z_n - q) + (1 - \xi_n)(u_n - q)\| \\
&\leq \xi_n \|P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)Sy_n] - P_C q\| + (1 - \xi_n)\|u_n - q\| \\
&\leq \xi_n \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)Sy_n - q\| + (1 - \xi_n)\|u_n - q\| \\
&= \xi_n \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(Sy_n - q)\| + (1 - \xi_n)\|u_n - q\| \\
&\leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\| + \xi_n(1 - \alpha_n \overline{\gamma})\|y_n - q\| + (1 - \xi_n)\|u_n - q\| \\
&\leq \xi_n \alpha_n \|\gamma f(x_n) - \gamma f(q)\| + \xi_n \alpha_n \|\gamma f(q) - Aq\| \\
&\quad + \xi_n(1 - \alpha_n \overline{\gamma})\|x_n - q\| + (1 - \xi_n)\|x_n - q\| \\
&\leq \xi_n \alpha_n \gamma \alpha \|x_n - q\| + \xi_n \alpha_n \|\gamma f(q) - Aq\| \\
&\quad + \xi_n(1 - \alpha_n \overline{\gamma})\|x_n - q\| + (1 - \xi_n)\|x_n - q\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\xi_n \alpha_n)\|x_n - q\| + \xi_n \alpha_n \|\gamma f(q) - Aq\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\xi_n \alpha_n)\|x_n - q\| + (\overline{\gamma} - \gamma\alpha)\xi_n \alpha_n \frac{\|\gamma f(q) - Aq\|}{\overline{\gamma} - \gamma\alpha} \\
&\leq \max\left\{\|x_n - q\|, \frac{\|\gamma f(q) - Aq\|}{\overline{\gamma} - \gamma\alpha}\right\}.
\end{aligned} \tag{3.9}$$

It follows from induction that

$$\|x_n - q\| \leq \max\left\{\|x_0 - q\|, \frac{\|\gamma f(q) - Aq\|}{\overline{\gamma} - \gamma\alpha}\right\}, \quad n \geq 0. \tag{3.10}$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Sy_n\}$, $\{Bx_n\}$, $\{f(x_n)\}$, and $\{ASy_n\}$.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+2} - x_{n+1}\| = 0$. From (3.2), we have

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\xi_{n+1} z_{n+1} + (1 - \xi_{n+1})u_{n+1} - \xi_n z_n - (1 - \xi_n)u_n\| \\
&= \|\xi_{n+1}(z_{n+1} - z_n) + (\xi_{n+1} - \xi_n)z_n + (1 - \xi_{n+1})(u_{n+1} - u_n) + (\xi_{n+1} - \xi_n)u_n\| \\
&\leq \xi_{n+1}\|z_{n+1} - z_n\| + (1 - \xi_{n+1})\|u_{n+1} - u_n\| + |\xi_{n+1} - \xi_n|(\|z_n\| + \|u_n\|).
\end{aligned} \tag{3.11}$$

We estimate $\|u_{n+1} - u_n\|$, so we have

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|T_r(x_{n+1} - r\Phi x_{n+1}) - T_r(x_n - r\Phi x_n)\| \\ &\leq \|(x_{n+1} - r\Phi x_{n+1}) - (x_n - r\Phi x_n)\| \\ &\leq \|x_{n+1} - x_n\|.\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11) that

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| \leq{}& \xi_{n+1}\|z_{n+1} - z_n\| + (1 - \xi_{n+1})\|x_{n+1} - x_n\| \\ &+ |\xi_{n+1} - \xi_n|(\|z_n\| + \|u_n\|).\end{aligned} \tag{3.13}$$

We note that

$$\begin{aligned}\|z_{n+1} - z_n\| &= \left\|P_C\left[\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)Sy_{n+1}\right] - P_C\left[\alpha_n\gamma f(x_n) + (I - \alpha_n A)Sy_n\right]\right\| \\ &\leq \left\|\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)Sy_{n+1} - \left(\alpha_n\gamma f(x_n) + (I - \alpha_n A)Sy_n\right)\right\| \\ &= \Big\|\alpha_{n+1}\gamma\left(f(x_{n+1}) - f(x_n)\right) + (\alpha_{n+1} - \alpha_n)\gamma f(x_n) \\ &\quad + (I - \alpha_{n+1}A)\left(Sy_{n+1} - Sy_n\right) + (\alpha_n - \alpha_{n+1})ASy_n\Big\| \\ &\leq \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\left\|\gamma f(x_n)\right\| + \left(1 - \alpha_{n+1}\overline{\gamma}\right)\left\|y_{n+1} - y_n\right\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\left\|ASy_n\right\| \\ &= \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\left(\left\|\gamma f(x_n)\right\| + \left\|ASy_n\right\|\right) \\ &\quad + \left(1 - \alpha_{n+1}\overline{\gamma}\right)\left\|y_{n+1} - y_n\right\|.\end{aligned} \tag{3.14}$$

Next, we estimate $\|y_{n+1} - y_n\|$, then we get

$$\begin{aligned}\left\|y_{n+1} - y_n\right\| &= \|J_{M,\lambda}(x_{n+1} - \lambda Bx_{n+1}) - J_{M,\lambda}(x_n - \lambda Bx_n)\| \\ &\leq \|(x_{n+1} - \lambda Bx_{n+1}) - (x_n - \lambda Bx_n)\| \\ &\leq \|x_{n+1} - x_n\|.\end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.14), we obtain that

$$\begin{aligned}\|z_{n+1} - z_n\| \leq{}& \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\left(\left\|\gamma f(x_n)\right\| + \left\|ASy_n\right\|\right) \\ &+ \left(1 - \alpha_{n+1}\overline{\gamma}\right)\|x_{n+1} - x_n\|.\end{aligned} \tag{3.16}$$

And substituting (3.12), (3.16) into (3.11), we get

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| \le{}& \xi_{n+1}\left\{\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\left(\|\gamma f(x_n)\| + \|ASy_n\|\right)\right.\\
&\left. + \left(1-\alpha_{n+1}\overline{\gamma}\right)\|x_{n+1}-x_n\|\right\} + (1-\xi_{n+1})\|x_{n+1}-x_n\|\\
&+ |\xi_{n+1}-\xi_n|\left(\|z_n\|+\|u_n\|\right)\\
\le{}& \left[1-(\overline{\gamma}-\gamma\alpha)\xi_{n+1}\alpha_{n+1}\right]\|x_{n+1}-x_n\| + \left(|\alpha_{n+1}-\alpha_n| + |\xi_{n+1}-\xi_n|\right)M,
\end{aligned} \tag{3.17}$$

where $M > 0$ is a constant satisfying

$$\sup_n\left\{\|\gamma f(x_n)\| + \|ASy_n\|,\ \|z_n\|+\|u_n\|\right\} \le M. \tag{3.18}$$

This together with (C1), (C2), and Lemma 2.5, implies that

$$\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\| = 0. \tag{3.19}$$

From (3.15), we also have $\|y_{n+1}-y_n\| \to 0$ as $n \to \infty$.

Step 3. We show the following:

(i) $\lim_{n\to\infty}\|Bx_n - Bq\| = 0$;

(ii) $\lim_{n\to\infty}\|\Phi x_n - \Phi q\| = 0$.

For $q \in \Omega$ and $q = J_{M,\lambda}(q-\lambda Bq)$, then we get

$$\begin{aligned}
\|y_n-q\|^2 &= \left\|J_{M,\lambda}(x_n-\lambda Bx_n) - J_{M,\lambda}(q-\lambda Bq)\right\|^2\\
&\le \left\|(x_n-\lambda Bx_n)-(q-\lambda Bq)\right\|^2\\
&\le \|x_n-q\|^2 + \lambda(\lambda-2\beta)\|Bx_n-Bq\|^2.
\end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned}
\|z_n-q\|^2 &= \left\|P_C\left(\alpha_n\gamma f(x_n) + (I-\alpha_nA)\,Sy_n\right) - P_C(q)\right\|^2\\
&\le \left\|\alpha_n(\gamma f(x_n)-Aq) + (I-\alpha_nA)(Sy_n-q)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|\gamma f(x_n)-Aq\|^2+(1-\alpha_n\overline{\gamma})\|y_n-q\|^2\\
&\quad+2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\|\\
&\leq \alpha_n\|\gamma f(x_n)-Aq\|^2+2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\|\\
&\quad+(1-\alpha_n\overline{\gamma})\left(\|x_n-q\|^2+\lambda(\lambda-2\beta)\|Bx_n-Bq\|^2\right)\\
&\leq \alpha_n\|\gamma f(x_n)-Aq\|^2+2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\|\\
&\quad+\|x_n-q\|^2+(1-\alpha_n\overline{\gamma})\lambda(\lambda-2\beta)\|Bx_n-Bq\|^2.
\end{aligned}
\tag{3.21}
$$

By the convexity of the norm $\|\cdot\|$, we have

$$
\begin{aligned}
\|x_{n+1}-q\|^2&=\|\xi_n z_n+(1-\xi_n)u_n-q\|^2\\
&=\|\xi_n(z_n-q)+(1-\xi_n)(u_n-q)\|^2\\
&\leq \xi_n\|z_n-q\|^2+(1-\xi_n)\|u_n-q\|^2.
\end{aligned}
\tag{3.22}
$$

Substituting (3.8), (3.21) into (3.22), we obtain

$$
\begin{aligned}
\|x_{n+1}-q\|^2&\leq \xi_n\Big\{\alpha_n\|\gamma f(x_n)-Aq\|^2+2\alpha_n\,(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\|+\|x_n-q\|^2\\
&\qquad+(1-\alpha_n\overline{\gamma})\lambda(\lambda-2\beta)\|Bx_n-Bq\|^2\Big\}+(1-\xi_n)\|x_n-q\|^2\\
&=\xi_n\alpha_n\|\gamma f(x_n)-Aq\|^2+2\xi_n\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\|+\xi_n\|x_n-q\|^2\\
&\quad+\xi_n(1-\alpha_n\overline{\gamma})\lambda(\lambda-2\beta)\|Bx_n-Bq\|^2+(1-\xi_n)\|x_n-q\|^2.
\end{aligned}
\tag{3.23}
$$

So, we obtain

$$
\begin{aligned}
\xi_n\,(1-\alpha_n\overline{\gamma})\lambda(2\beta-\lambda)\|Bx_n-Bq\|^2&\leq \xi_n\alpha_n\|\gamma f(x_n)-Aq\|^2+\epsilon_n\\
&\quad+\|x_n-x_{n+1}\|(\|x_n-q\|+\|x_{n+1}-q\|),
\end{aligned}
\tag{3.24}
$$

where $\epsilon_n=2\xi_n\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)Aq\|\|y_n-q\|$. Since condition (C1), (C2) and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ then we obtain that $\|Bx_n-Bq\|\to 0$ as $n\to\infty$. We consider this inequality in (3.21) that

$$
\begin{aligned}
\|z_n-q\|^2&\leq \alpha_n\|\gamma f(x_n)-Aq\|^2+(1-\alpha_n\overline{\gamma})\|y_n-q\|^2\\
&\quad+2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\|.
\end{aligned}
\tag{3.25}
$$

Substituting (3.20) into (3.25), we have

$$\begin{aligned}
\|z_n - q\|^2 &\le \alpha_n \|\gamma f(x_n) - Aq\|^2 + (1-\alpha_n\overline{\gamma})\left\{\|x_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bx_n - Bq\|^2\right\} \\
&\quad + 2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&= \alpha_n \|\gamma f(x_n) - Aq\|^2 + (1-\alpha_n\overline{\gamma})\|x_n - q\|^2 + (1-\alpha_n\overline{\gamma})\lambda(\lambda - 2\beta)\|Bx_n - Bq\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&\le \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 + (1-\alpha_n\overline{\gamma})\lambda(\lambda - 2\beta)\|Bx_n - Bq\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|.
\end{aligned} \tag{3.26}$$

Substituting (3.8) and (3.26) into (3.22), we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \xi_n\left\{\alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 + (1-\alpha_n\overline{\gamma})\lambda(\lambda - 2\beta)\|Bx_n - Bq\|^2\right. \\
&\qquad \left. +2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|\right\} \\
&\quad + (1-\xi_n)\left\{\|x_n - q\|^2 + r(r-2\sigma)\|\Phi x_n - \Phi q\|^2\right\} \\
&= \xi_n\alpha_n \|\gamma f(x_n) - Aq\|^2 + \xi_n\|x_n - q\|^2 + \xi_n(1-\alpha_n\overline{\gamma})\lambda(\lambda - 2\beta)\|Bx_n - Bq\|^2 \\
&\quad + 2\xi_n\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| + (1-\xi_n)\|x_n - q\|^2 \\
&\quad + (1-\xi_n)r(r-2\sigma)\|\Phi x_n - \Phi q\|^2.
\end{aligned} \tag{3.27}$$

So, we also have

$$\begin{aligned}
(1-\xi_n)r(2\sigma - r)\|\Phi x_n - \Phi q\|^2 &\le \xi_n\alpha_n\|\gamma f(x_n) - Aq\|^2 \\
&\quad + \epsilon_n + \|x_n - x_{n+1}\|(\|x_n - q\| + \|x_{n+1} - q\|) \\
&\quad + \xi_n(1-\alpha_n\overline{\gamma})\lambda(\lambda - 2\beta)\|Bx_n - Bq\|^2,
\end{aligned} \tag{3.28}$$

where $\epsilon_n = 2\xi_n\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|$. Since condition (C1), (C2), $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|Bx_n - Bq\| = 0$ then we obtain that $\|\Phi x_n - \Phi q\| \to 0$ as $n \to \infty$.

Step 4. We show the following:

(i) $\lim_{n\to\infty}\|x_n - u_n\| = 0$;

(ii) $\lim_{n\to\infty}\|x_n - y_n\| = 0$;

(iii) $\lim_{n\to\infty}\|y_n - Sy_n\| = 0$.

Since T_r is firmly nonexpansive, we observe that

$$\begin{aligned}
\|u_n - q\|^2 &= \|T_r(x_n - r\Phi x_n) - T_r(q - r\Phi q)\|^2 \\
&\le \langle (x_n - r\Phi x_n) - (q - r\Phi q), u_n - q \rangle \\
&= \frac{1}{2}\Big\{ \|(x_n - r\Phi x_n) - (q - r\Phi q)\|^2 + \|u_n - q\|^2 \\
&\qquad - \|(x_n - r\Phi x_n) - (q - r\Phi q) - (u_n - q)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - q\|^2 + \|u_n - q\|^2 - \|(x_n - u_n) - r(\Phi x_n - \Phi q)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - u_n\|^2 \\
&\qquad + 2r\langle \Phi x_n - \Phi q, x_n - u_n \rangle - r^2\|\Phi x_n - \Phi q\|^2 \Big\}.
\end{aligned} \tag{3.29}$$

Hence, we have

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r\|\Phi x_n - \Phi q\|\|x_n - u_n\|. \tag{3.30}$$

Since $J_{M,\lambda}$ is 1-inverse-strongly monotone, we have

$$\begin{aligned}
\|y_n - q\|^2 &= \|J_{M,\lambda}(x_n - \lambda B x_n) - J_{M,\lambda}(q - \lambda B q)\|^2 \\
&\le \langle (x_n - \lambda B x_n) - (q - \lambda B q), y_n - q \rangle \\
&= \frac{1}{2}\Big\{ \|(x_n - \lambda B x_n) - (q - \lambda B q)\|^2 + \|y_n - q\|^2 \\
&\qquad - \|(x_n - \lambda B x_n) - (q - \lambda B q) - (y_n - q)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - q\|^2 + \|y_n - q\|^2 - \|(x_n - y_n) - \lambda(B x_n - B q)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|x_n - q\|^2 + \|y_n - q\|^2 - \|x_n - y_n\|^2 \\
&\qquad + 2\lambda\langle x_n - y_n, B x_n - B q \rangle - \lambda^2\|B x_n - B q\|^2 \Big\},
\end{aligned} \tag{3.31}$$

which implies that

$$\|y_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - y_n\|^2 + 2\lambda\|x_n - y_n\|\|B x_n - B q\|. \tag{3.32}$$

Substituting (3.32) into (3.25), we have

$$\begin{aligned}
\|z_n - q\|^2 &\le \alpha_n \|\gamma f(x_n) - Aq\|^2 \\
&\quad + (1-\alpha_n\overline{\gamma})\left\{\|x_n - q\|^2 - \|x_n - y_n\|^2 + 2\lambda\|x_n - y_n\|\|Bx_n - Bq\|\right\} \\
&\quad + 2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&= \alpha_n\|\gamma f(x_n) - Aq\|^2 + (1-\alpha_n\overline{\gamma})\|x_n - q\|^2 - (1-\alpha_n\overline{\gamma})\|x_n - y_n\|^2 \\
&\quad + 2\lambda(1-\alpha_n\overline{\gamma})\|x_n - y_n\|\|Bx_n - Bq\| + 2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|.
\end{aligned} \tag{3.33}$$

Substituting (3.30) and (3.33) into (3.22), we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \xi_n\|z_n - q\|^2 + (1-\xi_n)\|u_n - q\|^2 \\
&\le \xi_n\Big\{\alpha_n\|\gamma f(x_n) - Aq\|^2 + (1-\alpha_n\overline{\gamma})\|x_n - q\|^2 - (1-\alpha_n\overline{\gamma})\|x_n - y_n\|^2 \\
&\qquad + 2\lambda(1-\alpha_n\overline{\gamma})\|x_n - y_n\|\|Bx_n - Bq\| \\
&\qquad + 2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|\Big\} \\
&\quad + (1-\xi_n)\left\{\|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r\|\Phi x_n - \Phi q\|\|x_n - u_n\|\right\} \\
&\le \xi_n\Big\{\alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_n - y_n\|^2 \\
&\qquad + 2\lambda(1-\alpha_n\overline{\gamma})\|x_n - y_n\|\|Bx_n - Bq\| \\
&\qquad + 2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|\Big\} \\
&\quad + (1-\xi_n)\left\{\|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r\|\Phi x_n - \Phi q\|\|x_n - u_n\|\right\} \\
&\le \xi_n\alpha_n\|\gamma f(x_n) - Aq\|^2 + \xi_n\|x_n - q\|^2 - \|x_n - y_n\|^2 \\
&\quad + 2\lambda\xi_n(1-\alpha_n\overline{\gamma})\|x_n - y_n\|\|Bx_n - Bq\| \\
&\quad + 2\xi_n\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&\quad + (1-\xi_n)\|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r(1-\xi_n)\|\Phi x_n - \Phi q\|\|x_n - u_n\|.
\end{aligned} \tag{3.34}$$

Then, we derive

$$\begin{aligned}
&\|x_n - u_n\|^2 + \|x_n - y_n\|^2 \\
&\quad = \xi_n\alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
&\qquad + 2\xi_n\lambda(1-\alpha_n\overline{\gamma})\|x_n - y_n\|\|Bx_n - Bq\|
\end{aligned}$$

$$\begin{aligned}
&+2\xi_n\alpha_n(1-\alpha_n\overline{\gamma})\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\| \\
&+2r(1-\xi_n)\left\|\Phi x_n-\Phi q\right\|\|x_n-u_n\| \\
\leq{}& \xi_n\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2+\|x_{n+1}-x_n\|\left(\left\|x_n-q\right\|+\left\|x_{n+1}-q\right\|\right) \\
&+2\xi_n\lambda(1-\alpha_n\overline{\gamma})\left\|x_n-y_n\right\|\left\|Bx_n-Bq\right\| \\
&+2\xi_n\alpha_n(1-\alpha_n\overline{\gamma})\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\| \\
&+2r(1-\xi_n)\left\|\Phi x_n-\Phi q\right\|\|x_n-u_n\|.
\end{aligned} \tag{3.35}$$

By condition (C1), (C2), $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, $\lim_{n\to\infty}\|\Phi x_n-\Phi q\|=0$, and $\lim_{n\to\infty}\|Bx_n-Bq\|=0$. So, we have $\|x_n-u_n\|\to 0$, $\|x_n-y_n\|\to 0$ as $n\to\infty$. It follows that

$$\left\|u_n-y_n\right\|\leq\|x_n-u_n\|+\left\|x_n-y_n\right\|\longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{3.36}$$

We note that $x_{n+1}-x_n=\xi_n(z_n-x_n)+(1-\xi_n)(u_n-x_n)$. From $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, and hence

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.37}$$

Since

$$\left\|z_n-y_n\right\|\leq\|z_n-x_n\|+\left\|x_n-y_n\right\|. \tag{3.38}$$

So, by (3.37) and $\lim_{n\to\infty}\|x_n-y_n\|=0$, we obtain

$$\lim_{n\to\infty}\left\|z_n-y_n\right\|=0. \tag{3.39}$$

Therefore, we observe that

$$\begin{aligned}
\left\|Sy_n-z_n\right\| &\leq \left\|Sy_n-P_C\left(\alpha_n\gamma f(x_n)+(I-\alpha_nA)Sy_n\right)\right\| \\
&\leq \left\|Sy_n-\alpha_n\gamma f(x_n)-(I-\alpha_nA)Sy_n\right\| \\
&\leq \alpha_n\left\|\gamma f(x_n)-ASy_n\right\|.
\end{aligned} \tag{3.40}$$

By condition (C1), we have $\|Sy_n-z_n\|\to 0$ as $n\to\infty$. Next, we observe that

$$\left\|Sy_n-y_n\right\|\leq\left\|Sy_n-z_n\right\|+\left\|z_n-y_n\right\|. \tag{3.41}$$

By (3.39) and (3.40), we have $\|Sy_n-y_n\|\to 0$ as $n\to\infty$.

Step 5. We show that $q \in \Theta := F(S) \cap \text{GMEP}(F, \varphi, \Phi) \cap I(B, M)$ and $\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy_n - q\rangle \leq 0$. It is easy to see that $P_{\Theta}(\gamma f + (I - A))$ is a contraction of H into itself. Indeed, since $0 < \gamma < \overline{\gamma}/\alpha$ we have

$$\begin{aligned}\|P_{\Theta}(\gamma f + (I - A))x - P_{\Theta}(\gamma f + (I - A))y\| &\leq \|(\gamma f + (I - A))x - (\gamma f + (I - A))y\| \\ &\leq \gamma\|f(x) - f(y)\| + |I - A|\|x - y\| \\ &\leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\ &= (1 - \overline{\gamma} + \gamma\alpha)\|x - y\|.\end{aligned} \tag{3.42}$$

Since H is complete, there exists a unique fixed point $q \in H$ such that $q = P_{\Theta}(\gamma f + (I - A))(q)$. By Lemma 2.2, we obtain that $\langle(\gamma f - A)q, w - q\rangle \leq 0$ for all $w \in \Theta$.

Next, we show that $\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy_n - q\rangle \leq 0$, where $q = P_{\Theta}(\gamma f + I - A)(q)$ is the unique solution of the variational inequality $\langle(\gamma f - A)q, p - q\rangle \geq 0$, for all $p \in \Theta$. We can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy_n - q\rangle = \lim_{i\to\infty}\langle(\gamma f - A)q, Sy_{n_i} - q\rangle. \tag{3.43}$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. We may assume without loss of generality that $y_{n_i} \rightharpoonup w$. We claim that $w \in \Theta$, since $\lim_{n\to\infty}\|y_n - Sy_n\| = 0$ and by Lemma 2.6, we have $w \in F(S)$.

Next, we show that $w \in \text{GMEP}(F, \varphi, \Phi)$. Since $u_n = T_r(x_n - r\Phi x_n)$, we know that

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle\Phi x_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.44}$$

It follows by (A2) that

$$\varphi(y) - \varphi(u_n) + \langle\Phi x_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C. \tag{3.45}$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \langle\Phi x_{n_i}, y - u_{n_i}\rangle + \frac{1}{r}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.46}$$

For $t \in (0, 1]$ and $y \in H$, let $y_t = ty + (1 - t)w$. From (3.46) we have

$$\begin{aligned}\langle y_t - u_{n_i}, \Phi y_t\rangle &\geq \langle y_t - u_{n_i}, \Phi y_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle\Phi x_{n_i}, y_t - u_{n_i}\rangle \\ &\quad - \frac{1}{r}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F(y_t, u_{n_i}) \\ &= \langle y_t - u_{n_i}, \Phi y_t - \Phi u_{n_i}\rangle + \langle y_t - u_{n_i}, \Phi u_{n_i} - \Phi x_{n_i}\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad - \frac{1}{r}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F(y_t, u_{n_i}).\end{aligned} \tag{3.47}$$

From $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|\Phi u_{n_i} - \Phi x_{n_i}\| \to 0$. Further, from (A4) and the weakly lower semicontinuity of φ, $(u_{n_i} - x_{n_i})/r \to 0$ and $u_{n_i} \rightharpoonup w$, we have

$$\langle y_t - w, \Phi y_t \rangle \geq -\varphi(y_t) + \varphi(w) + F(y_t, w). \tag{3.48}$$

From (A1), (A4), and (3.48), we have

$$\begin{aligned} 0 &= F(y_t, y_t) - \varphi(y_t) + \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[F(y_t, w) + \varphi(w) - \varphi(y_t)] \\ &\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t - w, \Phi y_t \rangle \\ &= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)t\langle y - w, \Phi y_t \rangle, \end{aligned} \tag{3.49}$$

and hence

$$0 \leq F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, \Phi y_t \rangle. \tag{3.50}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \varphi(y) - \varphi(w) + \langle y - w, \Phi w \rangle \geq 0. \tag{3.51}$$

This implies that $w \in \mathrm{GMEP}(F, \varphi, \Phi)$.

Lastly, we show that $w \in I(B, M)$. In fact, since B is a β-inverse-strongly monotone, B is monotone and Lipschitz continuous mapping. It follows from Lemma 2.3, that $M + B$ is a maximal monotone. Let $(v, g) \in G(M+B)$, since $g - Bv \in M(v)$. Again since $y_{n_i} = J_{M,\lambda}(x_{n_i} - \lambda B x_{n_i})$, we have $x_{n_i} - \lambda B x_{n_i} \in (I + \lambda M)(y_{n_i})$, that is, $(1/\lambda)(x_{n_i} - y_{n_i} - \lambda B x_{n_i}) \in M(y_{n_i})$. By virtue of the maximal monotonicity of $M + B$, we have

$$\left\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(x_{n_i} - y_{n_i} - \lambda B x_{n_i}) \right\rangle \geq 0, \tag{3.52}$$

and hence

$$\begin{aligned} \langle v - y_{n_i}, g \rangle &\geq \left\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(x_{n_i} - y_{n_i} - \lambda B x_{n_i}) \right\rangle \\ &= \langle v - y_{n_i}, Bv - By_{n_i} \rangle + \langle v - y_{n_i}, By_{n_i} - Bx_{n_i} \rangle \\ &\quad + \left\langle v - y_{n_i}, \frac{1}{\lambda}(x_{n_i} - y_{n_i}) \right\rangle. \end{aligned} \tag{3.53}$$

It follows from $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we have $\lim_{n\to\infty}\|Bx_n - By_n\| = 0$ and $y_{n_i} \rightharpoonup w$ that

$$\limsup_{n\to\infty} \langle v - y_n, g \rangle = \langle v - w, g \rangle \geq 0. \tag{3.54}$$

It follows from the maximal monotonicity of $B+M$ that $\theta \in (M+B)(w)$, that is, $w \in I(B,M)$. Therefore, $w \in \Theta$. It follows that

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy_n - q\rangle = \lim_{k\to\infty}\langle(\gamma f - A)q, Sy_{n_i} - q\rangle = \langle(\gamma f - A)q, w - q\rangle \le 0. \quad (3.55)$$

Step 6. We prove $x_n \to q$. By using (3.2) and together with Schwarz inequality, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\xi_n P_C[(\alpha_n\gamma f(x_n) + (I - \alpha_n A)Sy_n) - q] + (1-\xi_n)(u_n - q)\|^2 \\
&\le \xi_n\|P_C[(\alpha_n\gamma f(x_n) + (I - \alpha_n A)Sy_n) - P_C(q)]\|^2 + (1-\xi_n)\|u_n - q\|^2 \\
&\le \xi_n\|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(Sy_n - q)\|^2 + (1-\xi_n)\|x_n - q\|^2 \\
&\le \xi_n(I - \alpha_n A)^2\|Sy_n - q\|^2 + \xi_n\alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n\alpha_n\langle(I - \alpha_n A)(Sy_n - q), \gamma f(x_n) - Aq\rangle + (1-\xi_n)\|x_n - q\|^2 \\
&\le \xi_n(1 - \alpha_n\overline{\gamma})^2\|y_n - q\|^2 + \xi_n\alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n\alpha_n\langle Sy_n - q, \gamma f(x_n) - Aq\rangle \\
&\quad - 2\xi_n\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle + (1-\xi_n)\|x_n - q\|^2 \\
&= \xi_n(1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \xi_n\alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n\alpha_n\langle Sy_n - q, \gamma f(x_n) - \gamma f(q)\rangle + 2\xi_n\alpha_n\langle Sy_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\xi_n\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle + (1-\xi_n)\|x_n - q\|^2 \\
&\le \xi_n(1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \xi_n\alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n\alpha_n\|Sy_n - q\|\|\gamma f(x_n) - \gamma f(q)\| + 2\xi_n\alpha_n\langle Sy_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\xi_n\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle + (1-\xi_n)\|x_n - q\|^2 \\
&\le \xi_n(1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \xi_n\alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n\gamma\alpha\alpha_n\|y_n - q\|\|x_n - q\| + 2\xi_n\alpha_n\langle Sy_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\xi_n\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle + (1-\xi_n)\|x_n - q\|^2 \\
&\le \left(\xi_n - 2\xi_n\alpha_n\overline{\gamma} + \xi_n\alpha_n^2\overline{\gamma}^2\right)\|x_n - q\|^2 + \xi_n\alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n\gamma\alpha\alpha_n\|x_n - q\|^2 + 2\xi_n\alpha_n\langle Sy_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\xi_n\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle + (1-\xi_n)\|x_n - q\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \left(1-2\xi_n\alpha_n\overline{\gamma}+2\xi_n\gamma\alpha\alpha_n\right)\|x_n-q\|^2+\xi_n\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad+2\xi_n\alpha_n\langle Sy_n-q,\gamma f(q)-Aq\rangle \\
&\quad-2\xi_n\alpha_n^2\|A(Sy_n-q)\|\|\gamma f(x_n)-Aq\|+\xi_n\alpha_n^2\overline{\gamma}^2\|x_n-q\|^2 \\
&= \left(1-2(\overline{\gamma}-\gamma\alpha)\xi_n\alpha_n\right)\|x_n-q\|^2+\xi_n\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad+2\xi_n\alpha_n\langle Sy_n-q,\gamma f(q)-Aq\rangle \\
&\quad-2\xi_n\alpha_n^2\|A(Sy_n-q)\|\|\gamma f(x_n)-Aq\|+\xi_n\alpha_n^2\overline{\gamma}^2\|x_n-q\|^2 \\
&= \left(1-2(\overline{\gamma}-\gamma\alpha)\xi_n\alpha_n\right)\|x_n-q\|^2 \\
&\qquad+2(\overline{\gamma}-\gamma\alpha)\xi_n\alpha_n\Bigg\{\frac{\alpha_n}{2(\overline{\gamma}-\gamma\alpha)}\|\gamma f(x_n)-Aq\|^2 \\
&\qquad\qquad+\frac{1}{\overline{\gamma}-\gamma\alpha}\langle Sy_n-q,\gamma f(q)-Aq\rangle \\
&\qquad\qquad-\frac{\alpha_n}{\overline{\gamma}-\gamma\alpha}\|A(Sy_n-q)\|\|\gamma f(x_n)-Aq\| \\
&\qquad\qquad+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma\alpha)}\|x_n-q\|^2\Bigg\} \\
&= (1-\gamma_n)\|x_n-q\|^2+\gamma_n\delta_n,
\end{aligned}
\tag{3.56}
$$

where $\gamma_n = 2(\overline{\gamma}-\gamma\alpha)$ and $\delta_n = (\alpha_n/2\ (\overline{\gamma}-\gamma\alpha))\|\gamma f(x_n)-Aq\|^2 + (1/(\overline{\gamma}-\gamma\alpha))\langle Sy_n-q,\gamma f(q)-Aq\rangle - (\alpha_n/(\overline{\gamma}-\gamma\alpha))\|A\ (Sy_n-q)\|\|\gamma f(x_n)-Aq\| + (\alpha_n\overline{\gamma}^2/2(\overline{\gamma}-\gamma\alpha))\|x_n-q\|^2$. It is clear that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Hence, all conditions of Lemma 2.5, we can conclude that $x_n \to q$. This completes the proof.

□

Corollary 3.2. *Let H be a real Hilbert space and C be a closed convex subset of H. Let F be a bifunction of $C\times C$ into $\mathcal{R}$ satisfying (A1)–(A4) and $B,\Phi : C \to H$ be β, σ-inverse-strongly monotone mappings, let $\varphi : C \to \mathcal{R}$ be convex and lower semicontinuous function, $f : C \to C$ be a contraction with coefficient α $(0<\alpha<1)$, and $M : H \to 2^H$ be a maximal monotone mapping. Let S be a nonexpansive mapping of H into itself and assume that either (B1) or (B2) holds such that*

$$
\Theta := F(S)\cap \mathrm{GMEP}(F,\varphi,\Phi)\cap I(B,M)\neq\emptyset. \tag{3.57}
$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0\in C$ arbitrarily:

$$
x_{n+1} = \xi_n P_C\left[\alpha_n f(x_n) + (I-\alpha_n)SJ_{M,\lambda}(x_n-\lambda Bx_n)\right] + (1-\xi_n)T_r(x_n-r\Phi x_n), \tag{3.58}
$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0,2\beta)$ *such that* $0 < a \leq \lambda \leq b < 2\beta$ *and* $r \in (0,2\sigma)$ *with* $0 < c \leq d \leq 1-\sigma$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

(C2) $0 < \liminf_{n\to\infty}\xi_n < \limsup_{n\to\infty}\xi_n < 1$ *and* $\lim_{n\to\infty}((\xi_{n+1}-\xi_n)/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta$, *where* $q = P_\Theta\,(f+I)(q)$ *which solves the following variational inequality:*

$$\langle (f-I)q, p-q\rangle \leq 0, \quad \forall p \in \Theta. \tag{3.59}$$

Proof. Putting $A \equiv I$ and $\gamma \equiv 1$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Corollary 3.3. *Let* H *be a real Hilbert space and* C *be a closed convex subset of* H. *Let* F *be a bifunction of* $C \times C$ *into* $\mathcal{R}$ *satisfying (A1)–(A4) and* $B, \Phi : C \to H$ *be* β, σ*-inverse-strongly monotone mappings, let* $\varphi : C \to \mathcal{R}$ *be convex and lower semicontinuous function, and* $M : H \to 2^H$ *be a maximal monotone mapping. Let* S *be a nonexpansive mapping of* H *into itself and assume that either (B1) or (B2) holds such that*

$$\Theta := F(S) \cap \text{GMEP}(F,\varphi,\Phi) \cap I(B,M) \neq \emptyset. \tag{3.60}$$

Suppose $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$x_{n+1} = \xi_n P_C[\alpha_n u + (I-\alpha_n)SJ_{M,\lambda}\,(x_n - \lambda B x_n)] + (1-\xi_n)T_r(x_n - r\Phi x_n), \tag{3.61}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0,2\beta)$ *such that* $0 < a \leq \lambda \leq b < 2\beta$ *and* $r \in (0,2\sigma)$ *with* $0 < c \leq d \leq 1-\sigma$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

(C2) $0 < \liminf_{n\to\infty}\xi_n < \limsup_{n\to\infty}\xi_n < 1$ *and* $\lim_{n\to\infty}((\xi_{n+1}-\xi_n)/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta$, *where* $q = P_\Theta(q)$ *which solves the following variational inequality:*

$$\langle u-q, p-q\rangle \leq 0, \quad \forall p \in \Theta. \tag{3.62}$$

Proof. Putting $f \equiv u \in C$ in Corollary 3.2, we can obtain desired conclusion immediately. □

Corollary 3.4. *Let* H *be a real Hilbert space,* C *be a closed convex subset of* H. *Let* F *be a bifunction of* $C\times C$ *into* $\mathcal{R}$ *satisfying (A1)–(A4) and* $B, \Phi : C \to H$ *be* β, σ*-inverse-strongly monotone mappings,* $\varphi : C \to \mathcal{R}$ *is convex and lower semicontinuous function,* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$ *and* A *be a strongly positive linear bounded operator of* H *into itself with coefficient* $\overline{\gamma} > 0$. *Assume that* $0 < \gamma < \overline{\gamma}/\alpha$. *Let* S *be a nonexpansive mapping of* C *into itself and assume that either (B1) or (B2) holds such that*

$$\Theta := F(S) \cap \text{GMEP}(F,\varphi,\Phi) \cap \text{VI}(C,B) \neq \emptyset. \tag{3.63}$$

Suppose $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$x_{n+1} = \xi_n P_C\left[\alpha_n \gamma f(x_n) + (I - \alpha_n A) S P_C(x_n - \lambda B x_n)\right] + (1 - \xi_n) T_r(x_n - r\Phi x_n), \tag{3.64}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0, 2\beta)$ *such that* $0 < a \leq \lambda \leq b < 2\beta$ *and* $r \in (0, 2\sigma)$ *with* $0 < c \leq d \leq 1 - \sigma$ *satisfing the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

(C2) $0 < \liminf_{n\to\infty}\xi_n < \limsup_{n\to\infty}\xi_n < 1$ *and* $\lim_{n\to\infty}((\xi_{n+1} - \xi_n)/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta$, *where* $q = P_\Theta(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle(\gamma f -)q, p - q\rangle \leq 0, \quad \forall p \in \Theta. \tag{3.65}$$

Proof. Taking $J_{M,\lambda} = P_C$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Corollary 3.5. *Let* H *be a real Hilbert space,* C *be a closed convex subset of* H. *Let* F *be a bifunction of* $C \times C$ *into* $\mathcal{R}$ *satisfying (A1)–(A4) and* B, $\Phi : C \to H$ *be* β, σ*-inverse-strongly monotone mappings,* $\varphi : C \to \mathcal{R}$ *is convex and lower semicontinuous function,* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$. *Let* S *be a nonexpansive mapping of* C *into itself and assume that either (B1) or (B2) holds such that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F, \varphi, \Phi) \neq \emptyset. \tag{3.66}$$

Suppose $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$x_{n+1} = \xi_n P_C\left[\alpha_n f(x_n) + (I - \alpha_n) S x_n\right] + (1 - \xi_n) T_r(x_n - r\Phi x_n), \tag{3.67}$$

where $\{\alpha_n\}$, $\{\xi_n\} \subset (0,1)$ *and* $r \in (0, 2\sigma)$ *with* $0 < c \leq d \leq 1 - \sigma$ *satisfing the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

(C2) $0 < \liminf_{n\to\infty}\xi_n < \limsup_{n\to\infty}\xi_n < 1$ *and* $\lim_{n\to\infty}((\xi_{n+1} - \xi_n)/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta$, *where* $q = P_\Theta(f + I)(q)$ *which solves the following variational inequality:*

$$\langle(f - I)q, p - q\rangle \leq 0, \quad \forall p \in \Theta. \tag{3.68}$$

Proof. Taking $\gamma \equiv 1$, $A \equiv I$, $M = 0$, and $B \equiv 0$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Remark 3.6. Corollary 3.5 generalizes and improves the result of Yao and Liou [16].

4. Some Applications

In this section, we apply the iterative scheme (1.22) for finding a common fixed point of nonexpansive mapping and strictly pseudocontractive mapping and also apply Theorem 3.1 for finding a common fixed point of nonexpansive mappings and inverse-strongly monotone mappings.

Definition 4.1. A mapping $T : C \to C$ is called *strictly pseudocontraction* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{4.1}$$

If $\kappa = 0$, then S is nonexpansive. In this case, we say that $T : C \to C$ is a κ-strictly pseudocontraction. Putting $B = I - T$. Then, we have

$$\|(I - B)x - (I - B)y\|^2 \leq \|x - y\|^2 + \kappa\|Bx - By\|^2, \quad \forall x, y \in C. \tag{4.2}$$

Observe that

$$\|(I - B)x - (I - B)y\|^2 = \|x - y\|^2 + \|Bx - By\|^2 - 2\langle x - y, Bx - By\rangle, \quad \forall x, y \in C. \tag{4.3}$$

Hence, we obtain

$$\langle x - y, Bx - By\rangle \geq \frac{1-\kappa}{2}\|Bx - By\|^2, \quad \forall x, y \in C. \tag{4.4}$$

Then, B is $((1-\kappa)/2)$-inverse-strongly monotone mapping.

Using Theorem 3.1, we first prove a strongly convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontraction.

Theorem 4.2. *Let H be a real Hilbert space and C be a closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathcal{R}$ satisfying (A1)–(A4) and $B, \Phi : C \to H$ be β, σ-inverse-strongly monotone mapping, $\varphi : C \to \mathcal{R}$ be convex and lower semicontinuous function, $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$, and A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let S be a nonexpansive mapping of C into itself and let T be a κ-strictly pseudocontraction of C into itself. Assume that either (B1) or (B2) holds such that*

$$\Theta := F(S) \cap F(T) \cap \mathrm{GMEP}(F, \varphi, \Phi) \neq \emptyset. \tag{4.5}$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily:

$$x_{n+1} = \xi_n P_C\left[\alpha_n \gamma f(x_n) + (I - \alpha_n A)S((1-\lambda)u_n + \lambda T u_n)\right] + (1 - \xi_n)T_r(x_n - r\Phi x_n), \tag{4.6}$$

for all $n = 0, 1, 2, \ldots$, where $\{\alpha_n\}, \{\xi_n\} \subset (0, 1)$, $\lambda \in [0, 1-\kappa)$ and $r \in (0, 2\sigma)$. If $\lambda \in [a, b]$ for some a, b with $0 < a < b < 1-\kappa$ and $\{\sigma_n\}$ is chosen so that $r \in [c, d]$ for some c, d with $0 < c < d < 1-\sigma$ and satisfy the condition (C1)-(C2) in Theorem 3.1.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta$*, where* $q = P_\Theta(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle (\gamma f - A)q, p - q\rangle \leq 0, \quad \forall p \in \Theta, \tag{4.7}$$

which is the optimality condition for the minimization problem

$$\min_{q\in\Omega} \frac{1}{2}\langle Aq, q\rangle - h(q), \tag{4.8}$$

where h *is a potential function for* γf *(i.e.,* $h'(q) = \gamma f(q)$ *for* $q \in H$*).*

Proof. Put $B \equiv I - T$, then B is $((1-\kappa)/2)$-inverse-strongly monotone and $F(T) = I(B, M)$ and $J_{M,\lambda}(x_n - \lambda B x_n) = (1-\lambda)x_n + \lambda T x_n$. So by Theorem 3.1, we obtain the desired result. □

Corollary 4.3. *Let* H *be a real Hilbert space and* C *be a closed convex subset of* H*. Let* F *be a bifunction of* $C \times C$ *into* $\mathcal{R}$ *satisfying (A1)–(A4) and* B*,* $\Phi : C \to H$ *be* β*,* σ*-inverse-strongly monotone mapping,* $\varphi : C \to \mathcal{R}$ *is convex and lower semicontinuous function. Let* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$ *and* S *be a nonexpansive mapping of* C *into itself and let* T *be a* κ*-strictly pseudocontraction of* C *into itself. Assume that either (B1) or (B2) holds such that*

$$\Theta := F(S) \cap F(T) \cap \mathrm{GMEP}(F, \varphi, \Phi) \neq \emptyset. \tag{4.9}$$

Suppose $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$x_{n+1} = \xi_n P_C\left[\alpha_n f(x_n) + (I - \alpha_n) S((1-\lambda)u_n + \lambda T u_n)\right] + (1-\xi_n) T_r(x_n - r\Phi x_n), \tag{4.10}$$

for all $n = 0, 1, 2, \ldots$*, where* $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$*,* $\lambda \in [0, 1-\kappa)$ *and* $r \in (0, 2\sigma)$*. If* $\lambda \in [a, b]$ *is chosen for some* a, b *with* $0 < a < b < 1 - \kappa$ *and* $\{\sigma_n\}$ *is chosen so that* $r \in [c, d]$ *for some* c, d *with* $0 < c < d < 1 - \sigma$ *and satisfy the condition (C1)-(C2) in Theorem 3.1.*

Then $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(f + I)(q)$ which solves the following variational inequality:

$$\langle (f - I)q, p - q\rangle, \quad \forall p \in \Omega. \tag{4.11}$$

Proof. Put $A \equiv I$ and $\gamma \equiv 1$ in by Theorem 4.2, we obtain the desired result. □

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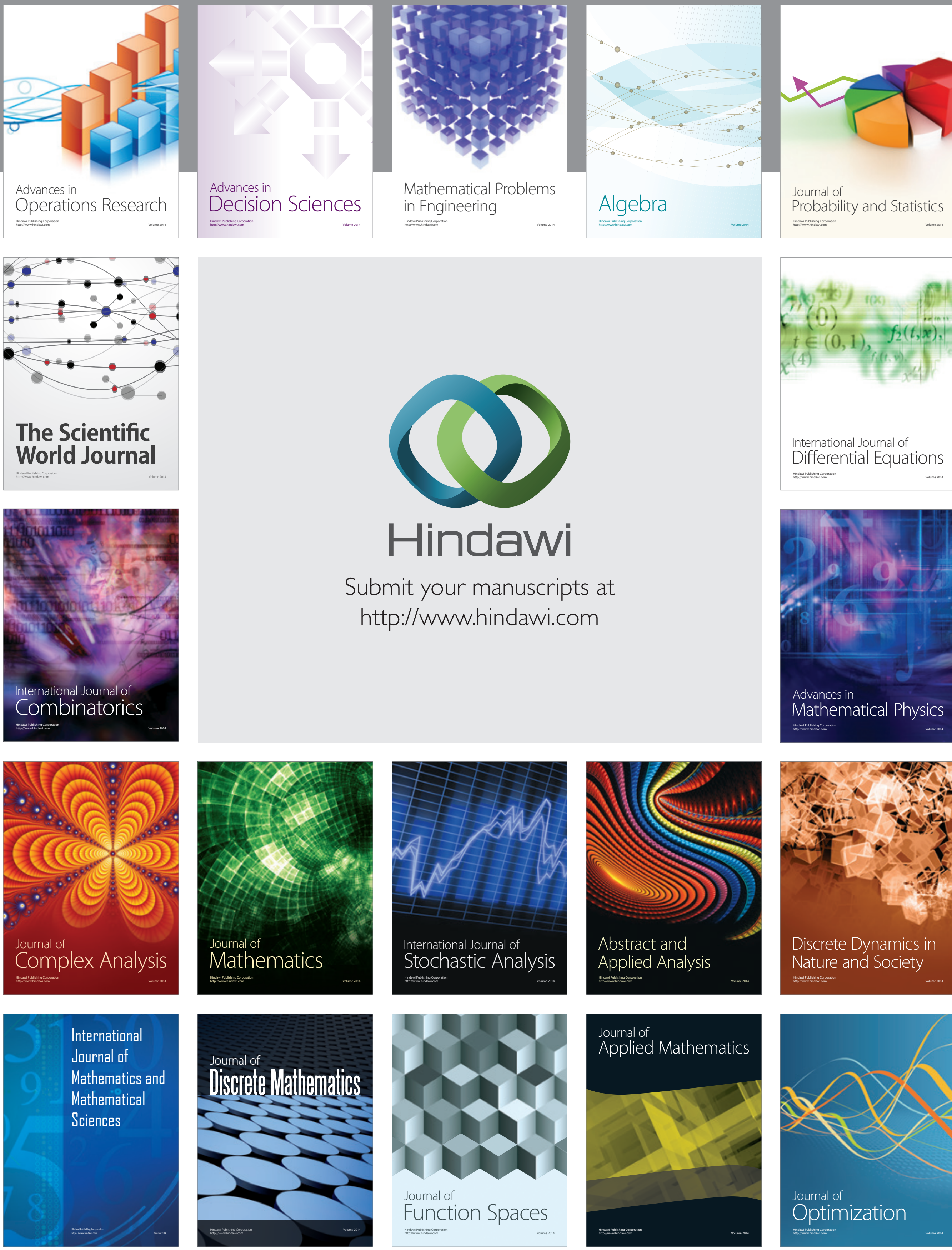
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