Research Article

One-Dimensional Problem of a Conducting Viscous Fluid with One Relaxation Time

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We introduce a magnetohydrodynamic model of boundary-layer equations for conducting viscous fluids. This model is applied to study the effects of free convection currents with thermal relaxation time on the flow of a viscous conducting fluid. The method of the matrix exponential formulation for these equations is introduced. The resulting formulation together with the Laplace transform technique is applied to a variety problems. The effects of a plane distribution of heat sources on the whole and semispace are studied. Numerical results are given and illustrated graphically for the problem.

1. Introduction

The modification of the heat-conduction equation from diffusive to a wave type may be affected either by a microscopic consideration of the phenomenon of heat transport or in a phenomenological way by modifying the classical Fourier law of heat conduction.

Many authors have considered various aspects of this problem and obtained similarity solutions. Samaan [1] investigated steady oscillating magnetohydrodynamic flow in a circular pipe. Analytical and numerical methods for the momentum and energy equations of a viscous incompressible fluid along a vertical plate have been considered by Samaan [2]. Chamkha [3] studied the magnetohydrodynamic flow of a uniformly stretched vertical permeable surface in the presence of heat generation/absorption and chemical reaction. Ishak et al. [4] investigated theoretically the unsteady mixed convection boundary layer flow and heat transfer due to a stretching vertical surface in a quiescent viscous and incompressible fluid.

Many authors presented some mathematical results, and a good amount of references can be found in the papers by Liao and Pop [5] and Nazar et al. [6]. Further, the stagnation region encounters the highest pressure, the highest heat transfer, and the highest rate of mass

deposition studied by Wang [7]. Singh et al. [8] investigated the problem of heat transfer in the flow of an incompressible fluid.

Samaan [9] investigated the heat and mass transfer over an accelerating surface with heat source in presence of suction and magnetic field. The flow of an unsteady, incompressible magnetohydrodynamics (MHDs) viscous fluid with suction is investigated by Muhammad et al. [10]. Heat and mass transfer over an accelerating surface with heat source in presence of magnetic field is derived by Samaan [11].

Recently, Samaan [12] studied the effects of variable viscosity and thermal diffusivity on the steady flow in the presence of the magnetic field. variable viscosity effects on hydrodynamic boundary layer flow along a continuously moving vertical plate were done by Mostafa [13]. Concerning the studies of state space formulation for MHDs and free convection flow with two relaxation times and the free convection effective perfectly conducting couple stress fluid, we may refer to Samaan [14], Ezzat et al. [15], and Hayat et al. [16]. Using differential transform method and Pade approximate for solving MHDs flow in a laminar liquid film from a horizontal stretching surfaces investigated by Rashidi et al. [17].

In the present work, we use a more general model of magnetohydrodynamic free convection flow, which also includes the relaxation time of heat conduction and the electric displacement current [18, 19]. An attempt to account for the time dependence of heat transfer, Cattaneo [20] and Vernotte [21] independently modified Fourier's law to include the relaxation time of the system. Generalized thermoelasticity stands for a hyperbolic thermoelasticity in which a thermomechanical load applied to a body is transmitted in a wave-like manner throughout the body, only transient thermoelastic waves are included in the survey by Hertnarski and Ignaczak [22]. The inclusion of the relaxation time and the electric displacement current modifies the governing thermal and electromagnetic field equations, changing them from the parabolic to a hyperbolic type, and thereby eliminating the unrealistic result that thermal and electromagnetic disturbances are realized instantaneously within a fluid.

The solution is obtained using a state-space approach. The importance of state-space analysis is recognized in fields where the time behavior of physical process is of interest. The first writer to introduce the state space approach in magnetohydrodynamic free convection flow was Ezzat [23, 24]. His works dealt with free convection flow in the absence of the applied magnetic field or when there are no heat sources. The present work is an attempt to generalize these results to include the effects of heat sources. The results obtained are used to solve a problem for the whole space with a plane distribution of heat sources. The solutions obtained are utilized in combination with the method of images to obtain the solution for a problem with heat sources distributed situated inside a semispace whose surface bounded by an infinite vertical plate.

The Laplace transform techniques is applied to one-dimensional problem. The inversion of the Laplace transforms is carried out using a numerical technique [25].

2. Formulation of the Problem

Let a constant magnetic field of strength H_0 act in the direction of the y-axis. Due to the effect of this magnetic field, there arises in the medium an induced magnetic field $\vec{h}$ and an induced electric field $\vec{E}$. All the considered functions will depend on y and the time t only.

The electromagnetic quantities satisfy Maxwell's equations [26]

$$\text{curl } \underset{\rightarrow}{h} = \underset{\rightarrow}{J} + \varepsilon_o \dot{E}, \tag{2.1}$$

$$\text{curl } \underset{\rightarrow}{E} = -\mu_o \dot{h}, \tag{2.2}$$

$$\text{div } \underset{\rightarrow}{h} = 0, \qquad \text{div } \underset{\rightarrow}{E} = 0, \tag{2.3}$$

$$\underset{\rightarrow}{B} = \mu_o \left(\underset{\rightarrow}{H_0} + \underset{\rightarrow}{h} \right), \qquad \underset{\rightarrow}{D} = \varepsilon_o \underset{\rightarrow}{E}, \tag{2.4}$$

where $\underset{\rightarrow}{J}$ is the electric current density, μ_o and ε_o are the magnetic and electric permeabilities, respectively, and $\underset{\rightarrow}{B}, \underset{\rightarrow}{D}$ are the magnetic and electric induction vectors, respectively, and dotte is the time variable.

These equations are supplemented by Ohm's law

$$\underset{\rightarrow}{J} = \sigma_o \left[\underset{\rightarrow}{E} + \mu_o \left(\underset{\rightarrow}{v} \times \underset{\rightarrow}{H_0} \right) \right], \tag{2.5}$$

where $\underset{\rightarrow}{v} = (u, 0, 0)$ is velocity vector of the fluid and σ_o is the electric conductivity.

As mentioned above, the applied field H_o and the induced magnetic field has the components

$$\underset{\rightarrow}{H_0} = (0, H_o, 0), \qquad \underset{\rightarrow}{h} = (h, 0, 0). \tag{2.6}$$

The vector $\underset{\rightarrow}{E}$ and $\underset{\rightarrow}{J}$ will have nonvanishing components only in the z-direction. That is,

$$\underset{\rightarrow}{E} = (0, 0, E), \qquad \underset{\rightarrow}{J} = (0, 0, J), \tag{2.7}$$

where from (2.5)

$$J = \sigma_o [E + \mu_o H_o u]. \tag{2.8}$$

The vector (2.1) and (2.2) reduce to the following scalar equations

$$h_{,y} = -(J + \varepsilon_o \dot{E}), \tag{2.9}$$

$$E_{,y} = -\mu_o \dot{h}. \tag{2.10}$$

Eliminating J between (2.8) and (2.9), we obtain

$$h_{,y} = -(\sigma_o E + \varepsilon_o \dot{E}) - \sigma_o \mu_o H_o u. \tag{2.11}$$

The pondermotive force $\underset{\rightarrow}{F} = \underset{\rightarrow}{J} \times \underset{\rightarrow}{B}$ has one nonvanishing component in the x-direction

$$F_x = \mu_o H_o(h_{,y} + \varepsilon_o \dot{E}), \tag{2.12}$$

where σ_o is the electric conductivity.

We investigate the free convective heat transfer in an incompressible hydromagnetic flow past an infinite vertical plate. The x-axis is taken along the plate in the direction of the flow and the y-axis normal to it. Let u be the component of the velocity in the x direction. All the fluid properties are assumed constant, except that the influence of the density variation with temperature is considered only in the body force term. In the energy equations, terms representing viscous and Joule's dissipation are neglected, as they are assumed to be very small in free convection flow [27]. Also, in the energy equation, the term representing the volumetric heat source is taken as a function of the space variables. In view of the assumptions, the equations that govern unsteady one-dimensional free convection flow in an incompressible conducting fluid through a porous medium bounded by an infinite nonmagnetic vertical plate in the presence of a constant magnetic field are (2.10)–(2.12), and the equations describing the flow in the boundary layer reduce to [28–30]

$$\rho \dot{u} = \rho \nu u_{,yy} - \rho \frac{\nu}{K} u + \rho \frac{\alpha^2}{H_o}(h_{,y} + \varepsilon_o \dot{E}) + \rho g \beta (T - T_\infty),$$

$$h_{,y} = -(\sigma_o E + \varepsilon_o \dot{E}) - \sigma_o \mu_o H_o u,$$

$$E_{,y} = -\mu_o \dot{h},$$

$$\rho c_p \dot{T} = \lambda T_{,yy} - \rho c_p v_o \ddot{T} + Q v_o \dot{Q}. \tag{2.13}$$

And the constitutive equation

$$(\rho_\infty - \rho) = \rho \beta (T - T_\infty). \tag{2.14}$$

In these equations, K is the permeability of the porous medium, α is the Alfven velocity, g is the acceleration due to gravity, β is the coefficient of volume expansion, T is the temperature distribution, T_∞ is the temperature of the fluid away from the plate, c_p is specific heat at constant pressure, λ is the thermal diffusivity, v_o is the relaxation time, ρ_∞ is the density of the fluid far from the surface, and ρ is the density of the fluid.

Let us introduce the following nondimensional variables

$$y^* = \frac{y\alpha}{\nu}, \quad t^* = \frac{t\alpha^2}{\nu}, \quad u^* = \frac{u}{\alpha}, \quad E^* = \frac{E}{\mu_o H_o \alpha}, \quad h^* = \frac{h}{H_o}, \quad v_o^* = \frac{\alpha^2 v_o}{\nu},$$

$$K^* = \frac{K\alpha^2}{\nu^2}, \quad \theta = \frac{T - T_\infty}{T_o - T_\infty}, \quad G_r = \frac{\nu \beta g(T_o - T_\infty)}{\alpha^3}, \quad Q^* = \frac{\nu^2 Q}{\lambda \alpha^2 (T_o - T_\infty)}. \tag{2.15}$$

In view of these transformations, (2.13)–(2.14) become

$$\dot{u} = u_{,yy} - \frac{1}{K}u + h_{,y} + a\dot{E} + G_r\theta,$$

$$h_{,y} = -\eta(E + u) - a\dot{E},$$

$$E_y = -\dot{h},$$

$$\theta_{,yy} - p_r(\dot{\theta} + v_o\ddot{\theta}) = -Q - v_o\dot{Q}, \tag{2.16}$$

where G_r is the Grashof number, c is the speed of light given by $c^2 = 1/\varepsilon_o\mu_o$, $a = \alpha^2/c^2$, $\eta = v\mu_o\sigma_o$ is a measure of the magnetic viscosity, and $p_r = \rho c_p v/\lambda$ is the Prandtl number. From now on, we will consider a heat source of the form

$$Q = Q_o\delta(y)H(t), \tag{2.17}$$

where $\delta(y)$ and $H(t)$ are the Dirac delta function and the Heaviside unit step function, respectively, Q is the strength of the applied heat source, and Q_o is a constant.

We will assume that the initial state of the medium is quiescent. Taking the Laplace transform, defined by the relation

$$\overline{f}(s) = \int_0^\infty e^{-st}f(t)dt, \tag{2.18}$$

of both sides of (2.16), we obtain

$$\frac{\partial^2 \overline{u}}{\partial y^2} - \left(\frac{1}{K} + s\right)\overline{u} = -G_r\overline{\theta} - \frac{\partial \overline{h}}{\partial y} - as\overline{E}, \tag{2.19}$$

$$\frac{\partial \overline{h}}{\partial y} = -(\eta + as)\overline{E} - \eta\overline{u}, \tag{2.20}$$

$$\frac{\partial \overline{E}}{\partial y} = -s\overline{h}, \tag{2.21}$$

$$\left[\frac{\partial^2}{\partial y^2} - p_r s(1 + \tau_o s)\right]\theta = -Q_o\frac{(1 + \tau_o s)}{s}\delta(y). \tag{2.22}$$

Eliminating E between (2.19)-(2.20), we get

$$\left(\frac{\partial^2}{\partial y^2} - F\right)\overline{u} = -G_r\overline{\theta} - n\frac{\partial \overline{h}}{\partial y},$$

$$\left(\frac{\partial^2}{\partial y^2} - \xi\right)\overline{h} = -\eta\frac{\partial u}{\partial y}, \tag{2.23}$$

where

$$F = s + \frac{1}{K} + \frac{as\eta}{\eta + as}, \qquad n = \frac{\eta}{\eta + as}, \qquad \xi = s(\eta + as). \tag{2.24}$$

We will choose as state variables the temperature θ, the velocity component u, and induced magnetic field h and their gradients. Equations (2.22)–(2.23) can be written as

$$\frac{\partial \overline{\theta}}{\partial y} = \overline{\theta}', \qquad \frac{\partial \overline{u}}{\partial y} = \overline{u}', \qquad \frac{\partial \overline{h}}{\partial y} = \overline{h}',$$

$$\frac{\partial \overline{\theta}'}{\partial y} = ps\overline{\theta} - Q_o\delta(y)\left(\frac{1 + v_o s}{s}\right),$$

$$\frac{\partial \overline{u}'}{\partial y} = -G_r\overline{\theta} + F\overline{u} - n\overline{h}',$$

$$\frac{\partial \overline{h}'}{\partial y} = -\eta\overline{u}' + \xi\overline{h}, \tag{2.25}$$

where $p = p_r(1 + v_o s)$. The above equations can be written in matrix form as

$$\frac{d\overline{v}(y, s)}{dy} = A(s)\overline{v}(y, s) + B(y, s), \tag{2.26}$$

where

$$A(s) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ ps & 0 & 0 & 0 & 0 & 0 \\ -G_r & F & 0 & 0 & 0 & -n \\ 0 & 0 & \xi & 0 & -\eta & 0 \end{bmatrix}, \qquad \overline{v}(y, s) = \begin{bmatrix} \overline{\theta}(y, s) \\ \overline{u}(y, s) \\ \overline{h}(y, s) \\ \overline{\theta}'(y, s) \\ \overline{u}'(y, s) \\ \overline{h}'(y, s) \end{bmatrix},$$

$$B(y, s) = -Q_o\delta(y)\left(\frac{1 + v_o s}{s}\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{2.27}$$

In order to solve the system (2.26), we need first to find the form of the matrix $\exp(A(s)y)$.

The characteristic equation of the matrix A has the form

$$k^6 - Z_1 k^4 + Z_2 k^2 - Z_3 = 0, \tag{2.28}$$

where

$$Z_1 = F + \xi + ps + \eta n,$$
$$Z_2 = F\xi + ps(F + \xi) + \eta nps, \tag{2.29}$$
$$Z_3 = psF\xi.$$

The roots $\pm k_1$, $\pm k_2$, and $\pm k_3$ of (2.28) satisfy the relations

$$Z_1 = k_1^2 + k_2^2 + k_3^2,$$
$$Z_2 = k_1^2 k_2^2 + k_2^2 k_3^2 + k_1^2 k_3^2, \tag{2.30}$$
$$Z_3 = k_1^2 k_2^2 k_3^2.$$

One of the roots, say k_1^2, has a simple expression given by

$$k_1^2 = ps. \tag{2.31}$$

The other two roots k_2^2 and k_3^2 satisfy the relation

$$k_2^2 + k_3^2 = F + \xi + \eta n,$$
$$k_2^2 k_3^2 = F\xi. \tag{2.32}$$

The Taylor series expansion of the matrix exponential has the form

$$\exp[A(s)y] = \sum_{n=0}^{\infty} \frac{1}{n!} [A(s)y]^n. \tag{2.33}$$

Using the well-known Cayley-Hamilton theorem, the infinite series can be truncated to the following form:

$$\exp[A(s)y] = L(s,y) = b_o I + b_1 A + b_2 A^2 + b_3 A^3 + b_4 A^4 + b_5 A^5, \tag{2.34}$$

where I is the unit matrix of order 6 and b_o–b_5 are some parameters depending on s and y. The characteristic roots $\pm k_i$, $i = 1, 2, 3$ of the matrix A must satisfy the equations

$$\exp(\pm k_1 y) = b_o \pm b_1 k_1 + b_2 k_1^2 \pm b_3 k_1^3 + b_4 k_1^4 \pm b_5 k_1^5,$$

$$\exp(\pm k_2 y) = b_o \pm b_1 k_2 + b_2 k_2^2 \pm b_3 k_2^3 + b_4 k_2^4 \pm b_5 k_2^5, \qquad (2.35)$$

$$\exp(\pm k_3 y) = b_o \pm b_1 k_3 + b a_2 k_3^2 \pm b_3 k_3^3 + b_4 k_3^4 \pm b_5 k_3^5.$$

The solution of this system of linear equations is given by

$$b_o = -R\left(k_2^2 k_3^2 C_1 + k_1^2 k_3^2 C_2 + k_2^2 k_1^2 C_3\right),$$

$$b_1 = -R\left(k_2^2 k_3^2 S_1 + k_3^2 k_1^2 S_2 + k_1^2 k_2^2\right),$$

$$b_2 = R\left[\left(k_2^2 + k_3^2\right)C_1 + \left(k_3^2 + k_1^2\right)C_2 + \left(k_1^2 + k_2^2\right)C_3\right],$$

$$b_3 = R\left[\left(k_2^2 + k_3^2\right)S_1 + \left(k_3^2 + k_1^2\right)S_2 + \left(k_1^2 + k_2^2\right)S_3\right], \qquad (2.36)$$

$$b_4 = -R(C_1 + C_2 + C_3),$$

$$b_5 = -R\ (S_1 + S_2 + S_3),$$

where

$$R = \frac{1}{\left(k_1^2 - k_2^2\right)\left(k_2^2 - k_3^2\right)\left(k_3^2 - k_1^2\right)},$$

$$C_1 = \left(k_2^2 - k_3^2\right)\cosh(k_1 y), \qquad S_1 = \frac{\left(k_2^2 - k_3^2\right)}{k_1}\sinh(k_1 y),$$

$$\qquad (2.37)$$

$$C_2 = \left(k_3^2 - k_1^2\right)\cosh(k_2 y), \qquad S_2 = \frac{\left(k_3^2 - k_1^2\right)}{k_2}\sinh(k_2 y),$$

$$C_3 = \left(k_1^2 - k_2^2\right)\cosh(k_3 y), \qquad S_3 = \frac{\left(k_1^2 - k_2^2\right)}{k_3}\sinh(k_3 y).$$

Substituting for the parameters b_o–b_5 from (2.36) into (2.34) and computing A^2, A^3, A^4, and A^5, we get the elements ($L_{ij} = 1, 2, 3, 4, 5, 6$) of the matrix $L\ (s, y)$ to be

$$L_{11} = R\left(k_1^2 - k_2^2\right)\left(k_3^2 - k_1^2\right)C_1,$$

$$L_{12} = L_{13} = 0,$$

$$L_{14} = R\left(k_1^2 - k_2^2\right)\left(k_3^2 - k_1^2\right)S_1,$$

$$L_{15} = L_{16} = 0,$$

$$L_{21} = G_r R\left[\left(k_1^2 - \xi\right)C_1 + \left(k_2^2 - \xi\right)C_2 + \left(k_3^2 - \xi\right)C_3\right],$$

$$L_{22} = R\left[\left(k_1^2 - k_2^2\right)\left(F - k_3^2\right)C_2 + \left(k_1^2 - k_3^2\right)\left(F - k_2^2\right)C_3\right],$$

$$L_{23} = n\xi R\left[\left(k_2^2 - k_1^2\right)S_2 + \left(k_3^2 - k_1^2\right)S_3\right],$$

$$L_{24} = G_r R\left[\left(k_1^2 - \xi\right)S_1 + \left(k_2^2 - \xi\right)S_2 + \left(k_3^2 - \xi\right)S_3\right],$$

$$L_{25} = R\left[\left(k_2^2 - \xi\right)\left(k_1^2 - k_2^2\right)S_2 + \left(k_3^2 - \xi\right)\left(k_1^2 - k_3^2\right)S_3\right],$$

$$L_{26} = nR\left[\left(k_2^2 - k_1^2\right)C_2 + \left(k_3^2 - k_1^2\right)C_3\right],$$

$$L_{31} = -G_r \eta R\left[k_1^2 S_1 + k_2^2 S_2 + k_3^2 S_3\right],$$

$$L_{32} = \eta F R\left[\left(k_2^2 - k_1^2\right)S_2 + \left(k_3^2 - k_1^2\right)S_3\right],$$

$$L_{33} = R\left[\left(n - k_3^2\right)\left(k_1^2 - k_2^2\right)C_2 + \left(n - k_2^2\right)\left(k_1^2 - k_3^2\right)C_3\right],$$

$$L_{34} = -G_r \eta R[\, C_1 + C_2 + C_3],$$

$$L_{35} = \eta R\left[\left(k_2^2 - k_1^2\right)C_2 + \left(k_3^2 - k_1^2\right)C_3\right],$$

$$L_{36} = R\left[\left(k_2^2 - F\right)\left(k_1^2 - k_2^2\right)S_2 + \left(k_3^2 - F\right)\left(k_1^2 - k_3^2\right)S_3\right],$$

$$L_{41} = -Rk_1^2\left(k_1^2 - k_2^2\right)\left(k_1^2 - k_3^2\right)S_1,$$

$$L_{42} = L_{43} = 0,$$

$$L_{44} = -R\left(k_1^2 - k_2^2\right)\left(k_1^2 - k_3^2\right)C_1,$$

$$L_{45} = L_{46} = 0,$$

$$L_{51} = GR\left[k_1^2\left(k_1^2 - \xi\right)S_1 + k_2^2\left(k_2^2 - \xi\right)S_2 + k_3^2\left(k_3^2 - \xi\right)S_3\right],$$

$$L_{52} = FR\left[\left(k_1^2 - k_2^2\right)\left(k_1^2 - \xi\right)S_1 + \left(k_1^2 - k_3^2\right)\left(k_3^2 - \xi\right)S_3\right],$$

$$L_{53} = \frac{n\xi R}{\eta}L_{35},$$

$$L_{54} = L_{21},$$

$$L_{55} = R\left[\left(k_1^2 - k_2^2\right)\left(k_2^2 - n\right)C_2 + \left(k_1^2 - k_3^2\right)\left(k_3^2 - n\right)C_3\right],$$

$$L_{56} = -nR\left[k_2^2\left(k_1^2 - k_2^2\right)S_2 + k_3^2\left(k_1^2 - k_3^2\right)S_3\right],$$

$$L_{61} = -\eta G_r R\left(k_1^2 C_1 + k_2^2 C_2 + k_3^2 C_3\right),$$

$$L_{62} = \frac{\eta F}{n} L_{26},$$

$$L_{63} = \xi L_{36},$$

$$L_{64} = L_{31},$$

$$L_{65} = \frac{\eta}{n} L_{56},$$

$$L_{66} = R\left[\left(k_2^2 - F\right)\left(k_1^2 - k_2^2\right)C_2 + \left(k_3^2 - F\right)\left(k_1^2 - k_3^2\right)C_3\right].$$

$$(2.38)$$

It is worth mentioning here that (2.32) have been used repeatedly in order to write the above entries in the simplest possible form. We will stress here that the above expression for the matrix exponential is a formal one. In the actual physical problem, the space is divided into two regions accordingly as $y \geq 0$ or $y \prec 0$. Inside the region $0 \leq y \leq \infty$, the positive exponential terms, not bounded at infinity, must be suppressed. Thus, for $y \geq 0$, we should replace each $\sinh(ky)$ by $-(1/2)\exp(-ky)$ and each $\cosh(ky)$ by $(1/2)\exp(-ky)$. In the region $y \leq 0$, the negative exponentials are suppressed instead.

We will now proceed to obtain the solution of the problem for the region $y \geq 0$. The solution for the other region is obtained by replacing each y by $-y$.

The formal solution of system (2.26) can be written in the form

$$\overline{v}(y,s) = \exp(A(s)y)\left[\overline{v}(0,s) + \int_0^y \exp(-A(s)z)B(z,s)dz\right]. \qquad (2.39)$$

Evaluating the integral in (2.39) using the integral properties of the Dirac delta function, we obtain

$$\overline{v}(y,s) = L(y,s)[\overline{v}(0,s) + H(s)], \qquad (2.40)$$

where

$$H(s) = \frac{-Q_0(1 + v_o s)}{2s}\begin{bmatrix} \dfrac{1}{2k_1} \\[2mm] \dfrac{G_r[k_1 k_2 k_3 + \xi(k_1 + k_2 + k_3)]}{2k_1 k_2 k_3 (k_1 + k_2)(k_1 + k_3)(k_2 + k_3)} \\[2mm] 0 \\[2mm] \dfrac{1}{2} \\[2mm] 0 \\[2mm] \dfrac{-\eta G_r}{2(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)} \end{bmatrix}. \qquad (2.41)$$

Equation (2.40) expresses the solution of the problem in the Laplace transform domain for $y \geq 0$ in terms of the vector $H(s)$ representing the applied heat source and the vector $\overline{v}(0, s)$ representing the conditions at the plate $y = 0$. To evaluate the components of this vector, we note first, due to the symmetry of the problem, that the velocity component and induced magnetic field component vanish at the plane source of heat, thus

$$h(0, t) = 0, \quad \text{or} \quad \overline{h}(0, s) = 0,$$

$$u(0, t) = 0, \quad \text{or} \quad \overline{u}(0, s) = 0. \tag{2.42}$$

Gauss's divergence theorem will now be used to obtain the thermal condition at the plane source. We consider a short cylinder of unit base, whose axis is perpendicular to the plane source of heat and whose bases lie on opposite sides of it. Taking the limit as the height of the cylinder tends to zero and noting that there is no heat flux through the lateral surface, we get

$$q(0, t) = \frac{Q_o}{2} H(t), \quad \text{or} \quad \overline{q}(0, s) = \frac{Q_o}{2s}. \tag{2.43}$$

We will use the generalized Fourier's law of heat conduction in the nondimensional form [31], namely,

$$q + v_o \frac{\partial q}{\partial t} = -\frac{\partial \theta}{\partial y}. \tag{2.44}$$

Taking the Laplace transform of both sides of this equation and using (2.43), we obtain

$$\overline{\theta}'(0, s) = \frac{-Q_o(1 + v_o s)}{2s}. \tag{2.45}$$

Equation (2.42) and (2.45) give three components of the $\overline{v}(0, s)$. To obtain the remaining three components, we substitute $y = 0$ on both sides of (2.40) getting a system of linear equations whose solution gives

$$\overline{\theta}(0, s) = \frac{-Q_o(1 + v_o s)}{2sk_1},$$

$$\overline{u}'(0, s) = \frac{wG_r Q_0(1 + v_o s)}{2sk_1(k_1 + k_2)(k_1 + k_3)(k_2 k_3 + \xi)},$$

$$\overline{h}'(0, s) = \frac{\eta G_r Q_o(1 + v_o s)}{2sk_1(k_1 + k_2)(k_1 + k_3)(k_2 k_3 + \xi)}. \tag{2.46}$$

Inserting the values from (2.42) and (2.46) into the right-hand side of (2.40) and performing the necessary matrix operations, we obtain

$$\overline{\theta}(y, s) = \frac{Q_o(1 + v_o s)}{2sk_1} e^{-k_1 y}, \tag{2.47}$$

$$\overline{u}(y, s) = \frac{-G_r Q_o(1 + v_o s)}{2s\beta\xi} \left[(k_2 - k_3)A_1 e^{-k_1 y} + (k_3 - k_1)A_2 e^{-k_2 y} + (k_1 - k_2)A_3 e^{-k_3 y} \right], \tag{2.48}$$

$$\overline{h}(y, s) = \frac{-G_r \eta Q_o(1 + v_o s)}{2s\beta} \left[(k_2 - k_3)(k_2 + k_3 - w)e^{-k_1 y} + (k_3 - k_1)(k_1 + k_3 - w)e^{-k_2 y} \right.$$
$$\left. + (k_1 - k_2)(k_1 + k_2 - w)e^{-k_3 y} \right], \tag{2.49}$$

where

$$w = k_1 k_2 k_3 + \xi(k_1 + k_2 + k_3),$$

$$\beta = k_1 \left(k_1^2 - k_2^2 \right) \left(k_3^2 - k_1^2 \right) (k_2 a_3 - k_3 a_2),$$

$$A_1 = w\xi(k_1 - k_2) - a_2 [a_3 - \xi k_3(k_1 + k_3)], \tag{2.50}$$

$$A_2 = -a_2 [a_3 - \xi k_3(k_1 + k_3)],$$

$$A_3 = -w\xi(k_2 - k_3) - a_2 [a_3 - \xi k_3(k_1 + k_3)],$$

$$a_2 = k_2^2 - \xi, \qquad a_3 = k_3^2 - \xi.$$

Also, substituting from (2.49) into (2.21) the induced electric field is given by

$$\overline{E}(y, s) = \frac{G_r \eta Q_o(1 + \tau_o s)}{2\beta} \left[(k_2 - k_3)(k_2 + k_3 - w)\frac{e^{-k_1 y}}{k_1} + (k_3 - k_1)(k_1 + k_3 - w)\frac{e^{-k_2\, y}}{k_2} \right.$$
$$\left. + (k_1 - k_2)(k_1 + k_2 - w)\frac{e^{-k_3 y}}{k_3} \right]. \tag{2.51}$$

Equations (2.47)–(2.51) determine completely the state of the fluid for $y \geq 0$. We mention in passing that these equations give also the solution to a semispace problem with a plane source of heat on its boundary that constitutes a rigid base. As mentioned before, the solution for the whole space when $y < 0$ is obtained from (2.47)–(2.51), by taking the symmetries under considerations.

We will show that the solution obtained above can be used as a set of building blocks from which the solutions to many interesting problems can be constructed. For future

reference, we will write down the solution to the problem in the case when the source of heat is located in the plane $y = c$, instead of the plane $y = 0$. In this case, we have

$$\overline{\theta}(y, s) = \frac{Q_o(1 + v_o s)}{2sk_1}e^{\pm k_1(y-c)},$$

(2.52)

$$\overline{u}(y, s) = \frac{-G_r Q_o(1 + v_o s)}{2s\beta\xi}\left[(k_2 - k_3)A_1 e^{\pm k_1(y-c)} + (k_3 - k_1)A_2 e^{\pm k_2(y-c)} + (k_1 - k_2)A_3 e^{\pm k_3(y-c)}\right],$$

(2.53)

$$\overline{h}(y, s) = \frac{G_r \eta Q_o(1 + v_o s)}{2s\beta}\left[(k_2 - k_3)(k_2 + k_3 - w)e^{\pm k_1(y-c)} + (k_3 - k_1)(k_1 + k_3 - w)e^{\pm k_2(y-c)}\right.$$

$$\left. + (k_1 - k_2)(k_1 + k_2 - w)e^{\pm k_3(y-c)}\right],$$

(2.54)

$$\overline{E}(y, s) = \frac{G_r \eta Q_o(1 + v_o s)}{2\beta}\left[(k_2 - k_3)(k_2 + k_3 - w)\frac{e^{\pm k_1(y-c)}}{k_1} + (k_3 - k_1)(k_1 + k_3 - w)\frac{e^{\pm k_2(y-c)}}{k_2}\right.$$

$$\left. + (k_1 - k_2)(k_1 + k_2 - w)\frac{e^{\pm k_3(y-c)}}{k_3}\right],$$

(2.55)

where the upper (plus) sign denotes the solution in the region $y \leq c$, while the lower (minus) sign denotes the solution in the region $y > c$.

3. Applications

We will now consider the problems of a semispace with a plane source of heat located inside the medium at the position $y = c$ and subject to the following boundary conditions.

(i) The shearing stress and the induced magnetic field are vanishing at the wall $(y = 0)$,

$$\frac{\partial u(0, t)}{\partial y} = 0, \quad \text{or} \quad \frac{\partial \overline{u}(0, s)}{\partial y} = 0,$$

$$h(0, t) = 0, \quad \text{or} \quad \overline{h}(0, s) = 0.$$

(3.1)

(ii) The temperature is kept at a constant value T_∞, which means that the temperature increment θ satisfies

$$\theta(0, t) = 0 \quad \text{or} \quad \overline{\theta}(0, s) = 0.$$

(3.2)

This problem can be solved in a manner analogous to the outlined above though the calculations become quite messy. We will instead use the reflection method together with

the solution obtained above for the whole space. This approach was proposed by Nowacki in the context of coupled thermoelasticity [31].

The boundary conditions of the problem can be satisfied by locating two heat sources in an infinite space, one positive at $y = c$ and the other negative of the same intensity at $y = -c$. The temperature increment θ is obtained as a superposition of the temperature for both plane distribution. Thus, $\theta = \theta_1 + \theta_2$, where θ_1 is the temperature due to the positive heat source, given by (2.52) and θ_2 is the temperature due to the negative heat source and is obtained from (2.52) by replacing c with $-c$, and noting that for all points of the semispace, we have $y + c > 0$. Thus, θ_2 is given by

$$\theta_2(y,s) = \frac{Q_o(1 + v_o s)}{2sk_1} e^{-k_1(y+c)}. \tag{3.3}$$

Combining (2.52) and (3.2), we obtain

$$\overline{\theta}(y,s) = \frac{Q_o(1 + v_o s)}{2sk_1} e^{-k_1 y} \sinh k_1 c \quad \text{for } y \geq c,$$

$$\overline{\theta}(y,s) = \frac{Q_o(1 + v_o s)}{2sk_1} e^{-k_1 c} \sinh k_1 y \quad \text{for } y < c. \tag{3.4}$$

Clearly, this distribution satisfies the boundary condition (3.2). We turn now to the problem of finding the distributions velocity, the induced magnetic field, and the induced electric field. Unfortunately, the above procedure of superposition cannot be applied to these fields as in the temperature fields. We define the scalar stream function ψ by the relation

$$u = \frac{\partial \psi}{\partial y}. \tag{3.5}$$

By integration (2.53) and using (3.5), we obtain the stream function due to the positive heat source at the position $y = c$ as

$$\psi = \frac{G_r Q_o(1 + v_o s)}{2s\beta\xi}\left[(k_2 - k_3)A_1\frac{e^{\pm k_1(y-c)}}{k_1} + (k_3 - k_1)A_2\frac{e^{\pm k_2(y-c)}}{k_2} + (k_1 - k_2)A_3\frac{e^{\pm k_3(y-c)}}{k_3}\right], \tag{3.6}$$

where the upper sign is valid for the region $0 \leq y < c$ and the lower sign is valid for the region $y \geq 0$. Similarly, the stream function for the negative heat source at $y = -c$ is given by

$$\psi = \frac{G_r Q_o(1 + v_o s)}{2s\beta\xi}\left[(k_2 - k_3)A_1\frac{e^{-k_1(y+c)}}{k_1} + (k_3 - k_1)A_2\frac{e^{-k_2(y+c)}}{k_2} + (k_1 - k_2)A_3\frac{e^{-k_3(y+c)}}{k_3}\right]. \tag{3.7}$$

Since ψ is a scalar field, we can use superposition to obtain the stream function for the semispace problem as

$$
\psi = \begin{cases}
\dfrac{G_r Q_o(1 + v_o s)}{s\beta\xi}\left[(k_2 - k_3)A_1\dfrac{e^{-k_1 y}\sinh k_1 c}{k_1} + (k_3 - k_1)A_2\dfrac{e^{-k_2 y}\sinh k_2 c}{k_2}\right. \\
\qquad\qquad \left. + (k_1 - k_2)A_3\dfrac{e^{-k_3 y}\sinh k_3 c}{k_3}\right] & \text{for } y \geq c, \\[2ex]
\dfrac{G_r Q_o(1 + v_o s)}{s\beta\xi}\left[(k_2 - k_3)A_1\dfrac{e^{-k_1 c}\sinh k_1 y}{k_1} + (k_3 - k_1)A_2\dfrac{e^{-k_2 c}\sinh k_2 y}{k_2}\right. \\
\qquad\qquad \left. + (k_1 - k_2)A_3\dfrac{e^{-k_3 c}\sinh k_3 y}{k_3}\right] & \text{for } y < c.
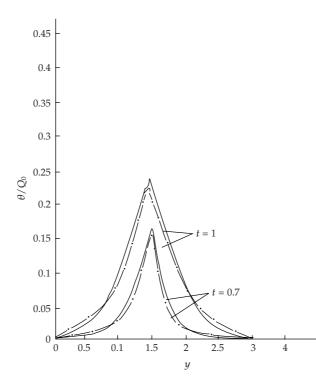\end{cases}
\tag{3.8}
$$

Using (3.8) and (3.5), we obtain the velocity distribution

$$
\overline{u} = \begin{cases}
\dfrac{-G_r Q_o(1 + v_o s)}{s\beta\xi}\left[(k_2 - k_3)A_1 e^{-k_1 y}\sinh k_1 c\right. \\
\qquad\qquad + (k_3 - k_1)A_2 e^{-k_2 y}\sinh k_2 c \\
\qquad\qquad \left. + (k_1 - k_2)A_3 e^{-k_3 y}\sinh k_3 c\right] & \text{for } y \geq c, \\[2ex]
\dfrac{G_r Q_o(1 + v_o s)}{s\beta\xi}\left[(k_2 - k_3)A_1 e^{-k_1 c}\cosh k_1 y\right. \\
\qquad\qquad + (k_3 - k_1)A_2 e^{-k_2 c}\cosh k_2 y \\
\qquad\qquad \left. + (k_1 - k_2)A_3 e^{-k_3 c}\cosh k_3 y\right] & \text{for } y < c.
\end{cases}
\tag{3.9}
$$

Differentiating (3.9) and using the resulting expressions together with (2.28), we obtain

$$
\overline{h} = \begin{cases}
\dfrac{G_r \eta Q_o(1 + v_o s)}{s\beta}\left[(k_2 - k_3)(k_2 + k_3 - w)e^{-k_1 y}\sinh k_1 c\right. \\
\qquad\qquad + (k_3 - k_1)(k_1 + k_3 - w)e^{-k_2 y}\sinh k_2 c \\
\qquad\qquad \left. + (k_1 - k_2)(k_1 + k_2 - w)e^{-k_3 y}\sinh k_3 c\right] & \text{for } y \geq c, \\[2ex]
\dfrac{G_r \eta Q_o(1 + v_o s)}{s\beta}\left[(k_2 - k_3)(k_2 + k_3 - w)\, e^{-k_1 c}\sinh k_1 y\right. \\
\qquad\qquad + (k_3 - k_1)(k_1 + k_3 - w)\, e^{-k_2 c}\sinh k_2 y \\
\qquad\qquad \left. + (k_1 - k_2)(k_1 + k_2 - w)\, e^{-k_3 c}\sinh k_3 y\right] & \text{for } y < c.
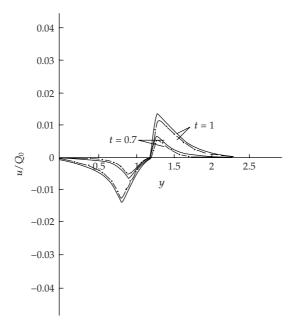\end{cases}
\tag{3.10}
$$

Figure 1: Variation of θ with y, (−) for $v_0 = 0.03$, (−·) for $v_0 = 0.4$, and $t = 0.7, 1$ at $P_r = 7$, $G_r = 4$, $K = 1.2$, $\alpha = 0.3$, and $c = 2$ representing onset stationary convection.

Substituting (3.10) and (2.21), we can get

$$
\overline{E} = \begin{cases}
\dfrac{G_r\eta Q_o(1+v_os)}{\beta}\bigg[(k_2-k_3)(k_2+k_3-w)e^{-k_1y}\dfrac{\sinh k_1c}{k_1} \\
\qquad + (k_3-k_1)(k_1+k_3-w)e^{-k_2y}\,\dfrac{\sinh k_2c}{k_2} \\
\qquad + (k_1-k_2)(k_1+k_2-w)e^{-k_3y}\dfrac{\sinh k_3c}{k_3}\bigg] \qquad \text{for } y \geq c, \\[2ex]
\dfrac{-G_r\eta\, Q_o(1+v_os)}{\beta}\bigg[(k_2-k_3)(k_2+k_3-w)e^{-k_1c}\dfrac{\cosh k_1y}{k_1} \\
\qquad + (k_3-k_1)(k_1+k_3-w)e^{-k_2c}\dfrac{\cosh k_2y}{k_2} \\
\qquad + (k_1-k_2)(k_1+k_2-w)e^{-k_3c}\dfrac{\cosh k_3y}{k_3}\bigg] \quad \text{for } y < c.
\end{cases} \tag{3.11}
$$

Clearly, $\partial\overline{u}(0,s)/\partial y = \overline{h}(0,s) = 0$ in agreement with (3.1).

Figure 2: Variation of u with y, $(-)$ for $v_0 = 0.03$, $(-\cdot)$ for $v_0 = 0.4$, and $t = 0.7, 1$ at $P_r = 7, G_r = 4, K = 1.2$, $\alpha = 0.3$, and $c = 2$ representing onset stationary convection.

4. Inversion of the Laplace Transform

In order to invert the Laplace transforms in the above equations, we will use a numerical technique based on Fourier expansions of functions.

Let $\overline{g}(s)$ be the Laplace transform of a given function $g(t)$. The inversion formula of Laplace transforms states that

$$g(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st}\overline{g}(s)ds, \tag{4.1}$$

where d is an arbitrary positive constant greater than all the real parts of the singularities of $\overline{g}(s)$. Taking $s = d + iy$, we get

$$g(t) = \frac{e^{dt}}{2\pi} \int_{-\infty}^{\infty} e^{ity}\overline{g}(d + iy)dy. \tag{4.2}$$

This integral can be approximated by

$$g(t) = \frac{e^{dt}}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikt\Delta y}\overline{g}(d + ik\Delta y)\Delta y. \tag{4.3}$$

Figure 3: Variation of h with y, (−) for $v_0 = 0.03$, (−·) for $v_0 = 0.4$, and $t = 0.7, 1$ at $P_r = 7$, $G_r = 4$, $K = 1.2$, $\alpha = 0.3$, and $c = 2$ representing onset stationary convection.

Taking $\Delta y = \pi/t_1$, we obtain

$$g(t) = \frac{e^{dt}}{t_1}\left[\frac{1}{2}\overline{g}(d) + \mathrm{Re}\left(\sum_{k=1}^{\infty} e^{ik\pi t/t_1}\overline{g}\left(d + \frac{ik\pi}{t_1}\right)\right)\right]. \tag{4.4}$$

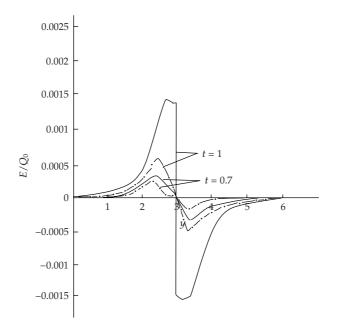For numerical purposes, this is approximated by the function

$$g_N(t) = \frac{e^{dt}}{t_1}\left[\frac{1}{2}\overline{g}(d) + \mathrm{Re}\left(\sum_{k=1}^{N} e^{ik\pi t/t_1}\overline{g}\left(d + \frac{ik\pi}{t_1}\right)\right)\right], \tag{4.5}$$

where N is a sufficiently large integer chosen such that

$$\frac{e^{dt}}{t_1}\mathrm{Re}\left[e^{iN\pi(t/t_1)}\overline{g}\left(d + \frac{iN\pi}{t_1}\right)\right] \prec \eta, \tag{4.6}$$

where η is a preselected small positive number that corresponds to the degree of accuracy to be achieved, Formula (3.11) is the numerical inversion formula valid for $0 \leq t \leq 2t_1$ [22]. In particular, we choose $t = t_1$, getting

$$g_N(t) = \frac{e^{dt}}{t}\left[\frac{1}{2}\overline{g}(d) + \mathrm{Re}\left(\sum_{k=1}^{N} (-1)^k \overline{g}\left(d + \frac{ik\pi}{t}\right)\right)\right]. \tag{4.7}$$

Figure 4: Variation of E with y, $(-)$ for $v_0 = 0.03$, $(-\cdot)$ for $v_0 = 0.4$, and $t = 0.7, 1$ at $P_r = 7$, $G_r = 4$, $K = 1.2$, $\alpha = 0.3$, and $c = 2$ representing onset stationary convection.
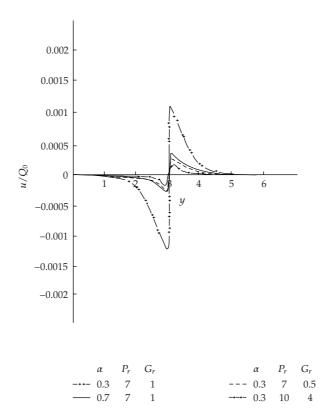
5. Numerical Results

The constants of the problem were taken as $\varepsilon_o = 0.003$, $\alpha = 0.3$, $G_r = 4$, $K = 1.2$, and $c = 2$. All constants are given in SI units. The computations were carried out for the two different values of time, namely, $t = 0.7$ and 1. The functions θ, u, h, and E are evaluated. The results are shown in Figures 1, 2, 3, and 4. In these figures, solid lines represent the solution corresponding to using the generalized Fourier equation of heat conduction ($v_o = 0.4$), while dashed lines represent the solution corresponding to using the classical Fourier heat equation ($v_o = 0.03$).

The important phenomenon observed in all computations is that the solution of any of the considered functions vanishes identically outside a bounded region of space surrounding the heat source at a distance from it equal to $x^*(t)$, and say that $x^*(t)$ is a particular value of y depending only on the choice of t and is the location of the wave front. This demonstrates clearly the difference between the solution corresponding to using classical Fourier heat equation ($v_o = 0.03$) and to using the non-Fourier case ($v_o = 0.4$). In the first and older theory, the waves propagate with infinite speeds, so the value of any of the functions is not identically zero (though it may be very small) for any large value of y. In the non-Fourier theory, the response to the thermal and mechanical effects does not reach infinity instantaneously but remains in a bounded region of space given by $0 \prec y \prec y^*(t)$ for the semispace problem and by $\text{Min}(0, y^*(t) - c) \prec y \prec y + y^*(t)$ for the whole space problem.

We notice that results for all functions considered in the semispace problem when the relaxation time is appeared in heat equation are distinctly different from those when the relaxation time disappeared.

We also notice that for small values of time, the solution is localized in a finite region near the plane of heat sources. This region grows with increasing time until it fills the whole boundary-layer region.

	α	P_r	G_r		α	P_r	G_r
−·+·−	0.3	7	1	− − −	0.3	7	0.5
——	0.7	7	1	−·+·−	0.3	10	4

Figure 5: Velocity distribution u with y, for different values of α, P_r, and G_r, with $t = 0.7$, $v_0 = 0.4$, $K = 1$, and $c = 2$ representing onset stationary convection.

Also, we observe from Figure 5. that the effect of heating by free convention currents when $G_r > 0$. In this case, it is noticed that as Alfven velocity α increases, the velocity is found to decrease. This is mainly due to the fact that the effect of the magnetic field corresponds to a term signifying a positive force that tends to decelerate the fluid particles. Also, it is noticed that the velocity increases as G_r increases, while it decreases when the Prandtl number p_r increases.

We notice that results for the temperature distribution when the relaxation time is appears in the heat equation are distinctly different from those when the relaxation time is not mentioned in the heat equation. This is due to the fact that thermal waves in the Fourier theory of heat equation travel with an infinite speed of propagation as opposed to finite speed in the non-Fourier case. We also notice that for small values of time, the solution is localized in a finite region near the plane surface. This region grows with increasing time until it fills the whole boundary-layer region. As time t increases, results for both theories of heat equation almost coincide which is expected, since magnetohydrodynamic free convection flow effects are short lived. At all values of time, the velocity distributions for both theories coincide.

The effect of the relaxation time on the variation temperature and velocity distribution for this problem is shown in Figures 1 and 2, respectively.

The values of Grashof number G_r have been chosen as they are interesting from physical point of view.

6. Concluding Remarks

Many metallic materials are manufactured after they have been redefined sufficiently in the molten state. Therefore, it is a central problem in metallurgical chemistry to study the free convection effects on conducting fluid metal. For instance, liquid sodium Na (100°C) exhibit very small electrical resistivity.

The effects of Grashof number, Alfven velocity, Prandtl number, and relaxation time on the oneset temperature and velocity distribution are discussed. A discussion is provided for the effects of heating on a viscous conducting fluid is given.

The importance of state space analysis is recognized in fields where the time behavior of any physical process is of interest.

The state-space approach is more general than the classical Laplace and Fourier transform techniques. Consequently, state space is applicable to all systems that can be analyzed by integral transforms in time and is applicable to many systems for which transform theory breaks down [32].

Owing to the complicated nature of the governing equations for the unsteady magnetohydrodynamic with gradient pressure flow, few attempts have been made to solve problems in this field. These attempts utilized approximate methods valid for only a specific range of some parameters.

In this work, the method of direct integration by means of the matrix exponential, which is a standard approach in modern control theory and developed in detail in many texts such as Ogata [33], and Ezzat et al. [34, 35], is introduced in the field of magnetohydrodynamics and applied to two specific problems in which the temperature and velocity are coupled. This method gives exact solutions in the Laplace transform domain without any assumed restrictions on either the applied magnetic field or the velocity, temperature distributions, and viscoelastic parameter.

The method used in the present work is applicable to a wide range of problems. It can be applied to problems which are described by the linearized Navier-Stokes equations. The same approach was used quite successfully in dealing with problems in thermoelasticity theory [14, 15, 19, 34, 35].

Nomenclature

t: Time
x, y: Coordinates system
v: $=(u, 0, 0)$ velocity vector of the fluid
T: Temperature distribution
T_o: Temperature of the plate
T_∞: Temperature of the fluid away from the plate
p_r: Prandtl number
G_r: Grashof number
c_p: Specific heat at constant pressure
g: Acceleration due to gravity
ρ: Density
λ: Thermal diffusivity
h: Induced magnetic field
Q_o: Constant
Q: The strength of the applied heat source

E: Electric field
c: Spead of light
F: Magnetic viscosity
J: Electric current density
μ_o: Magnetic permeability
β: Coefficient of volume expansion
ε_o: Electric permeability
H_o: Magnetic intensity vector
υ_0:   Relaxation time
$\delta(y)$: Dirac delta function
$H(t)$: Heaviside unit step function
K: Permeability of the porous medium
α: Alfven velocity
σ_o: Electric conductivity.

Acknowledgment

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