Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 102848, 10 pages
doi:10.1155/2012/102848

Research Article

Fast Detection of Weak Singularities in a Chaotic Signal Using Lorenz System and the Bisection Algorithm

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Received 1 March 2012; Accepted 1 May 2012

Academic Editor: Cristian Toma

Signals with weak singularities are important for condition monitoring, fault forecasting, and medicine diagnosis. However, the weak singularity in a signal is usually hidden by strong noise. A novel fast method is proposed for detecting a weak singularity in a noised signal by determining a critical threshold towards chaos for the Lorenz system. First, a rough critical threshold value is calculated by local Lyapunov exponents with a step size 0.1. Second, the exact threshold value is calculated by the bisection algorithm. The advantage of the method will not only reduce the computation costs, but also show the weak singular signal which can be accurately identified from strong noise. When the variance of an external signal method embeds into a Lorenz system, according to the parametric equivalent relation between the Lorenz system and the original system, the critical threshold value of the parameter in a Lorenz system is determined.

1. Introduction

In engineering, most weak singular information often is submerged into strong signals, such as the peaks, the discontinuities, and so forth. Moreover, when the some weak singular points are magnified slowly with time, at the moment when the fault occurs, the output signals usually contain jump points that are often singular points. Therefore, weak singular detection has played an important role in condition monitoring, fault forecast and medicine diagnosis [1, 2]. For example, some weak singular vibration signals in machine processes are important for fault forecasting.

The weak-signal detection is a central problem in the general field of signal processing and the use of chaos theory in weak-signal detection, and it is also a topic of interest in chaos

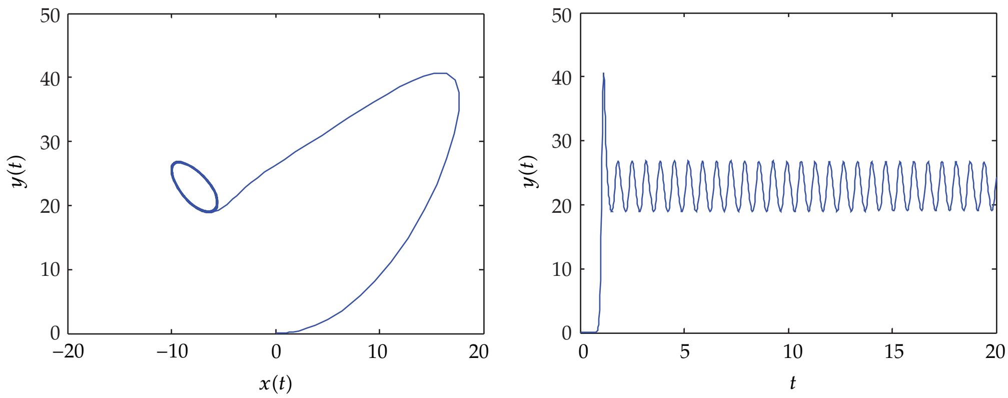


Figure 1: $r = 24$, phase plane of periodic state (sampling time 20 s).

control. At present, however, this research is mainly theory and simulation with MATLAB in terms of the Duffing-Holmes oscillator [3–6]. Whether other chaos system had better to characteristic than the Duffing-Holmes oscillator for detecting weak a singular signal. In this paper, a weak singular signal embedded in the strong signal is detected by Lorenz system. In 1963, an atmospheric scientist named E.N. Lorenz of M.I.T. proposed a simple model for thermally induced fluid convection in the atmosphere [7]. In Lorenz's mathematical model of convection, three state variables are used (x, y, z). The variable x is proportional to the amplitude of the fluid velocity circulation in the fluid ring, while y and z measure the distribution of temperature around the ring. The so-called Lorenz systems may be derived formally from the Navier-Stokes partial differential equations of fluid mechanics. The Lorenz model reads in standard notation as follows:

$$\begin{aligned} x' &= a(x - y), \\ y' &= -xz + rx - y, \\ z' &= xy - bz. \end{aligned} \tag{1.1}$$

For $a = 10$ and $b = 8/3$ (a favorite set of parameters for experts in the field, integrated with fourth-order Runge-Kutta method with a fixed step size, $t = 0.01$ s, there is an attractor for $r = 24$ for which the origin is of periodic state (Figure 1). The $r = 24.5$ gives the phase plane of chaotic state in which the other two attractors on the x-y plane become unstable spirals which is called a strange attractor (sometimes called the "butterfly attractor") and $y(t)$ take on a complex chaotic trajectory as shown in Figure 2.

Many researchers [8–10] analyze the Lorenz characteristic using r as a control variable. Upwards, in terms of $r = 24$ and $r = 25$, a Lorenz system has proved that there is a huge difference in the phase space trajectories between the chaotic state and the periodic state, and this difference can be used for the detection of weak singular signals in strong noise. Meanwhile, if Lyapunov exponents are adopted as the threshold value evaluated roughly

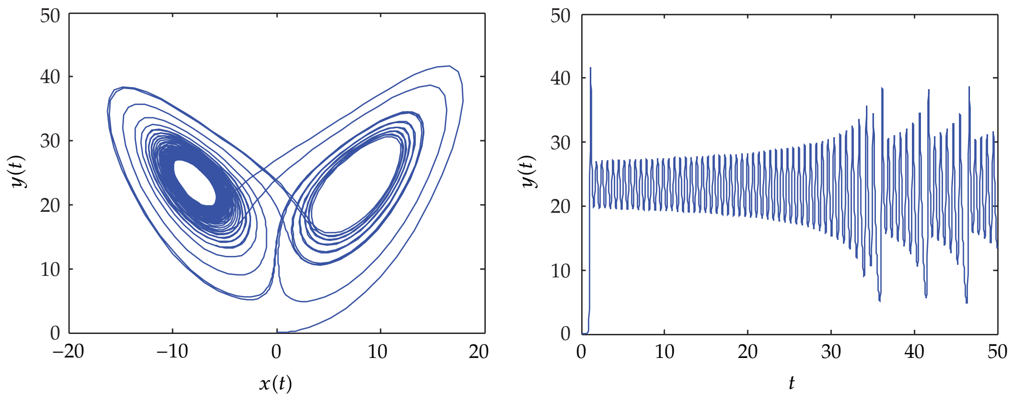


Figure 2: $r = 24.5$, phase plane of chaotic state (sampling time 50 s).

for a chaotic critical state, the bisection algorithm can fast approach any accurate threshold value. Thus, when an external signal is embedded into parameter r, one chaotic threshold is determined conveniently which can detect a weak singular signal in strong noise.

2. The Chaotic Behavior of the Detecting Lorenz by Lyapunov Exponent

The Lyapunov exponent (LE) is frequently computed measure for characterizing of chaotic dynamics. It describes a method for diagnosing whether or not a system is chaotic. For a discrete mapping $x(t+1) = f[x(t)]$, we calculate the local expansion of a flow by considering the difference of two trajectories as follows:

$$x(t+1) - y(t+1) = f(x(t)) - f(y(t)) \approx \frac{\partial f}{\partial x}[x(t)] \cdot [x(t) - y(t)]. \tag{2.1}$$

If this grows like

$$|x(t+1) - y(t+1)| \approx e^{\lambda}|x(t) - y(t)|, \tag{2.2}$$

then the exponent λ is called the Lyapunov exponent. If it is positive, bounded flows will generally be chaotic. We can solve for this exponent, asymptotically,

$$\lambda \approx \ln\left(\frac{|x_{n+1} - y_{n+1}|}{|x_n - y_n|}\right). \tag{2.3}$$

Since the Lorenz system is in three dimensions, it has three Lyapunov exponents. How efficient and reliable can algorithms to compute Lyapunov exponents be? For three-dimensional mapping as Lorenz system

$$\begin{aligned} x_{n+1} &= f_1(x_n, y_n, z_n) = a(x_n - y_n), \\ y_{n+1} &= f_2(x_n, y_n, z_n) = -x_n z_n + r x_n - y_n, \\ z_{n+1} &= f_3(x_n, y_n, z_n) = x_n y_n + b z_n. \end{aligned} \tag{2.4}$$

We get a Jacobian matrix for Lorenz flow

$$J(x_n, y_n, z_n) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_n}, \dfrac{\partial f_1}{\partial y_n}, \dfrac{\partial f_1}{\partial z_n} \\ \dfrac{\partial f_2}{\partial x_n}, \dfrac{\partial f_2}{\partial y_n}, \dfrac{\partial f_2}{\partial z_n} \\ \dfrac{\partial f_3}{\partial x_n}, \dfrac{\partial f_3}{\partial y_n}, \dfrac{\partial f_3}{\partial z_n} \end{bmatrix} = \begin{bmatrix} a & -a & 0 \\ r - z_n & -1 & -x_n \\ y_n & x_n & -b \end{bmatrix}. \tag{2.5}$$

Algorithms to compute eigenvalues of matrices are remarkably efficient: supposing the point successive mapping from the initial point $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2), \ldots$, $P_n(x_n, y_n, z_n)$, the Jackobian matrix of the previous $(n-1)$th point is $J_0 = J(x_0, y_0, z_0)$, $J_1 = J(x_1, y_1, z_1)$, $J_2 = J(x_2, y_2, z_2), \ldots$, and $J_{n-1} = J(x_{n-1}, y_{n-1}, z_{n-1})$. Defining $J^{(n)} = J_{n-1} J_{n-2} \cdots J_1 J_0$, the module of eigenvalue for $J^{(n)}$ is ${J_1}^{(n)}$, ${J_2}^{(n)}$, and ${J_3}^{(n)}$, and ${J_1}^{(n)} > {J_2}^{(n)} > J_3^{(n)}$, the Lyapunov exponents are defined as follows:

$$\lambda_1 = \lim_{n \to \infty} \sqrt[n]{J_1^{(n)}}, \qquad \lambda_2 = \lim_{n \to \infty} \sqrt[n]{J_2^{(n)}}, \qquad \lambda_3 = \lim_{n \to \infty} \sqrt[n]{J_3^{(n)}}. \tag{2.6}$$

When (1.1) is in the chaotic state, at least one of the three Lyapunov exponents in (2.6) is positive. The value is called maximum Lyapunov exponent. The chaotic behavior of the detection (1.1) is established on the basis of maximum Lyapunov exponents. If the system is not a point attractor, then the largest exponent cannot be negative. The Lyapunov exponent links with self-similarity of fractal dimension [11, 12].

3. Threshold Calculated Based on Lyapunov Exponents

To confirm the existence of the weak singular signal, we need to define a proper index for denoting the change in the states of Lorenz system. The index should be sensitive to a weak singular signal, but insensitive to the random noise from the viewpoint of statistical characteristics. Thus, the dynamic properties of Lorenz system are reflected statistically by Lyapunov exponents which are described in the following as [13–15]:

Let initial condition: [0.00001, 0.00001, 0.00001], with about typically 30 points in the region $r = [20, 30]$ chosen to calculate the Lyapunov exponents (LE), the computation's precision of r is two digits after the decimal point, shown in Table 1. The LE curve is plotted in Figure 3. Obviously, when $r = 24.05$, (1.1) takes on the chaotic state, and when $r = 24.10$, (1.1) takes on the periodic state.

Table 1: Lyapunov exponents in Lorenz.

No.	r	Max LE	No.	r	Max LE
1	20	−0.15	16	24.4	0.782
2	20.4	−0.141	17	24.8	0.82
3	20.8	−0.127	18	25.2	0.833
4	21.2	−0.114	19	25.6	0.836
5	21.6	−0.099	20	26	0.844
6	22	−0.087	21	26.4	0.857
7	22.4	−0.074	22	26.8	0.873
8	22.8	−0.06	23	27.2	0.881
9	23.2	−0.047	24	27.6	0.892
10	23.6	−0.033	25	28	0.907
11	23.8	−0.027	26	28.4	0.909
12	24	−0.018	27	28.8	0.922
13	24.05	−0.014	28	29.2	0.923
14	24.1	0.736	29	29.6	0.939
15	24.2	0.758	30	30	0.945

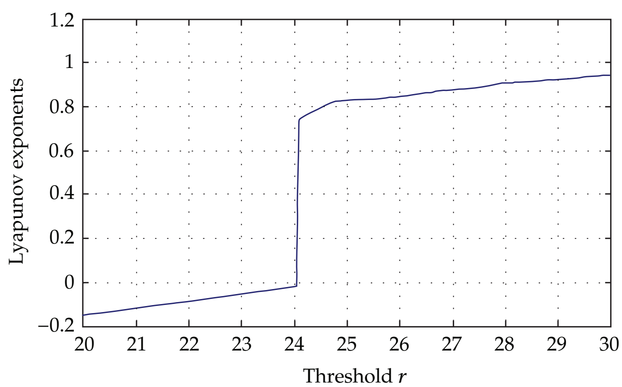


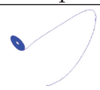
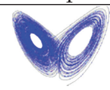
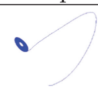
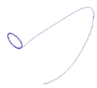
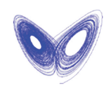
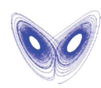
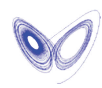
Figure 3: The relational curve between max. LE and r.

The LE changes from positive to negative corresponding to the region $r = [24.05, 24.1]$, and denotes the chaotic system's extreme sensitivity to the changed parameters. If the threshold r is equal to 24.05, and computation precision of r is only three effective digits after decimal point, as the critical threshold between a chaotic and periodic state, the sensitivity property is not precise enough.

4. Quickly Approaching Critical Threshold with the Bisection Algorithm

First, the rough region of the system threshold r is estimated by Lyapunov exponents with computation precision to be one digit after decimal point. Whatever the region of $r = [24.05, 24.1]$ is always sensitivity region changed from chaotic state to large periodic state. Since the bisection algorithm can converge to an optimizing solution quickly [16],

Table 2: Threshold r based on the bisection algorithm in the region $r = [24.05, 24.1]$.

Step	Periodic, r_1	Phase plane	Chaos, r_2	Phase plane	$(r_1 + r_2)/2$	Phase plane
1	24.05		24.1		24.075	
2	24.078		24.079		24.0785	
3	24.0782		24.0783		24.07825	
4	24.07820		24.07821			

the threshold value is determined by the bisection algorithm in the region $r = [24.05, 24.1]$. For the initial condition $[0.00001, 0.00001, 0.00001]$, in order to improve the sensitivity of (1.1), the computation precision of r has risen from five digits after decimal point. The steps are as follows:

(1) Because 24.1 corresponds to the chaotic state and 24.05 corresponds to the periodic state, $r = 24.075$ is the midpoint value between 24.05 (chaotic) and 24.1 (periodic).

(2) Because $r = 24.075$ corresponds to periodic states, the region of r is [24.075, 24.1]. Then r is accumulated from 24.075 to 24.1 with the step 0.001 up to 24.079 which corresponds to the chaotic state and 24.078 which corresponds to the periodic state. The 24.0785 is the middle value between 24.078 (periodic) and 24.079 (chaotic).

(3) Because $r = 24.0785$ corresponds to chaotic state, the region of r is taken [24.078, 24.0785]. Then r is accumulated from 24.078 to 24.0785 with the step 0.0001 up to $r = 24.0783$ which corresponds to chaotic state and $r = 24.0782$ which corresponds to periodic state. The 24.07825 is the middle value between 24.0782 (periodic) and 24.0783 (chaotic).

(4) Because $r = 24.07825$ corresponds to the chaotic state, the interval of r is [24.0782, 24.07825]. Then r is accumulated from 24.0782 to 24.07825 with the step 0.00001 up to 24.07821 which corresponds to chaotic state.

(5) Finally, the threshold value calculated is 24.07820. When a weak noisy signal is merged into (1.1), it takes on the large-scale chaotic state. The calculating process is shown Table 2.

4.1. How Is an External Signal Merged into Lorenz System

The Lorenz system of differential equations contains the item of the power two, where a, b are constants, parameter r can be motivated by exterior stimulations to generate a chaotic trajectory or periodic trajectory. We can adjust the amplitude r of the reference signal to the special value as in the chaotic critical state. The value is called the threshold value. How will the external signal be embedded into the control variable r in (1.1)?

The variance of a random signal is a measure of its statistical dispersion, indicating how far from the expected value its values typically are. The variance of a real-valued random

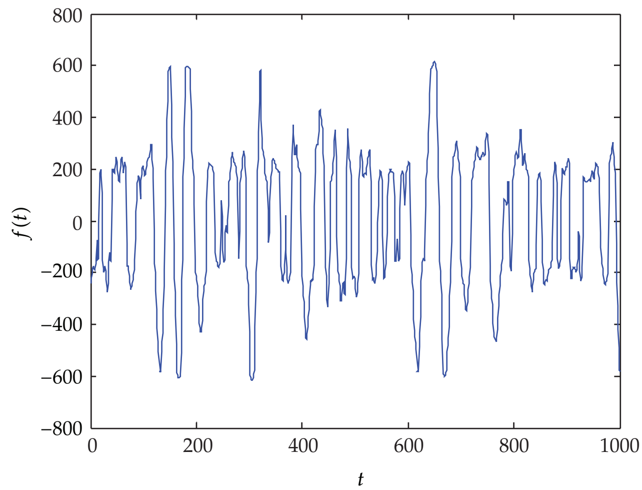


Figure 4: Random signal with variance 7.59×10^4 ($r = 24.0782$, sampling time 20 s).

signal is its second central moment, and the variance is simply the square of the standard deviation and also happens to be its second cumulant.

Let the time sequence of a random signal $f(t)$ be $x_1, x_2, x_3, \ldots, x_N$.

The mean value is $\overline{x} = 1/N \sum_{i=1}^{N} x_i$, and the variance of the time sequence is

$$\sigma = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2, \tag{4.1}$$

Since variance determines within what range values concentrated in a series fluctuate around the series mean and provides a quantitative measure of these fluctuations [17], the variance of an external signal $f(t)$ is merged into r, that is as follows:

$$r = r_0 + \text{var}(f(t)). \tag{4.2}$$

$\text{Var}(f(t))$ is variance function in MATLAB.

So long as the threshold is adjusted appropriately, the behavior of the Lorenz system will be changed dramatically from chaotic states to periodic states. Then Lorenz equation becomes

$$\begin{aligned} x' &= a(x-y), \\ y' &= -xz + [r_0 + \text{var}(f(t))]x - y, \\ z' &= xy - bz. \end{aligned} \tag{4.3}$$

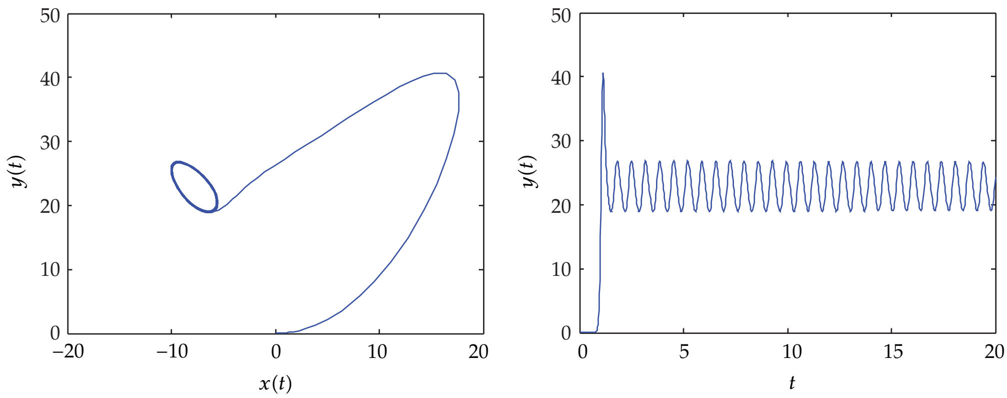


Figure 5: x-y plane of $f(t)$ merged into (4.3) ($r = 24.0782$, sampling time 20 s).

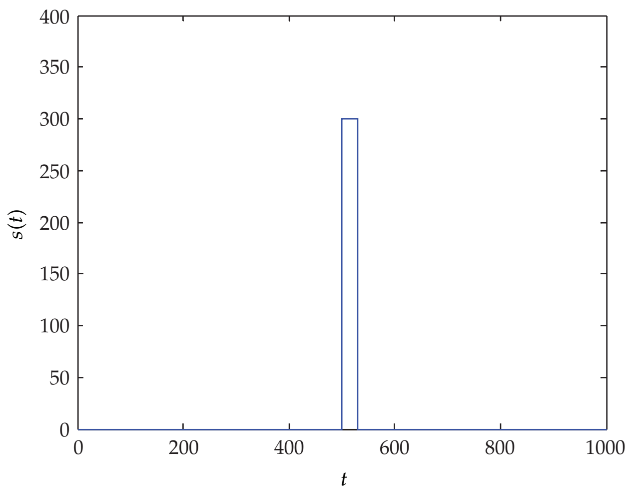


Figure 6: One-pulse signal (sampling time 50 s).

In (4.3), random signal $f(t)$ is shown as Figure 4, its variance is $\text{var}(f(t)) = 7.59 \times 10^4$, then $r_0 = r - \text{var}(f(t)) = 24.07820 - 7.59 = 16.4882$, (4.3) takes on periodic state (Figure 5).

When one weak noisy signal $s(t)$ (Figure 6) is merged into input $f(t)$ (Figure 7), that is $f_1(t) = S(m) + f(t)$, the variance of $f_1(t)$ is $\text{Var}(f_1(t)) = 8.04 \times 10^4 = 8.04$, $r = r_0 + \text{var}(S(m)) = 16.4882 + 8.04 = 24.5282$, then (4.3) takes on chaotic state (Figure 8).

5. Conclusion

Since the bisection algorithm can quickly converge to the critical threshold whose precision can be changed freely, searching any precision grade of the critical threshold of a Lorenz system will spent less time. If the variance of a random signal has a constant or a limited range band, in case weak-singularities signal happens and arouses the variance of the random signal to change infinitely small, the weak singularities signal can be detected.

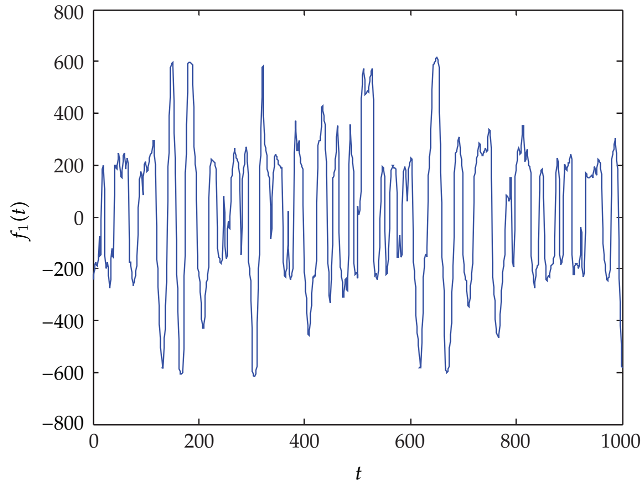


Figure 7: $f_1(t) = s(t) + f(t)$ (sampling time 50 s).

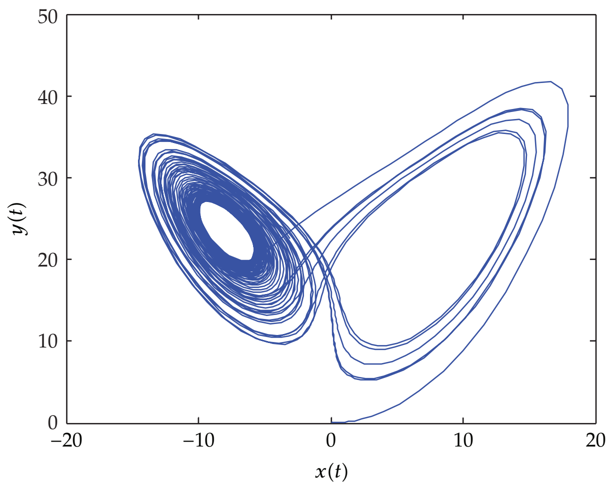


Figure 8: x-y plane of $r = 24.5282$ (sampling time 50 s).

Since the Runge-Kutta method of fourth-order is one kind of approximate solution method for dynamic equations, a difference time step size will impact the computation's precision for the threshold value.

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