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Research Article

Conforming Finite Element Approximations for a Fourth-Order Steklov Eigenvalue Problem

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This paper characterizes the spectrum of a fourth-order Steklov eigenvalue problem by using the spectral theory of completely continuous operator. The conforming finite element approximation for this problem is analyzed, and the error estimate is given. Finally, the bounds for Steklov eigenvalues on the square domain are provided by Bogner-Fox-Schmit element and Morley element.

1. Introduction

Steklov eigenvalue problems, in which the eigenvalue parameter appears in the boundary condition, have several deep applications both in maths and physics. For instance, they are found in the study of surface waves (see [1]), the analysis of stability of mechanical oscillators immersed in a viscous fluid (see [2]), and the study of the vibration modes of a structure in contact with an incompressible fluid (see, e.g., [3]), and the first eigenvalue also plays a crucial role in the positivity preserving property for the biharmonic operator Δ^2 under the boundary conditions $u = \Delta u - \lambda u_\nu = 0$ on $\partial\Omega$ (see [4]) and so forth.

Thus, numerical methods for approximate Steklov eigenvalues become a concerned problem by mathematics and engineering community. Many scholars have investigated the finite element methods for second-order Steklov eigenvalue problem and achieved many results; for example, see [5–12] and so on.

However, for fourth-order Steklov eigenvalue problems the existing references are mostly qualitative analysis: [13] studied the bound for the first eigenvalue on the square and proved that the first eigenvalue is simple and its eigenfunction does not change sign, [14] discussed the smallest nonzero Steklov eigenvalue by the method of a posteriori-a priori inequalities, [15, 16] studied the spectrum of a fourth-order Steklov eigenvalue problem on a bounded domain in $\mathbb{R}^n$ and gave the explicit form of the spectrum in the case where the domain is a ball, and [17] provided bounds for the first non-zero Steklov eigenvalues when Ω is isometric to an n-dimensional Euclidean ball. More recently, [18] proved the existence of an optimal convex shape among domains of given measure, and [19] by a new method established Weyl-type asymptotic formula for the counting function of the biharmonic Steklov eigenvalues. But as for the finite element methods for fourth-order Steklov eigenvalue problem, to the best of our knowledge, there are no reports.

This paper discusses conforming finite element approximations for a fourth-order Steklov eigenvalue problem, and the main work is as follows.

(1) We define the operator $T : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow H^2(\Omega) \cap H_0^1(\Omega)$ and prove that T is completely continuous thus characterize the spectrum of a fourth-order Steklov eigenvalue problem by the spectrum of T. Note that [16] analyzed the spectrum of this problem by introducing an orthogonal decomposition, $H^2(\Omega) \cap H_0^1(\Omega) = W \bigoplus H_0^2(\Omega)$, and conducted the research on the space W. Compared with the argument used in [16], our approach is more direct and lays the foundation for further discussion of the finite element approximation.

(2) We study for the first time conforming finite element approximations for a fourth-order Steklov eigenvalue problem by using the spectral approximation theory (e.g., see [20, 21]) and prove a priori error estimates of finite element eigenvalues and eigenfunctions. Hence, in principle, we can compute approximate eigenvalues and eigenfunctions of a fourth-order Steklov eigenvalue problem on any bounded domain by finite element methods. As an example, we compute the approximate eigenvalues by the conforming Bogner-Fox-Schmit element, and numerical results indicate that the numerical eigenvalues of Bogner-Fox-Schmit element approximate the exact eigenvalues from above. We also compute the nonconforming Morley element eigenvalues, and the numerical results show that the Morley element eigenvalues approximate the exact eigenvalues from below. Thus, we provide bounds for the exact Steklov eigenvalues on a square, which are more precise than those given in [16].

The rest of this paper is organized as follows. In the next section, the spectrum of the fourth-order Steklov eigenvalue problem is characterized by using the spectral theory of completely continuous operator. In Section 3, the error estimate of conforming finite element approximation for the fourth-order Steklov eigenvalue problem is proved. Numerical experiments of the Bogner-Fox-Schmit element and Morley element are presented in Section 4 to give bounds for Steklov eigenvalues on the square.

Let $W^{s,q}(\Omega)$ denote the usual Sobolev space with real-order s with norm $\|\cdot\|_{s,q}$. For simplicity, we write $H^s(\Omega)$ for $W^{s,2}(\Omega)$ with norm $\|\cdot\|_s$ and $H^s(\partial\Omega)$ for $W^{s,2}(\partial\Omega)$ with norm $\|\cdot\|_{s,\partial\Omega}$. $H^0(\Omega) = L_2(\Omega)$, $H^0(\partial\Omega) = L_2(\partial\Omega)$. Throughout this paper, C denotes a generic positive constant independent of h, which may not be the same at each occurrence.

2. The Spectrum of the Fourth-Order Steklov Eigenvalue Problem

We consider the fourth-order Steklov eigenvalue problem

$$\Delta^2 u = 0, \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{2.2}$$

$$\Delta u = \lambda u_\nu, \quad \text{on } \partial\Omega, \tag{2.3}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain and $\partial\Omega$ is smooth, or $\Omega \subset \mathbb{R}^2$ is a convex domain, u_ν denotes the outer normal derivative of u on $\partial\Omega$.

Denote $V = H^2(\Omega) \cap H_0^1(\Omega)$.

The weak form of (2.1)–(2.3) is given by the following. Find $\lambda \in \mathbb{R}$, $0 \neq u \in V$, such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in V, \tag{2.4}$$

where

$$a(u,v) = \int_\Omega \Delta u \Delta v dx, \qquad b(u,v) = \int_{\partial\Omega} u_\nu v_\nu ds. \tag{2.5}$$

Lemma 2.1. *Assume that Ω is a Lipschitz bounded domain which satisfies the uniform outer ball condition. Then the space V becomes a Hilbert space when endowed with the scalar product $a(u,v) = \int_\Omega \Delta u \Delta v dx$, $\forall u,v \in V$, and $\|u\|_a = \sqrt{a(u,u)}$ is equivalent to the norm $\|\cdot\|_2$ induced by $H^2(\Omega)$.*

Proof. See Lemma 1 and its proof in [15]. □

It is obvious that the condition of Lemma 2.1 holds for $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in \mathbb{C}^2$ or a convex domain $\Omega \subset \mathbb{R}^2$.

The source problem associated with (2.4) is as follows. Find $u \in V$, such that

$$a(u,v) = b(f,v), \quad \forall v \in V. \tag{2.6}$$

It follows from Lemma 2.1 that $a(\cdot,\cdot)$ is a symmetric, continuous, and V-elliptic bilinear form on V. By Schwarz inequality and trace theorem we have

$$|b(f,v)| = \left|\int_{\partial\Omega} f_\nu v_\nu ds\right| \leq \|f_\nu\|_{0,\partial\Omega}\|v_\nu\|_{0,\partial\Omega} \leq \|f_\nu\|_{0,\partial\Omega}\|v\|_2, \quad \forall v \in V. \tag{2.7}$$

Hence, from Lax-Milgram Theorem we know that (2.6) has one and only one solution.

Therefore, according to the source problem (2.6) we define the operator $T : V \to V$ by

$$a(Tf,v) = b(f,v), \quad \forall v \in V. \tag{2.8}$$

From [20], we know that (2.4) has the equivalent operator form:

$$Tu = \frac{1}{\lambda}u. \tag{2.9}$$

Lemma 2.2. *The operator $T : V \to V$ is self-adjoint and completely continuous.*

Proof. By the definition of T, for any $u,v \in V$, there holds

$$a(Tu,v) = b(u,v) = b(v,u) = a(Tv,u) = a(u,Tv), \tag{2.10}$$

that is, T is self-adjoint with respect to $a(\cdot,\cdot)$.

Next we will prove that T is completely continuous. By Schwarz inequality and trace theorem, for any $f \in V$ we have

$$a(Tf,Tf) = b(f,Tf) \le \|f_\nu\|_{0,\partial\Omega}\|(Tf)_\nu\|_{0,\partial\Omega} \le C\|f_\nu\|_{0,\partial\Omega}\|Tf\|_2; \tag{2.11}$$

it follows from the fact that $a(\cdot,\cdot)$ is V-elliptic that

$$\|Tf\|_2 \le C\|f_\nu\|_{0,\partial\Omega}. \tag{2.12}$$

Let $\{f_m\}$ be a bounded sequence in $\|\cdot\|_2$; then by trace theorem we have $\{(f_m)_\nu\}$ is a bounded sequence in $\|\cdot\|_{1/2,\partial\Omega}$. And, by compact embedding $H^{1/2}(\partial\Omega) \subset L_2(\partial\Omega)$, we know that there is a subsequence $\{(f_{m_l})_\nu\}$ that is a Cauchy sequence in $\|\cdot\|_{0,\partial\Omega}$. From (2.12) we conclude that $\{Tf_{m_l}\}$ is a Cauchy sequence in $\|\cdot\|_2$, which implies that T is completely continuous. □

Let (λ, u) be an eigenpair of (2.4); then $\|u\|_a \ne 0$. Since $a(u,u) = \lambda b(u,u)$, we see that $\|u_\nu\|_{0,\partial\Omega} \ne 0$, $\lambda \ne 0$. And, by trace theorem we have

$$\|u_\nu\|_{0,\partial\Omega} \le C\|u\|_2 \le C\|u\|_a; \tag{2.13}$$

thus,

$$\lambda = \frac{a(u,u)}{b(u,u)} = \frac{\|u\|_a^2}{\|u_\nu\|_{0,\partial\Omega}^2} \ge \frac{1}{C^2} > 0. \tag{2.14}$$

Therefore, from the spectral theory of completely continuous operator we know that all eigenvalues of T are real and have finite algebraic multiplicity. We arrange the eigenvalues of T by increasing order:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \nearrow +\infty. \tag{2.15}$$

The eigenfunctions corresponding to two arbitrary different eigenvalues of T must be orthogonal. And there must exist a standard orthogonal basis with respect to $\|\cdot\|_a$ in eigenspace corresponding to the same eigenvalue. Hence, we can construct a complete orthonormal system of V by using the eigenfunctions of T corresponding to $\{\lambda_j\}$:

$$u_1, u_2, u_3, \ldots. \tag{2.16}$$

Remark 2.3. Reference [16] first discussed the property of the spectrum and obtained the above results by compact embedding $H^{1/2}(\partial\Omega) \subset L_2(\partial\Omega)$. But [16] conducted the study in such a space W: $H^2(\Omega) \cap H_0^1(\Omega) = W \bigoplus H_0^2(\Omega)$, while we define the operator T on the space $V = H^2(\Omega) \cap H_0^1(\Omega)$ directly. Our method is convenient for constructing conforming finite element space $V_h \subset V$ and analyzing the finite element error estimates.

3. Finite Element Method and Its Error Estimates

Let $V_h \subset V$ be a conforming finite element space; for example, V_h is the finite element space associated with one of the Argyris element, Bell element, and Bogner-Fox-Schmit element (see [22]).

The conforming finite element approximation of (2.4) is given by the following. Find $\lambda_h \in \mathbb{R}$, $0 \neq u_h \in V_h$, such that

$$a(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in V_h. \tag{3.1}$$

The source problem associated with (3.1) is as follows. Find $u_h \in V_h$, such that

$$a(u_h, v) = b(f, v), \quad \forall v \in V_h. \tag{3.2}$$

Likewise, from Lax-Milgram theorem we know that (3.2) has a unique solution.

Thus, we can define the operator $T_h : V \to V_h$ by

$$a(T_h f, v) = b(f, v), \quad \forall v \in V_h. \tag{3.3}$$

From [20], we know that (3.1) has the equivalent operator form:

$$T_h u_h = \frac{1}{\lambda_h} u_h. \tag{3.4}$$

Let $P_h : V \to V_h$ be Ritz projection operator; then

$$a(u - P_h u, v) = 0, \quad \forall v \in V_h. \tag{3.5}$$

Combining (2.8) with (3.3), we deduce that, for any $u \in V, v \in V_h$, there holds

$$a(P_h T u - T_h u, v) = a(P_h T u - T u, v) + a(T u - T_h u, v) = 0, \tag{3.6}$$

hence,

$$\|T_h u - P_h T u\|_a = 0, \quad \forall u \in V; \tag{3.7}$$

thus we get $T_h = P_h T$. It is clear that

$$T_h|_{V_h} : V_h \to V_h \tag{3.8}$$

is a self-adjoint finite rank operator with respect to the inner product $a(\cdot, \cdot)$, and the eigenvalues of (3.1) can be arranged as

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \lambda_{3,h} \leq \cdots \lambda_{N_h,h} \quad (N_h = \dim V_h). \tag{3.9}$$

As for the regularity of source problem (2.6) it has been reported in the literatures; for example, see [4]. Here, we prove the following regular estimates which will be used in the sequel.

Lemma 3.1. *If* $\Omega \subset \mathbb{R}^n (n \geq 2)$ *with* $\partial\Omega \in \mathbb{C}^r$ $(r \geq 3)$, $f_\nu \in H^{r-(5/2)}(\partial\Omega)$, *and* Tf *is the solution of* (2.6), *then* $Tf \in H^r(\Omega)$ *and*

$$\|Tf\|_r \leq C_p \|f_\nu\|_{r-(5/2),\partial\Omega}. \tag{3.10}$$

Proof. Let $u = Tf$, $\Delta u = v$; then the boundary value problem (2.6) is transformed to

$$\Delta v = 0, \quad \text{in } \Omega, \tag{3.11}$$

$$v = f_\nu, \quad \text{on } \partial\Omega, \tag{3.12}$$

$$\Delta u = v, \quad \text{in } \Omega, \tag{3.13}$$

$$u = 0, \quad \text{on } \partial\Omega. \tag{3.14}$$

Note that (3.11)-(3.12) and (3.13)-(3.14) are two second-order problems. From [23, 24], we know that, when $f_\nu \in H^{r-(5/2)}(\partial\Omega)$, there exists a weak solution $v \in H^{r-2}(\Omega)$ to (3.11)-(3.12) and

$$\|v\|_{r-2} \leq C_p \|f_\nu\|_{r-(5/2),\partial\Omega}. \tag{3.15}$$

From [25] we have that, when $\partial\Omega \in \mathbb{C}^r$, there exists a weak solution $u \in H^r(\Omega)$ to (3.13)-(3.14) and

$$\|u\|_r \leq C_p \|v\|_{r-2}. \tag{3.16}$$

By combining the above two inequalities we obtain $Tf \in H^r(\Omega)$ and (3.10). □

In this paper, C_p denotes the prior constant dependent on the equation and Ω and independent of the right-hand side of the equation and h. Clearly, constants C_p that appeared in Lemma 3.1 are not the same.

Lemma 3.2. *Let* (λ, u) *and* (λ_h, u_h) *be the* k*th eigenpair of* (2.4) *and* (3.1), *respectively. Then,*

$$\lambda_h - \lambda = \frac{\|u_h - u\|_a^2}{\|(u_h)_\nu\|_{0,\partial\Omega}^2} - \lambda \frac{\|(u_h - u)_\nu\|_{0,\partial\Omega}^2}{\|(u_h)_\nu\|_{0,\partial\Omega}^2}. \tag{3.17}$$

Proof. See [20, 26]. □

Denote

$$\eta_h = \sup_{f \in V, \|f\|_a = 1} \inf_{v \in V_h} \|Tf - v\|_a. \tag{3.18}$$

Theorem 3.3. *Suppose that $\eta_h \to 0(h \to 0)$; then there holds*

$$\|T - T_h\|_a \longrightarrow 0 \quad (h \longrightarrow 0). \tag{3.19}$$

Proof. By the definition of operator norm we have

$$\begin{aligned}\|T - T_h\|_a &= \sup_{f\in V, \|f\|_a=1} \|(T - T_h)f\|_a = \sup_{f\in V, \|f\|_a=1} \|Tf - P_hTf\|_a \\ &= \sup_{f\in V, \|f\|_a=1} \inf_{v\in V_h} \|Tf - v\|_a = \eta_h \longrightarrow 0 \quad (h \longrightarrow 0).\end{aligned} \tag{3.20}$$

□

Remark 3.4. It is satisfied naturally in conforming finite elements that $\eta_h \to 0$ $(h \to 0)$; which is not a restriction. Since T_h is a finite rank operator, it follows from the operator theory that the limit T of T_h must be completely continuous. Thus, we have provided another proof that $T : V \to V$ is completely continuous.

Let $M(\lambda)$ denote the eigenfunctions space of (2.4) corresponding to the eigenvalue λ.

Theorem 3.5. *Suppose that $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with $\partial\Omega \in \mathbb{C}^r (r \geq 3)$. Then, $M(\lambda) \subset H^r(\Omega)$.*

Proof. Let $u \in M(\lambda)$; then $(\lambda, u) \in \mathbb{R} \times V$ satisfying (2.4). In (2.4) let $\lambda u = f$; then $f_\nu = (\lambda u)_\nu \in H^{1/2}(\partial\Omega)$. Therefore, from Lemma 3.1 we know that $u \in H^3(\Omega)$. And $u \in H^3(\Omega)$ leads to $f_\nu = (\lambda u)_\nu \in H^{1+1/2}(\partial\Omega)$; again from Lemma 3.1 it follows that $u \in H^4(\Omega)$. By using Lemma 3.1 repeatedly we deduce $u \in H^r(\Omega)$. Thus, $M(\lambda) \subset H^r(\Omega)$. □

Theorem 3.6. *Suppose that $\partial\Omega \in \mathbb{C}^r (r \geq 3)$, and $V_h \subset V$ is a piecewise m-degree finite element space. Let (λ_h, u_h) be the kth conforming finite element eigenpair of* (3.1) *and λ the kth eigenvalue of* (2.4). *Then, there exists $u \in M(\lambda)$ such that*

$$\|u - u_h\|_a \leq Ch^t, \tag{3.21}$$

$$|\lambda_h - \lambda| \leq Ch^{2t}, \tag{3.22}$$

where $t = \min\{r, m + 1\} - 2$.

Proof. By the interpolation error estimate of m-degree finite element and Lemma 3.1, we have

$$\begin{aligned}\eta_h &= \sup_{f\in V, \|f\|_a=1} \inf_{v\in V_h} \|Tf - v\|_a \leq \sup_{f\in V, \|f\|_a=1} Ch\|Tf\|_3 \\ &\leq \sup_{f\in V, \|f\|_a=1} Ch\|f_\nu\|_{1/2,\partial\Omega} \leq \sup_{f\in V, \|f\|_a=1} Ch\|f\|_a \\ &\leq Ch \longrightarrow 0 \quad (h \longrightarrow 0).\end{aligned} \tag{3.23}$$

Then, from Theorem 3.3 we know that $\|T - T_h\|_a \to 0$ $(h \to 0)$. Thus, according to Theorem 7.4 in [20] we have

$$\|u - u_h\|_a \le C\left\|(T - T_h)|_{M(\lambda)}\right\|_a. \tag{3.24}$$

From Theorem 3.5 we have $M(\lambda) \subset H^r(\Omega)$. Therefore, for any $u \in M(\lambda)$, $\|u\|_a = 1$, we deduce that

$$\begin{gathered}\|(T - T_h)u\|_a = \|Tu - P_h Tu\|_a = \frac{1}{\lambda}\|u - P_h u\|_a \le Ch^t\|u\|_{t+2},\\ \left\|(T - T_h)|_{M(\lambda)}\right\|_a = \sup_{u\in M(\lambda),\|u\|_a=1}\|(T - T_h)u\|_a;\end{gathered} \tag{3.25}$$

combining the above two relations with (3.24), we get the desired result (3.21).

By Lemma 3.2, we get

$$|\lambda_h - \lambda| \le C\frac{\|u_h - u\|_a^2}{\|(u_h)_\nu\|_{0,\partial\Omega}^2}, \tag{3.26}$$

which together with (3.21) yields (3.22). □

Corollary 3.7. *Suppose that $\partial\Omega \in \mathbb{C}^6$. Let (λ_h, u_h) be the kth eigenpair of the Argyris element. Then, there exists the kth eigenpair (λ, u) of* (2.4) *such that*

$$\begin{gathered}\|u - u_h\|_a \le Ch^4,\\ |\lambda_h - \lambda| \le Ch^8.\end{gathered} \tag{3.27}$$

Proof. Since the Argyris element contains the complete polynomials of degree ≤ 5, that is, $m = 5$. From the assumption $r = 6$, we have $t = 4$. Then, by Theorem 3.6 we get the desired results. □

Corollary 3.8. *Suppose that $\partial\Omega \in \mathbb{C}^5$. Let (λ_h, u_h) be the kth eigenpair of the Bell element. Then, there exists the kth eigenpair (λ, u) of* (2.4) *such that*

$$\begin{gathered}\|u - u_h\|_a \le Ch^3,\\ |\lambda_h - \lambda| \le Ch^6.\end{gathered} \tag{3.28}$$

Proof. Since the Bell element contains the complete polynomials of degree ≤ 4, that is, $m = 4$. From the assumption $r = 5$, we have $t = 3$. Then, by Theorem 3.6 we get the desired results. □

Corollary 3.9. *Suppose that $\partial\Omega \in \mathbb{C}^4$. Let (λ_h, u_h) be the kth eigenpair of Bogner-Fox-Schmit element. Then, there exists the kth eigenpair (λ, u) of* (2.4) *such that*

$$\|u - u_h\|_a \leq Ch^2,$$
$$|\lambda_h - \lambda| \leq Ch^4. \tag{3.29}$$

Proof. Applying Theorem 3.6 with $m = 3$ and noting that $r = 4$ from the assumption and $t = 2$, we complete the proof immediately. □

Remark 3.10. Next we will discuss, when $\Omega \subset \mathbb{R}^2$ is convex, the regularities of the boundary value problem (2.6) and the eigenvalue problem (2.4) and the error estimates of finite element approximations.

To complete the discussion we need the following regular estimate. Suppose that $u \in W^{3,q}(\Omega) \cap H_0^1(\Omega)$; then

$$\|u\|_{3,q} \leq C_p \|\Delta u\|_1, \tag{3.30}$$

where $q < 2/(3 - \pi/\omega)$ while q can be arbitrarily close to $2/(3 - \pi/\omega)$, and ω is the largest inner angle of Ω.

Reference [27] gave this estimate (see (1.2.9) in [27]) and used it as a fundamental result. Although we have not seen the proof of it, we believe that this estimate is correct.

Suppose that $\Omega \subset \mathbb{R}^2$ is convex, (3.30) holds, and $f_\nu \in H^{1/2}(\partial\Omega)$. Let $u = Tf$, $\Delta u = v$; then the boundary value problem (2.6) is transformed to (3.11)-(3.12) and (3.13)-(3.14). From [23, 24], we know that there exists a weak solution $v \in H^1(\Omega)$ to (3.11)-(3.12) and

$$\|v\|_1 \leq C_p \|f_\nu\|_{1/2,\partial\Omega}. \tag{3.31}$$

Reference [28] proved that there exists a weak solution $u \in W^{3,q}(\Omega)$ to (3.13)-(3.14); and from (3.30) we get

$$\|u\|_{3,q} \leq C_p \|v\|_1. \tag{3.32}$$

By combining the above two inequalities we obtain $Tf \in W^{3,q}(\Omega)$ and the following regular estimate:

$$\|Tf\|_{3,q} \leq C_p \|f_\nu\|_{1/2,\partial\Omega}. \tag{3.33}$$

From the fact that the weak solution, to the boundary value problem, $Tf \in W^{3,q}(\Omega)$, it is easy to know that $M(\lambda) \subset W^{3,q}(\Omega)$, namely, $M(\lambda) \subset H^{s+2}(\Omega)$, where $s < (\pi/\omega) - 1$ while s can be arbitrarily close to $\pi/\omega - 1$.

When Ω is a rectangle, it can be deduced that $M(\lambda) \subset H^{s+2}(\Omega)$, where $s < 1$ while s can be arbitrarily close to 1.

Similar to Theorem 3.6 and Corollaries 3.7–3.9, by using (3.33) we can prove the following error estimates. Let (λ_h, u_h) be the kth eigenpair of the Argyris element, Bell

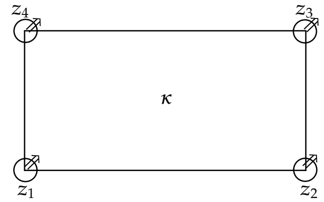

Figure 1

element, or Bogner-Schmit element. Then, there exists the kth eigenpair (λ, u) of (2.4) such that

$$\|u - u_h\|_a \leq Ch^s,$$
$$|\lambda_h - \lambda| \leq Ch^{2s}. \tag{3.34}$$

Remark 3.11. In this section we give a priori estimates of finite element approximations (see Theorem 3.6, Corollaries 3.7–3.9, and (3.34)). These estimates indicate that when the mesh size h is small enough we can obtain sufficiently precise approximations of fourth-order Steklov eigenvalues and eigenfunctions (biharmonic function).

4. Numerical Examples

Consider the eigenvalue problem (2.1)–(2.3), where $\Omega = (0, \pi/2) \times (0, \pi/2)$.

We illustrate the Bogner-Fox-Schmit element by Figure 1.

The degrees of freedom (interpolation conditions) of Bogner-Fox-Schmit element are function values and gradients $(\partial/\partial x_1, \partial/\partial x_2)$ and the second derivatives $\partial^2/\partial x_1 \partial x_2$ at the four vertices of a rectangle. We adopt a uniform square partition π_h with mesh diameter h for Ω, and the Bogner-Fox-Schmit finite element space defined on π_h is

$$V_h = \left\{ v \in C^1(\Omega) : v|_\kappa \in Q_3(\kappa), \forall \kappa \in \pi_h, v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial^2 v}{\partial x_1 \partial x_2} \right.$$
$$\left. \text{are continuous at element vertices, and } v \text{ vanishes on boundary nodes} \right\}, \tag{4.1}$$

where $Q_3(\kappa)$ is the bicubic polynomial space on an element κ.

It is well known that the Bogner-Fox-Schmit element is a conforming plate element. We compute the first four eigenvalues of (2.1)–(2.3) by the Bogner-Fox-Schmit element by using MATLAB and list the numerical results in Table 1.

It can be seen from Table 1 that the eigenvalues of Bogner-Fox-Schmit element decrease with the decrease of h. This is not an accident. In fact, for conforming finite element approximations for many eigenvalue problems, the minimum-maximum principle is valid; therefore it insures that numerical eigenvalues approximate exact eigenvalues from above (see [20, 26]). Our numerical results coincide with this principle.

Table 1: Numerical eigenvalues on the square domain $\Omega = (0, \pi/2) \times (0, \pi/2)$ using the Bogner-Fox-Schmit element.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}\pi}{10}$	2.2127407	4.4166105	4.4166105	6.0842699
$\frac{\sqrt{2}\pi}{20}$	2.2127002	4.4162026	4.4162026	6.0819058
$\frac{\sqrt{2}\pi}{40}$	2.2126976	4.4161741	4.4161741	6.0817349
$\frac{\sqrt{2}\pi}{60}$	2.2126974	4.4161725	4.4161725	6.0817253
$\frac{\sqrt{2}\pi}{80}$	2.2126974	4.4161723	4.4161723	6.0817236

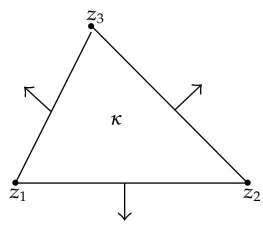

Figure 2

Is it possible to compute the lower bound of the eigenvalues of (2.1)–(2.3)? Reference [29] proved theoretically that the nonconforming Morley element can produce the lower bound for the eigenvalues of plate vibration problem, and [30] provided numerical example. These works inspire us to compute approximate eigenvalues of the fourth-order Steklov problem (2.1)–(2.3) by using the Morley element. We illustrate the Morley element by Figure 2.

The degrees of freedom (interpolation conditions) of Morley element are function values at the three vertices and outer normal derivatives at the three midpoints of the three edges. We adopt a uniform isosceles right triangulation π_h along three directions for Ω (each triangle is divided into four congruent triangles), and the Morley finite element space defined on π_h is

$$\begin{aligned} V_h = \{v \in L_2(\Omega) : v|_\kappa \in P_2(\kappa), \forall \kappa \in \pi_h, v, v_\nu \\ \text{are continuous at element vertices and midpoints} \\ \text{of three edges, resp., and } v \text{ vanishes on boundary nodes}\}, \end{aligned} \tag{4.2}$$

where $P_2(\kappa)$ is the quadratic polynomial space on an element κ.

The Morley element is a nonconforming plate element. We compute the first four eigenvalues of (2.1)–(2.3) by the Morley element by using MATLAB, and list the numerical results in Table 2.

From Table 2 it can be seen that the eigenvalues of Morley element increase with the decrease in h. We have the reason to conjecture that the eigenvalues by Morley element approximate the exact ones from below.

Table 2: Numerical eigenvalues on the square domain $\Omega = (0, \pi/2) \times (0, \pi/2)$ using Morley element.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}\pi}{16}$	2.1526851	3.9593995	4.1045601	5.1088362
$\frac{\sqrt{2}\pi}{32}$	2.1961067	4.2871754	4.3250009	5.7806630
$\frac{\sqrt{2}\pi}{64}$	2.2083493	4.3822829	4.3917069	5.9986854
$\frac{\sqrt{2}\pi}{128}$	2.2115857	4.4075238	4.4098509	6.0600499
$\frac{\sqrt{2}\pi}{256}$	2.2124164	4.4139906	4.4145665	6.0761982
$\frac{\sqrt{2}\pi}{512}$	2.2126268	4.4156245	4.4157676	6.0803290

From Tables 1 and 2 we can provide bounds for the exact eigenvalues:

$$\begin{aligned} &\lambda_1 \in (2.2126268, 2.2126974), \qquad \lambda_2 \in (4.4156245, 4.4161723), \\ &\lambda_3 \in (4.4157676, 4.4161723), \qquad \lambda_4 \in (6.0803290, 6.0817236). \end{aligned} \tag{4.3}$$

Remark 4.1. Reference [16] gave bounds for the smallest eigenvalue of (2.1)–(2.3) on $\Omega = (0, \pi/2) \times (0, \pi/2)$: $\lambda_1 \in (2.2118,\ 2.2133)$. By comparison, the bounds we give here are more precise; furthermore, we give the upper and lower bounds for the first four eigenvalues, and it is also confirmed by the numerical experiments that the smallest eigenvalue λ_1 is simple [13].

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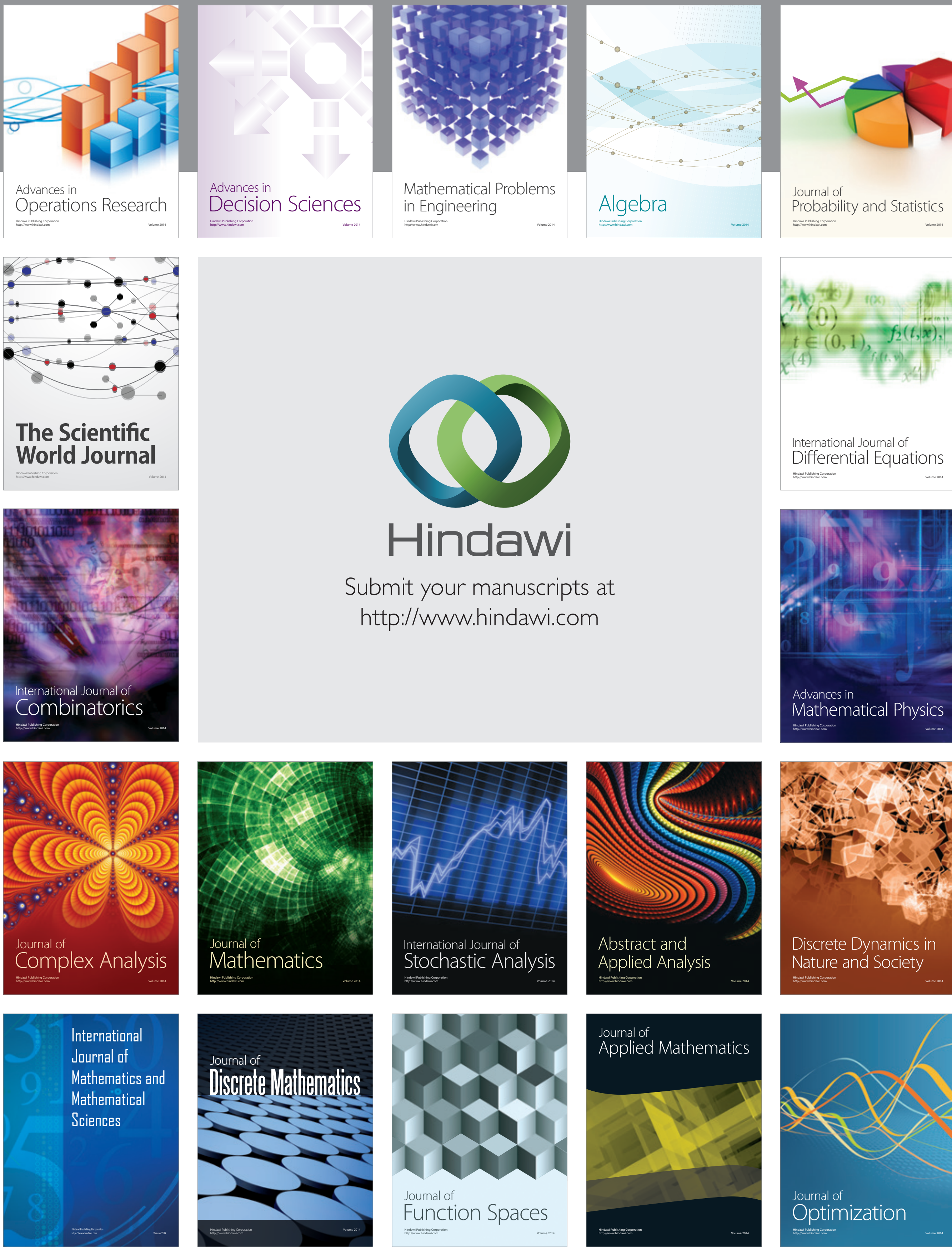
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