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Research Article

State Feedback Guaranteed Cost Repetitive Control for Uncertain Discrete-Time Systems

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This paper considers the problem of guaranteed cost repetitive control for uncertain discrete-time systems. The uncertainty in the system is assumed to be norm-bounded and time-varying. The objective is to develop a novel design method so that the closed-loop repetitive control system is quadratically stable and a certain bound of performance index is guaranteed for all admissible uncertainties. The state feedback control technique is used in the paper. While for the case that the states are not measurable, an observer-based control scheme is adopted. Sufficient conditions for the existence of guaranteed cost control law are derived in terms of linear matrix inequality (LMI). The control and observer gains are characterized by the feasible solutions to these LMIs. The optimal guaranteed cost control law is obtained efficiently by solving an optimization problem with LMI constraints using existing convex optimization algorithms. A simulation example is provided to illustrate the validity of the proposed method.

1. Introduction

In practice, many tracking systems have to deal with periodic reference and/or disturbance signals, for example, industrial robots, computer disk drives, and rotating machine tools. Repetitive control, which is based on the internal model principle proposed by Francis and Wonham [1], has been proved to be a useful control strategy for this class of systems. Up to date, researchers have devoted considerable efforts to the analysis and design of repetitive control systems. For the continuous-time case, Weiss and Häfele [2] discussed the repetitive control of MIMO systems using $H\infty$ design; Tsai and Yao [3] derived upper and lower bounds of the repetitive controller parameters that ensure stability and desired performance;

Doh and Chung [4] presented a linear matrix inequality- (LMI-) based repetitive controller design method for systems with norm-bounded uncertainties, while for the discrete-time case, Osburn and Franchek [5] developed a method for designing repetitive controllers using nonparametric frequency response plant models; Freeman et al. [6] proposed an optimality-based repetitive control algorithm for time-invariant systems; Pipeleers et al. [7] proposed a novel design approach for SISO high-order repetitive controllers.

It is well known that in many practical systems, the system model always contains some uncertain elements due to poor plant knowledge, reduced-order models, and nonlinearities such as hysteresis or friction, slowly varying parameters, and so forth, and the uncertainties frequently lead to deterioration of system performance and instability of systems. Hence, robust stability and stabilization for uncertain systems have been the focus of much research in the recent years. However, for the repetitive control of uncertain discrete-time systems, to the best of our knowledge, there are no previous results reported in the literature. This motivates our research.

When controlling a system involving uncertainties, it is often desirable to design a robust controller that not only stabilizes the closed-loop system but also guarantees an ideal level of performance for all admissible uncertainties. One way to address this problem is the so-called guaranteed cost control technique (see, e.g., [8, 9]). Furthermore, LMI approach is a powerful tool in the control theory and applications and has been applied to a wide range of control problems, such as the output feedback control [9] and filter design of time-delayed systems [10]. In this paper, we will adopt these two useful methodologies (i.e., guaranteed cost control technique and LMI approach) to discuss the state feedback repetitive control for discrete-time systems with norm-bounded and time-varying uncertainties. The objective is to develop a novel design method that not only provides an ideal level of performance while preserving system stability but also can be efficiently implemented using existing software. The approach taken in this paper is as follows: we first combine the state vectors of the repetitive controller and the uncertain system and derive the sufficient condition in the form of LMI for the existence of guaranteed cost control law. Next, for the case that the states of a system are not available for measurement, we present an observer-based control scheme. The control and observer gains are characterized by the feasible solutions to some LMIs. Finally, a convex optimization problem with LMI constraints is introduced to solve the optimal guaranteed cost control law using existing LMI software [11].

Notation. R^n denotes the n-dimensional Euclidean space; $R^{n\times m}$ is the set of all $n \times m$ real matrices; I is the identity matrix; null matrix or null vector of appropriate dimension is denoted by $\mathbf{0}$; the superscript "T" stands for the transpose of a matrix; the notation $P > 0$ and $P \geq 0$ for $P \in R^{n\times n}$ means that the matrix P is real symmetric positive definite or positive semidefinite, respectively; the symmetric terms in a symmetric matrix are denoted by $*$, for example, $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$.

2. Preliminaries of Guaranteed Cost Control

Consider an uncertain discrete-time system described by the following state equation:

$$x(t+1) = (A + HF(t)E)x(t), \tag{2.1}$$

where $x(t) \in R^n$ is the state vector with initial condition $x(0)$, A, H and E are known real constant matrices with appropriate dimensions, and $F(t)$ is a real uncertain matrix function satisfying $F^T(t)F(t) \le I$.

Associated with the uncertain system (2.1) is the following quadratic cost function with a given weighting matrix $Q > 0$:

$$J = \sum_{t=0}^{\infty} x^T(t)Qx(t). \tag{2.2}$$

Definition 2.1 (see [8]). A positive definite real matrix P is said to be a quadratic cost matrix for the system (2.1) and cost function (2.2) if

$$(A + HF(t)E)^T P(A + HF(t)E) - P + Q < 0 \tag{2.3}$$

for all $F(t)$ satisfying the bound $F^T(t)F(t) \le I$.

Lemma 2.2 (see [8]). *Suppose that $P > 0$ is a quadratic cost matrix for the uncertain system* (2.1) *and cost function* (2.2)*. Then the system is quadratically stable and the cost function satisfies the bound $J \le x^T(0)Px(0)$.*

The following theorem shows that the existence of a quadratic matrix is equivalent to the feasibility of an LMI.

Theorem 2.3. *Consider the system* (2.1) *and cost function* (2.2)*. There exists a quadratic cost matrix if and only if there exist a scalar $\varepsilon > 0$ and matrix $X > 0$ such that*

$$\begin{bmatrix} -X + \varepsilon HH^T & AX & \mathbf{0} & \mathbf{0} \\ * & -X & XE^T & X \\ * & * & -\varepsilon I & \mathbf{0} \\ * & * & * & -Q^{-1} \end{bmatrix} < 0. \tag{2.4}$$

Moreover, the cost function (2.2) *satisfies the bound $J \le x^T(0)X^{-1}x(0)$.*

To prove the theorem, we need the following lemma.

Lemma 2.4 (see [12]). *Let Σ_1 and Σ_2 be real constant matrices of compatible dimensions and $M(t)$ a real matrix function satisfying $M^T(t)M(t) \le I$. Then the following inequality holds:*

$$\Sigma_1 M(t)\Sigma_2 + \Sigma_2^T M^T(t)\Sigma_1^T \le \varepsilon\Sigma_1\Sigma_1^T + \varepsilon^{-1}\Sigma_2^T\Sigma_2, \quad \textit{for any } \varepsilon > 0. \tag{2.5}$$

Proof of Theorem 2.3. By an obvious application of Schur's complement formula [13], the inequality (2.3) is equivalent to

$$\begin{bmatrix} -P^{-1} & A + HF(t)E \\ * & -P + Q \end{bmatrix} < 0. \tag{2.6}$$

The inequality (2.6) can be further written as

$$\begin{bmatrix} -P^{-1} & A \\ * & -P+Q \end{bmatrix} + \begin{bmatrix} H \\ \mathbf{0} \end{bmatrix} F(t) \begin{bmatrix} \mathbf{0} & E \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ E^T \end{bmatrix} F^T(t) \begin{bmatrix} H^T & \mathbf{0} \end{bmatrix} < 0. \quad (2.7)$$

In light of Lemma 2.4, the inequality (2.7) holds for any $F(t)$ satisfying $F^T(t)F(t) \le I$ if and only if there exists a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} -P^{-1} + \varepsilon H H^T & A \\ * & -P + Q + \varepsilon^{-1} E^T E \end{bmatrix} < 0, \quad (2.8)$$

which is further equivalent to

$$\begin{bmatrix} -P^{-1} + \varepsilon H H^T & A & \mathbf{0} \\ * & -P+Q & E^T \\ * & * & -\varepsilon I \end{bmatrix} < 0. \quad (2.9)$$

Premultiplying and postmultiplying the inequality (2.9) by the matrix $\mathrm{diag}\{I, P^{-1}, I\}$ yield

$$\begin{bmatrix} -P^{-1} + \varepsilon H H^T & A P^{-1} & \mathbf{0} \\ * & -P^{-1} + P^{-1} Q P^{-1} & P^{-1} E^T \\ * & * & -\varepsilon I \end{bmatrix} < 0. \quad (2.10)$$

By denoting $X := P^{-1}$ and using Schur complements again, it is straightforward to verify that the inequality (2.10) is equivalent to (2.4). This completes the proof. □

3. State Feedback Repetitive Control

In this paper, we will consider the uncertain discrete-time SISO system described by

$$\Sigma_p : \begin{cases} x_p(t+1) = (A_p + \Delta A_p) x_p(t) + (B_p + \Delta B_p) u_p(t), \\ y_p(t) = (C_p + \Delta C_p) x_p(t) + (D_p + \Delta D_p) u_p(t), \end{cases} \quad (3.1)$$

where $x_p(t)$, $u_p(t)$, and $y_p(t)$ are the state vector, control input, and measured output, respectively; A_p, B_p, C_p, and D_p are real constant matrices with appropriate dimensions; the pairs (A_p, B_p) and (A_p, C_p) are stabilizable and detectable, respectively; $\Delta A_p, \Delta B_p, \Delta C_p$, and ΔD_p are parameter uncertainties which are norm-bounded and can be described by

$$\begin{bmatrix} \Delta A_p & \Delta B_p \\ \Delta C_p & \Delta D_p \end{bmatrix} = \begin{bmatrix} \overline{H}_1 \\ \overline{H}_2 \end{bmatrix} \Delta(t) \begin{bmatrix} \overline{E}_1 & \overline{E}_2 \end{bmatrix}, \quad (3.2)$$

where $\overline{H}_1$, $\overline{H}_2$, $\overline{E}_1$, and $\overline{E}_2$ are known constant matrices with appropriate dimensions, and $\Delta(t)$ is an uncertain matrix satisfying the bound $\Delta^T(t)\Delta(t) \leq I$.

According to the internal model principle, in order to achieve zero tracking error in steady state, it is necessary to include in the loop the generator of periodic reference and/or disturbance signal, which is usually known as the repetitive controller. The transfer function of digital periodic signal generator with period L is [14]

$$\Sigma_r = \frac{1}{1 - z^{-L}}. \tag{3.3}$$

As can be seen from (3.3), the periodic signal generator introduces L open-loop poles uniformly distributed over a circumference of unit radius, which makes great differences between the design of repetitive control system and that of conventional feedback control system, and increases the difficulty of design work.

The state-space description of Σ_r can be written as

$$\Sigma_r : \begin{cases} x_r(t+1) = A_r x_r(t) + B_r u(t), \\ u_p(t) = C_r x_r(t) + D_r u(t), \end{cases} \tag{3.4}$$

where

$$\begin{gathered} A_r = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \\ B_r = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T, \\ C_r = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \\ D_r = 1. \end{gathered} \tag{3.5}$$

Remark 3.1. To enhance the robust stability, additional filtering is usually added to the repetitive controller. Selecting $\Sigma_r = 1/(1 - \gamma z^{-L})$ with $\gamma \in (0,1)$ yields a commonly used repetitive control scheme which sacrifices the high-frequency performance for system stability. All the results in this section can be extended with elements of A_r and C_r modified to include this scheme.

By using the augmented state vector $x = [x_p^T, x_r^T]^T$, we combine (3.1) and (3.4) to yield the following system:

$$\begin{aligned} x(t+1) &= (A + H_1\Delta(t)E_1)x(t) + (B + H_1\Delta(t)E_2)u(t), \\ y(t) &= (C + H_2\Delta(t)E_1)x(t) + (D + H_2\Delta(t)E_2)u(t), \end{aligned} \tag{3.6}$$

where

$$A = \begin{bmatrix} A_p & B_pC_r \\ \mathbf{0} & A_r \end{bmatrix}, \qquad B = \begin{bmatrix} B_pD_r \\ B_r \end{bmatrix},$$
$$C = \begin{bmatrix} C_p & D_pC_r \end{bmatrix}, \qquad D = D_pD_r, \tag{3.7}$$
$$H_1 = \begin{bmatrix} \overline{H}_1 \\ \mathbf{0} \end{bmatrix}, \qquad H_2 = \overline{H}_2, \qquad E_1 = \begin{bmatrix} \overline{E}_1 & \overline{E}_2C_r \end{bmatrix}, \qquad E_2 = \overline{E}_2D_r.$$

Associated with the system (3.6) is the quadratic cost function with given weighting matrices $Q > 0$ and $R > 0$:

$$J = \sum_{t=0}^{\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t) \right). \tag{3.8}$$

Remark 3.2. For square $m \times m$ MIMO linear systems, by selecting the repetitive controller as

$$\Sigma_r = \frac{1}{1 - z^{-L}} \times I_{m\times m}, \tag{3.9}$$

the design technique proposed in this paper is also applicable by just rewriting the state-space description of Σ_r to obtain the corresponding state-space matrices A_r, B_r, C_r, and D_r.

The problem in this section is to design a memoryless state feedback control law

$$u(t) = Kx(t) \tag{3.10}$$

such that for any admissible uncertain matrix $\Delta(t)$, the resulting closed-loop system

$$x(t+1) = (A + BK + H_1\Delta(t)(E_1 + E_2K))x(t) \tag{3.11}$$

is not only stable, but also gives an upper bound for the closed-loop cost function

$$J = \sum_{t=0}^{\infty} x^T(t)\left(Q + K^TRK\right)x(t). \tag{3.12}$$

Remark 3.3. By combining the state vectors of the repetitive controller and the uncertain discrete-time system, the resulting closed-loop system with state feedback control law has a form similar to that of (2.1). Although similar problems have been investigated by some researchers for conventional uncertain systems without the repetitive controller, it is the merit of the paper that the simultaneous consideration of robust stability and performance for the repetitive control of uncertain discrete-time systems is achieved for the first time, and an optimal guaranteed cost control law, which not only preserves system stability but also ensures an adequate level of performance, can be obtained by the approach presented in the paper.

Definition 3.4. Consider the uncertain system (3.6) and cost function (3.8). The controller of the form (3.10) is said to be a state feedback guaranteed cost controller with cost matrix $P > 0$ if the matrix $P > 0$ is a quadratic cost matrix for the closed-loop system (3.11) and cost function (3.12).

Remark 3.5. Using the results of last section, it follows that if (3.10) is a guaranteed cost control law with cost matrix $P > 0$, then the resulting closed-loop system will be quadratically stable. Furthermore, the closed-loop system guarantees an adequate level of performance.

The following theorem provides an efficient way to solve the guaranteed cost state feedback control law (3.10) by existing convex optimization algorithms.

Theorem 3.6. *If there exist a scalar $\varepsilon > 0$ and matrices $X > 0, Y$ such that the following LMI holds:*

$$\begin{bmatrix} -X + \varepsilon H_1 H_1^T & AX + BY & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -X & Y^T & X & XE_1^T + Y^T E_2^T \\ * & * & -R^{-1} & \mathbf{0} & \mathbf{0} \\ * & * & * & -Q^{-1} & \mathbf{0} \\ * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{3.13}$$

then $u(t) = YX^{-1}x(t)$ is a guaranteed cost control law for the uncertain system (3.6).

Proof. According to Theorem 2.3, the existence of a quadratic cost matrix for the closed-loop system (3.11) and cost function (3.12) is equivalent to

$$\begin{bmatrix} -X + \varepsilon H_1 H_1^T & (A + BK)X & \mathbf{0} \\ * & -X + X(Q + K^T RK)X & X(E_1 + E_2 K)^T \\ * & * & -\varepsilon I \end{bmatrix} < 0. \tag{3.14}$$

By using Schur complements, the inequality (3.14) is further equivalent to

$$\begin{bmatrix} -X + \varepsilon H_1 H_1^T & AX + BKX & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -X & XK^T & X & XE_1^T + XK^T E_2^T \\ * & * & -R^{-1} & \mathbf{0} & \mathbf{0} \\ * & * & * & -Q^{-1} & \mathbf{0} \\ * & * & * & * & -\varepsilon I \end{bmatrix} < 0. \tag{3.15}$$

Now setting $Y = KX$, it is ready to see that (3.15) yields (3.13). Moreover, the guaranteed cost control gain is $K = YX^{-1}$. This completes the proof. □

In this paper, we are interested in designing a controller of the form (3.10) to minimize the upper bound of (3.8). However, this bound is dependent on the initial condition $x(0)$.

To remove this dependence on the initial condition, we adopt the approach proposed by Petersen et al. [8]. Suppose that the initial condition is arbitrary but belongs to the set

$$\Omega := \left\{x(0) \in R^n \mid x(0) = \Psi\nu,\ \nu^T\nu \leq 1\right\}, \tag{3.16}$$

where Ψ is a given matrix. Then, the cost bound (3.8) leads to

$$J \leq \lambda_{\max}\left(\Psi^T X^{-1}\Psi\right), \tag{3.17}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue.

Furthermore, introduce a scalar λ satisfying

$$\begin{aligned} &\lambda_{\max}\left(\Psi^T X^{-1}\Psi\right) < \lambda, \\ &\Longleftrightarrow -\lambda I + \Psi^T X^{-1}\Psi < 0, \\ &\Longleftrightarrow \begin{bmatrix} -\lambda I & \Psi^T \\ * & -X \end{bmatrix} < 0. \end{aligned} \tag{3.18}$$

Consequently, the design problem of the optimal guaranteed cost state feedback control law (3.10) can be formulated as the following optimization problem:

$$\begin{aligned} &\underset{\varepsilon>0, X>0, Y}{\text{minimize}} \quad \lambda \\ &\text{subject to} \quad \text{LMIs (3.13), (3.18)}, \end{aligned} \tag{3.19}$$

which is a convex optimization problem with LMI constraints and can be effectively solved by MATLAB LMI Toolbox.

4. Observer-Based Controller Design

In many practical control systems and applications, the states of a system are not always available for measurement. Hence, it is very necessary to introduce a state observer to reconstruct the states of the system. In the following work, we will focus on the design of an observer-based controller.

The dynamic observer-based control for the system (3.6) is constructed as

$$\Sigma_o : \begin{cases} \hat{x}(t+1) = A\hat{x}(t) + Bu(t) + \Gamma\left(y(t) - \hat{y}(t)\right), \\ \hat{y}(t) = C\hat{x}(t) + Du(t), \end{cases} \tag{4.1}$$

$$u(t) = K\hat{x}(t), \tag{4.2}$$

where $\hat{x}$ is the estimation of $x, \hat{y}$ is the observer output, K and Γ are the control gain and observer gain, respectively.

Define the state estimation error as

$$e(t) = x(t) - \hat{x}(t). \tag{4.3}$$

By applying the observer-based controller (4.1) and (4.2) to the system (3.6), we obtain the closed-loop system of the form

$$\begin{aligned} \hat{x}(t+1) &= (A + BK + \Gamma H_2 \Delta(E_1 + E_2 K))\hat{x}(t) + (\Gamma C + \Gamma H_2 \Delta E_1)e(t), \\ e(t+1) &= (H_1 - \Gamma H_2)\Delta(E_1 + E_2 K)\hat{x}(t) + (A - \Gamma C + (H_1 - \Gamma H_2)\Delta E_1)e(t), \end{aligned} \tag{4.4}$$

which can be further written as

$$\begin{bmatrix} \hat{x}(t+1) \\ e(t+1) \end{bmatrix} = (\Phi + M\Delta N) \begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix}, \tag{4.5}$$

where

$$\Phi = \begin{bmatrix} A + BK & \Gamma C \\ \mathbf{0} & A - \Gamma C \end{bmatrix}, \qquad M = \begin{bmatrix} \Gamma H_2 \\ H_1 - \Gamma H_2 \end{bmatrix}, \qquad N = \begin{bmatrix} E_1 + E_2 K & E_1 \end{bmatrix}. \tag{4.6}$$

As can be seen from (4.5), the expression of the closed-loop system with state observer is identical with that of (2.1). Therefore, we can utilize the results given in Theorem 2.3 to design the control gain K and observer gain Γ. Associated with the system (4.5) is the following cost function with $Q_1 > 0$ and $Q_2 > 0$:

$$J = \sum_{t=0}^{\infty} \left(x^T(t) Q_1 x(t) + e^T(t) Q_2 e(t) \right). \tag{4.7}$$

The following theorem gives the main result on observer-based controller design by which the control gain K and observer gain Γ could be solved.

Theorem 4.1. *The uncertain system* (3.6) *is quadratically stable by the observer-based control* (4.1) *and* (4.2) *provided that there exist a scalar* $\varepsilon > 0$ *and matrices* $P_1 > 0$, Y, Γ, *such that*

$$\begin{bmatrix} -P_1 & \mathbf{0} & AP_1 + BY & \Gamma C & \Gamma H_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -I & \mathbf{0} & A - \Gamma C & H_1 - \Gamma H_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & -P_1 & \mathbf{0} & \mathbf{0} & P_1 E_1^T + Y^T E_2^T & P_1 \left(Q_1^{1/2}\right)^T & \mathbf{0} \\ * & * & * & -I & \mathbf{0} & E_1^T & \mathbf{0} & \left(Q_2^{1/2}\right)^T \\ * & * & * & * & -I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -I & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\varepsilon I & \mathbf{0} \\ * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0. \tag{4.8}$$

Moreover, the stabilizing observer and control gains are given by Γ *and* $K = YP_1^{-1}$, *respectively.*

Proof. Define the Lyapunov function as $V(\hat{x}(t), e(t)) = \hat{x}^T(t)P_1\hat{x}(t) + e^T(t)P_2 e(t)$, where $P_1 > 0$ and $P_2 > 0$. Then according to Lemma 2.2, the closed-loop system (4.5) is quadratically stable if the following inequality holds:

$$(\Phi + M\Delta(t)N)^T P(\Phi + M\Delta(t)N) - P + Q < 0, \tag{4.9}$$

where $P = \mathrm{diag}\{P_1, P_2\}$, $Q = \mathrm{diag}\{Q_1, Q_2\}$.

By applying Schur complements and some basic matrix manipulations to the LMI (2.4), the stability condition for system (4.5) can be equivalently written as

$$\begin{bmatrix} -P & \Phi P & M & \mathbf{0} & \mathbf{0} \\ * & -P & \mathbf{0} & PN^T & PS^T \\ * & * & -I & \mathbf{0} & \mathbf{0} \\ * & * & * & -I & \mathbf{0} \\ * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{4.10}$$

where $S = \mathrm{diag}\{Q_1^{1/2}, Q_2^{1/2}\}$.

Note that

$$\Phi P = \begin{bmatrix} AP_1 + BKP_1 & \Gamma C P_2 \\ \mathbf{0} & AP_2 - \Gamma C P_2 \end{bmatrix},$$
$$PN^T = \begin{bmatrix} P_1 E_1^T + P_1 K^T E_2^T \\ P_2 E_1^T \end{bmatrix}, \tag{4.11}$$
$$PS^T = \begin{bmatrix} P_1 \left(Q_1^{1/2}\right)^T & \mathbf{0} \\ \mathbf{0} & P_2 \left(Q_2^{1/2}\right)^T \end{bmatrix}.$$

Then it is straightforward to prove that (4.10) is equivalent to (4.8) with $P_2 = I$, $K = YP_1^{-1}$. This completes the proof. □

The optimal control gain K and observer gain Γ can be obtained by solving the following optimization problem:

$$\begin{aligned} &\underset{\varepsilon>0,\ P_1>0, Y, \Gamma}{\text{minimize}} \quad \lambda, \\ &\text{subject to} \quad \begin{bmatrix} -\lambda I & \Psi^T \\ * & -\begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \end{bmatrix} < 0, \quad \text{LMI (4.8)}. \end{aligned} \tag{4.12}$$

The LMI (4.8) of Theorem 4.1 provides an efficient way to solve the observer and control gains by existing LMI software. However, it will undoubtedly yield conservative results in view of the proof with $P_2 = I$. As can be seen from the proof, since the entries in (4.10) occur in nonlinear fashion with respect to its arguments, it would be impossible to employ the standard LMI optimization approach to find the solutions if not letting $P_2 = I$. The conservativeness brought by Theorem 4.1 may rest in the sense that in some cases it will fail to produce a feasible solution when one actually exists.

To reduce the conservativeness induced by setting $P_2 = I$, in what follows, an alternative approach, which can be divided into two steps, will be presented. Firstly, the LMI result for the stability of closed-loop system (4.5), by which the suitable control gain K and observer gain Γ could be obtained, is derived under the assumption that the original system described by (3.1) is with no perturbations in the output equation (i.e., $H_2 = \overline{H}_2 = \mathbf{0}$). Secondly, the observer gain Γ, which is solved in the first step, is supposed to be known a prior. Then sufficient condition for the existence of guaranteed cost control gain K is derived in terms of LMI, and a convex optimization problem is formulated to solve the optimal control gain K by minimizing the upper bound of the cost function.

First we present an LMI result for the stability of closed-loop system (4.5) with no perturbations in the output equation. Before proceeding, we need to introduce the following lemma.

Lemma 4.2 (see [15]). *For a given full row rank* $C \in R^{m\times n}$ *with singular value decomposition* $C = U[C_0 \quad \mathbf{0}]V^T$, *where* $U \in R^{m\times m}$ *and* $V \in R^{n\times n}$ *are unitary matrices and* $C_0 \in R^{m\times m}$ *is a diagonal matrix with positive diagonal elements in decreasing order, assume that* $X \in R^{n\times n}$ *is a symmetric matrix, then there exists a matrix* $\overline{X} \in R^{m\times m}$ *such that* $CX = \overline{X}C$ *if and only if*

$$X = V\begin{bmatrix} X_1 & \mathbf{0} \\ \mathbf{0} & X_2 \end{bmatrix} V^T, \tag{4.13}$$

where $X_1 \in R^{m\times m}$, $X_2 \in R^{(n-m)\times(n-m)}$. *Moreover, the matrix* $\overline{X}$ *is given by* $\overline{X} = UC_0X_1C_0^{-1}U^T$.

The suitable control gain K and observer gain Γ for system (4.5) with $H_2 = \mathbf{0}$ could be solved by the following theorem.

Theorem 4.3. *The uncertain system* (3.6) *with* $H_2 = \mathbf{0}$ *is quadratically stable by the observer-based control* (4.1) *and* (4.2) *provided that there exist a scalar* $\varepsilon > 0$ *and matrices* $P_1 > 0$, $P_{21} > 0$, $P_{22} > 0, Y, W$, *such that*

$$\begin{bmatrix} -P_1 & \mathbf{0} & AP_1 + BY & WC & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -P_2 & \mathbf{0} & AP_2 - WC & H_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & -P_1 & \mathbf{0} & \mathbf{0} & P_1E_1^T + Y^TE_2^T & P_1\left(Q_1^{1/2}\right)^T & \mathbf{0} \\ * & * & * & -P_2 & \mathbf{0} & P_2E_1^T & \mathbf{0} & P_2\left(Q_2^{1/2}\right)^T \\ * & * & * & * & -I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -I & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\varepsilon I & \mathbf{0} \\ * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{4.14}$$

where the singular value decomposition of full row rank matrix C *is* $C = U[C_0 \quad \mathbf{0}]V^T$, $P_2 = V\left[\begin{smallmatrix} P_{21} & \mathbf{0} \\ \mathbf{0} & P_{22} \end{smallmatrix}\right]V^T$. *Moreover, the suitable control and observer gains are given by* $K = YP_1^{-1}$ *and* $\Gamma = WUC_0P_{21}^{-1}C_0^{-1}U^T$, *respectively.*

Proof. For SISO linear systems considered in this paper, it is obvious that the matrix C is full row rank. For $m \times m$ MIMO systems, without loss of generality, we suppose that $\operatorname{rank}(C_p) = m$, which implies $\operatorname{rank}(C = [C_p \quad D_pC_r]) = m$.

Since P_2 can be expressed as $P_2 = V\left[\begin{smallmatrix} P_{21} & \mathbf{0} \\ \mathbf{0} & P_{22} \end{smallmatrix}\right]V^T$, then according to Lemma 4.2, there exists a matrix $\overline{P}_2$ such that the equality $CP_2 = \overline{P}_2C$ holds. The matrix $\overline{P}_2$ and its inverse are given by $\overline{P}_2 = UC_0P_{21}C_0^{-1}U^T$ and $\overline{P}_2^{-1} = UC_0P_{21}^{-1}C_0^{-1}U^T$, respectively.

Furthermore, noting that

$$\Phi P = \begin{bmatrix} AP_1 + BKP_1 & \Gamma C P_2 \\ \mathbf{0} & AP_2 - \Gamma C P_2 \end{bmatrix} = \begin{bmatrix} AP_1 + BKP_1 & \Gamma \overline{P}_2 C \\ \mathbf{0} & AP_2 - \Gamma \overline{P}_2 C \end{bmatrix}, \tag{4.15}$$

then it is ready to see that (4.10) is equivalent to (4.14) with $\Gamma = W\overline{P}_2^{-1}$, $K = YP_1^{-1}$. This completes the proof. □

Once the observer gain Γ has been yielded from Theorem 4.3, we may now proceed to design the optimal control gain K which minimizes the upper bound of the cost function (4.7). The feasible control gain could be solved by the following theorem.

Theorem 4.4. *Suppose that the observer gain Γ in* (4.1) *is solved a priori by Theorem 4.3. Then the closed-loop system* (4.5) *is quadratically stable provided that there exist a scalar $\varepsilon > 0$ and matrices $P_1 > 0, P_2 > 0, Y$ satisfying the following LMI. Moreover, if this condition holds, then the control gain is given by $K = YP_1^{-1}$*

$$\begin{bmatrix} -P_1 & \mathbf{0} & AP_1 + BY & \Gamma C P_2 & \Gamma H_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -P_2 & \mathbf{0} & AP_2 - \Gamma C P_2 & H_1 - \Gamma H_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & -P_1 & \mathbf{0} & \mathbf{0} & P_1 E_1^T + Y^T E_2^T & P_1\left(Q_1^{1/2}\right)^T & \mathbf{0} \\ * & * & * & -P_2 & \mathbf{0} & P_2 E_1^T & \mathbf{0} & P_2\left(Q_2^{1/2}\right)^T \\ * & * & * & * & -I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -I & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\varepsilon I & \mathbf{0} \\ * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0. \tag{4.16}$$

Proof. It can be completed immediately from (4.10) by setting $Y = KP_1$. □

Hence, the optimal guaranteed cost control gain K which minimizes the upper bound of the cost function (4.7) can be obtained by solving the following LMI-constrained optimization problem:

$$\begin{aligned} &\underset{\varepsilon>0, P_1>0, P_2>0, Y}{\text{minimize}} \quad \lambda, \\ &\text{subject to} \quad \begin{bmatrix} -\lambda I & \Psi^T \\ * & -\begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix} \end{bmatrix} < 0, \quad \text{LMI (4.17)}. \end{aligned} \tag{4.17}$$

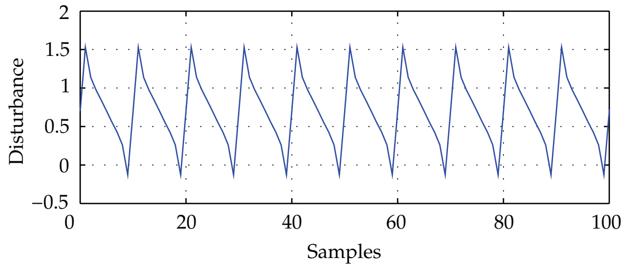

Figure 1: Disturbance signal used in simulation.

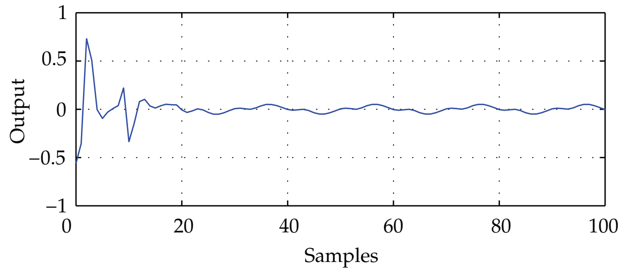

Figure 2: Time response of system output.

5. Simulation Example

Consider the uncertain system (3.1) with the following parameters:

$$A_p = \begin{bmatrix} 1.57 & -0.776 \\ 0.776 & 0 \end{bmatrix}, \qquad B_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C_p = \begin{bmatrix} 1.056 & -1.105 \end{bmatrix}, \qquad D_p = 4.4,$$
$$\overline{H}_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \qquad \overline{H}_2 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \qquad \overline{E}_1 = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \qquad \overline{E}_2 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \tag{5.1}$$
$$\Delta(t) = \begin{bmatrix} \sin 4\pi t & 0 \\ 0 & \sin 2\pi t \end{bmatrix}.$$

The control performance specification is to design an observer-based state feedback controller which stabilizes the closed-loop system and rejects a disturbance signal defined by $d(t) = 0.7 + 0.5\sin(\omega t) + 0.3\sin(2\omega t) + 0.2\sin(3\omega t) + 0.1\sin(4\omega t)$, as shown in Figure 1, where $\omega = 2\pi/L$ and $L = 10$.

Choose the weighting matrices as $Q_1 = Q_2 = I_{12\times 12}$ and $R = 0.2$. Now, we are in a position to solve the control gain K and observer gain Γ by the approach presented in Section 4. Firstly, we consider utilizing the result given in Theorem 4.1. However, it is found that the LMI (4.8) is infeasible, which, in some sense, validates the conservativeness induced by the proof with $P_2 = I$. Next, we turn to the results given in Theorems 4.3 and 4.4. The feasible solution of the observer gain Γ obtained by solving the LMI (4.14) is

$$\begin{aligned}\Gamma = [&0.3138, 0.0201, 0.2089, -0.0007, -0.0013, -0.0007,\\ &-0.0002, 0.0003, 0.0007, 0.0009, 0.0007, -0.0005]^T.\end{aligned} \tag{5.2}$$

Then, with the observer gain Γ known a prior, the optimal guaranteed cost control gain K is obtained as follows by solving the optimization problem (4.17), and the corresponding cost bound is $J \leq 31.33$,

$$\begin{aligned}K = [&-1.1577, 0.4636, 0.0158, 0.0006, -0.0027,\\ &-0.0014, -0.0001, 0.0002, 0, -0.0001, 0, -1].\end{aligned} \tag{5.3}$$

Figure 2 shows the response of the system output. It can be seen that the disturbance is attenuated to about 2.76 percent in four-sample periods when considering the amplitude of the disturbance and the output, although biggish amplitude oscillations occur as the output tends to steady state.

6. Conclusion

In this paper, a solution to the problem of repetitive control for uncertain discrete-time systems is presented. The state feedback control and guaranteed cost control techniques are adopted. Sufficient conditions for the existence of guaranteed cost control law are derived in terms of LMI, and it is shown that the control and observer gains can be characterized by the feasible solutions to the LMIs. A convex optimization problem is introduced to solve the optimal guaranteed cost control law. The validity of the proposed method is verified by a simulation example.

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