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Research Article

Multi-Period Mean-Variance Portfolio Selection with Uncertain Time Horizon When Returns Are Serially Correlated

Ling Zhang[1, 2] and Zhongfei Li[3]

[1] *School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou 510275, China*
[2] *Department of Applied Mathematics, Guangdong University of Finance, Guangzhou 510521, China*
[3] *Lingnan (University) College/Business School, Sun Yat-Sen University, Guangzhou 510275, China*

Correspondence should be addressed to Zhongfei Li, lnslzf@mail.sysu.edu.cn

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We study a multi-period mean-variance portfolio selection problem with an uncertain time horizon and serial correlations. Firstly, we embed the nonseparable multi-period optimization problem into a separable quadratic optimization problem with uncertain exit time by employing the embedding technique of Li and Ng (2000). Then we convert the later into an optimization problem with deterministic exit time. Finally, using the dynamic programming approach, we explicitly derive the optimal strategy and the efficient frontier for the dynamic mean-variance optimization problem. A numerical example with AR(1) return process is also presented, which shows that both the uncertainty of exit time and the serial correlations of returns have significant impacts on the optimal strategy and the efficient frontier.

1. Introduction

The portfolio selection problem, which is one of great importance from both theoretical and practical perspectives, aims to find the best allocation of wealth among different assets in financial market. The mean-variance analysis pioneered by Markowitz [1] is one of the most widely used frameworks dealing with portfolio selection problems. In the past few decades, the mean-variance model stimulates a great deal of extensions and applied researches under single-period setting. Due to the nonseparability of multi-period mean-variance models, only up to 2000, Li and Ng [2] develop the embedding technique and solve a multi-period mean-variance portfolio selection problem analytically. In their work, the returns of risky assets are assumed to be independent and identically distributed, and this assumption is also adopted by lots of the later literature, such as Guo and Hu [3].

A large number of empirical analyses of the assets price dynamics show that there exist salient serial correlations in the returns of financial assets, and the correlation structure is very complicated. The ARMA model is developed to study the feature of the financial assets returning with serial correlations in the field of econometrics, and it is widely used in the empirical research of financial market. Hakansson [4, 5] had already taken the impact of serial correlations into account on his portfolio selection problems and had investigated the myopic optimal portfolio strategies when there existed serial correlations of yields and not. However, due to the complexity of multi-period portfolio selection problem with serial correlations of returns, there are little relevant literature and results focused on the impact of serial correlations on the optimal portfolio selection strategy. Balvers and Mitchell [6] first derived an analytical solution for a dynamic portfolio selection problem with autocorrelation assets returns, where the utility function was a negative exponential function, and the assets returns were subject to the normal ARMA(1,1) process. Dokuchaev [7] analyzed a discrete-time portfolio selection model with serial correlations and found the correlation structure which ensured the optimal strategy being myopic for both the power and the log utility functions. Çelikyurt and Özekici [8] studied such models with the assumption that the market evolution followed a Markov chain and the states were observable, whose objective functions depended on the mean and the variance of the terminal wealth. Çanakoğlu and Özekici [9] considered the utility maximization problem with imperfect information modulated by a hidden Markov chain, and obtained the explicit characterization of the optimal strategy and the value function. Wei and Ye [10] extended the work of [8] to take the risk control over bankruptcy into consideration. Xu and Li [11] investigated a multi-period mean-variance portfolio selection problem with one risky asset whose returns were serially correlated. By using the embedding technique of Li and Ng [2] and the dynamic programming approach, they obtained the explicit optimal strategy and proposed a measure of the risky asset value. To our knowledge, up to now, quite a few papers consider serial correlations of returns under dynamic portfolio selection framework.

On the other hand, the literature mentioned above makes an important hypothesis, implicitly or explicitly, that an investor knows her/his final exit time exactly at the moment of entering the market and making investment decisions, that is, the investment horizon is deterministic, either finite or infinite. In practice, however, the investor's exit time may be impacted by many exogenous and endogenous factors. An investor may exit from the market when she/he faces an unexpected need of huge consumption, sudden death, job loss, early retirement, investment target achieved, and so forth. Thus, it is more practical to weaken the restrictive assumption that the investment horizon is deterministic. If the exit time is uncertain, it is a random variable. As far as we know, study on the uncertain exit time can be dated back to Yarri [12], who studied an optimal consumption problem with an uncertain investment horizon. Hakansson [13] extended the work of [12] to a multi-period setting with a risky asset and an uncertain time horizon. Merton [14] addressed a dynamic optimal investment and consumption problem, and the uncertain retiring time was defined as the first jump of an independent Poisson process. Karatzas and Wang [15] considered an optimal investment problem in complete markets with the assumption that the exit time was a stopping time of the asset price filtration. Martellini and Urošević [16] extended the original model of [1] to a static mean-variance model in which the exit time was dependent on asset returns. Guo and Hu [3] analyzed a multi-period mean-variance investment problem with uncertainty time of exiting. Huang et al. [17] dealt with the portfolio selection problem with uncertain time horizon by adopting the worst-case CVaR methodology. Blanchet-Scalliet et al. [18] extended the optimal investment problem of [14] to allow the investor's time horizon to be stochastic and correlate to the returns of risky assets. Yi et al. [19] considered a multiperiod asset-liability management problem with an uncertain investment horizon under the mean-variance framework.

To the best of our knowledge, there is no work that considers the multiperiod mean-variance portfolio selection with an uncertain investment horizon and serially correlate returns at the same time. In the present paper, we try to tackle such problem. We assume that the distribution of the exit time is known, and the serial correlations of risky asset returns are settled the same as Hakansson [5] and Xu and Li [11]. We first embed our nonseparable problem, in the sense of dynamic programming, into a separable one by employing the embedding technique of Li and Ng [2]; then transform the separable problem with uncertain exit time into one with deterministic time horizon; finally solve the problem with deterministic time horizon by using the dynamic programming approach.

The rest of the paper is organized as follows. Section 2 formulates our problem and embeds it into a separable auxiliary problem. In Section 3, we solve the tractable auxiliary problem. In Section 4, we derive the optimal strategy and the efficient frontier of the original problem. Section 5 extends the results to the case of multiple risky assets. Section 6 gives a numerical simulation to show the impacts of exit time and serial correlations on the mean-variance efficient frontier. Finally, we conclude the paper in Section 7.

2. Modeling

We consider a financial market consisting of a risky asset and a riskless asset. The return rates of the riskless asset and risky asset at period $t+1$ (the time interval from time t to time $t+1$) are denoted by r_t^0 and r_t, respectively. It is assumed that r_t^0 is a constant and r_t is a $(t+1)$-measurable random variable. The risky asset will not degenerate into the riskless asset at any period, and its return rates $\{r_t, t = 0, 1, \ldots\}$ are correlated, that is, the value of r_t is dependent on the values of r_s, $s < t$, which are the realized returns of risky asset at the past periods. Thus, at time t, the expectation of a random variable, denoted by E_t, is a conditional expectation based on all of the history information up to time t.

We assume that an investor, who joins the market at time 0 with the initial wealth x_0, may invest her/his wealth among the risky asset and the riskless asset within a time horizon of T periods. At the beginning of each period t ($t = 1, \ldots, T$), the investor may adjust the amount invested in the risky and riskless assets by transaction. However, she/he may be forced to leave the financial market at time τ before T by some uncontrollable reasons. The uncertain exit time τ is supposed to be an exogenously random variable with the discrete probability distribution $\tilde{p}_t = \Pr\{\tau = t\}$, $t = 1, 2, \ldots$. Therefore, the actual exit time of the investor is $T \wedge \tau := \min\{T, \tau\}$, and its probability distribution is

$$p_t := \Pr\{T \wedge \tau = t\} = \begin{cases} \tilde{p}_t, & t = 1, \ldots, T-1, \\ 1 - \sum_{t=1}^{T-1} \tilde{p}_t, & t = T. \end{cases} \tag{2.1}$$

Let u_t be the amount invested in the risky asset at the beginning of period $t+1$. The investment series over T periods, $u := \{u_0, u_1, \ldots, u_{T-1}\}$, is called an investment strategy.

Define the excess return of risky asset at period $t+1$ ($t = 0,1,\ldots,T-1$) as $R_t = r_t - r_t^0$, which is assumed to be nondegenerated before time $t+1$, that is, the risky asset will not degenerate into the riskless asset at period $t+1$. Let x_t be the wealth of the investor at time t ($t = 0,1,\ldots,T$). If the investment strategy u is used in a self-financing way, the wealth dynamics can be described mathematically as

$$x_{t+1} = r_t^0 x_t + R_t u_t, \qquad t = 0,1,\ldots,T-1. \tag{2.2}$$

The multi-period mean-variance portfolio selection problem with uncertain exit time and serially correlate returns now can be formulated as

$$P(\omega)\begin{cases} \max\limits_{u} & E_0(x_{T\wedge\tau}) - \omega \mathrm{Var}_0(x_{T\wedge\tau}) \\ \text{s.t.} & x_{t+1} = r_t^0 x_t + R_t u_t, \quad t = 0,1,\ldots,T-1, \end{cases} \tag{2.3}$$

where ω is a given positive constant, representing the degree of the investor's risk aversion, and Var_0 is the variance conditional on the information available at time 0. There are some other assumptions with respect to model $P(\omega)$, which are summarized as follows: (a) short selling is permitted at any periods for the risky asset; (b) transaction costs and fees are negligible; (c) the investor can borrow and lend the riskless asset at any periods without limitation.

Recall that the mean-variance model $P(\omega)$ is difficult to solve due to its nonseparable structure in the sense of dynamic programming, which is one of the most powerful and universal methodologies for optimization problems with separable nature. Fortunately, Li and Ng [2] propose an embedding technique, and this technique is also applicable to solve the current problem with uncertain exit time and serially correlate returns. Instead of solving problem $P(\omega)$ directly, we first consider the following auxiliary problem:

$$A(\lambda,\omega)\begin{cases} \max\limits_{u} & E_0\left(\lambda x_{T\wedge\tau} - \omega x_{T\wedge\tau}^2\right) \\ \text{s.t.} & x_{t+1} = r_t^0 x_t + R_t u_t, \quad t = 0,1,\ldots,T-1, \end{cases} \tag{2.4}$$

for a given constant $\lambda > 0$.

Let $\Psi_A(\lambda,\omega)$ and $\Psi_P(\omega)$ be the optimal solution sets of problem $A(\lambda,\omega)$ and $P(\omega)$, respectively, namely,

$$\begin{aligned} \Psi_A(\lambda,\omega) &= \{u \mid u \text{ is an optimal solution of } A(\lambda,\omega)\}, \\ \Psi_P(\omega) &= \{u \mid u \text{ is an optimal solution of } P(\omega)\}. \end{aligned} \tag{2.5}$$

The following two theorems can be proven by a similar method to that described in Li and Ng [2], and so their proofs are omitted.

Theorem 2.1. *For any optimal solution u^* of $\Psi_P(\omega)$, u^* is the optimal solution of $\Psi_A(\lambda^*,\omega)$ with $\lambda^* = 1 + 2\omega E_0(x_{T\wedge\tau})|_{u^*}$.*

Theorem 2.2. *If $u^* \in \Psi_A(\lambda^*,\omega)$, a necessary condition for $u^* \in \Psi_P(\omega)$ is $\lambda^* = 1 + 2\omega E_0(x_{T\wedge\tau})|_{u^*}$.*

3. Analytical Solution of Auxiliary Problem $A(\lambda, \omega)$

In this section, we translate the auxiliary problem $A(\lambda, \omega)$ into a portfolio selection problem with certain exit time and then solve it by using the dynamic programming approach.

Since

$$\begin{aligned} E_0\left[\lambda x_{T\wedge\tau} - \omega x_{T\wedge\tau}^2\right] &= \sum_{t=1}^{T} E_0\left[\lambda x_{T\wedge\tau} - \omega x_{T\wedge\tau}^2 \mid T\wedge\tau = t\right] P(T\wedge\tau = t) \\ &= E_0\left[\sum_{t=1}^{T}\left(\lambda x_t - \omega x_t^2\right)p_t\right], \end{aligned} \tag{3.1}$$

problem $A(\lambda, \omega)$ can be written equivalently as

$$A(\lambda,\omega)\begin{cases} \max\limits_{u} & E_0\left[\sum\limits_{t=1}^{T}(\lambda x_t - \omega x_t^2)p_t\right] \\ \text{s.t.} & x_{t+1} = r_t^0 x_t + R_t u_t, \qquad t = 0,1,\ldots,T-1. \end{cases} \tag{3.2}$$

Define the value function

$$\begin{aligned} f_t^*(x_t) &= \max_{u_t} f_t(x_t) \\ &= \max_{u_t} E_t\left[\sum_{s=t}^{T}\left(\lambda x_s - \omega x_s^2\right)p_s\right] \end{aligned} \tag{3.3}$$

as the optimal expected utility using the optimal strategy conditional on the information available at time t $(t = 0,1,\ldots,T-1)$, and the boundary condition is

$$f_T^*(x_T) = \left(\lambda x_T - \omega x_T^2\right)p_T. \tag{3.4}$$

According to the dynamic programming principle, we have the Bellman equation

$$\begin{aligned} f_t^*(x_t) &= \max_{u_t} f_t(x_t) \\ &= \max_{u_t} E_t\left[\left(\lambda x_t - \omega x_t^2\right)p_t + f_{t+1}^*(x_{t+1})\right], \end{aligned} \tag{3.5}$$

for $t = 0,1,\ldots,T-1$.

First, we give the following notations:

$$\theta_t = \frac{E_t^2(\lambda_{t+1}R_t)}{E_t\left(\omega_{t+1}R_t^2\right)}, \qquad t = 0, 1, \ldots, T-1, \tag{3.6}$$

$$\Xi_t = E_t\left[\frac{\lambda^2}{4\omega}\sum_{s=t}^{T-1}\theta_s\right], \qquad t = 0, 1, \ldots, T-1, \tag{3.7}$$

$$\omega_t = p_t + \left(r_t^0\right)^2\left[E_t(\omega_{t+1}) - \frac{E_t^2(\omega_{t+1}R_t)}{E_t\left(\omega_{t+1}R_t^2\right)}\right], \qquad t = 0, 1, \ldots, T-1, \quad \omega_T = p_T, \tag{3.8}$$

$$\lambda_t = p_t + r_t^0\left[E_t(\lambda_{t+1}) - \frac{E_t(\omega_{t+1}R_t)E_t(\lambda_{t+1}R_t)}{E_t\left(\omega_{t+1}R_t^2\right)}\right], \qquad t = 0, 1, \ldots, T-1, \; \lambda_T = p_T. \tag{3.9}$$

For notational simplicity, we define $\sum_{j=s}^{t}(\cdot)_j = 0$ and $\prod_{j=s}^{t}(\cdot)_j = 1$ if $s > t$.

Note that R_t and R_{t+1} are not independent of each other for $t = 0, 1, \ldots, T-1$, so both ω_{t+1} and λ_{t+1} are dependent on the risky asset return at period $t+1$, R_t. Then, for $t = 0, 1, \ldots, T-1$,

$$E_t(\omega_{t+1}R_t) \neq E_t(\omega_{t+1})E_t(R_t), \qquad E_t(\lambda_{t+1}R_t) \neq E_t(\lambda_{t+1})E_t(R_t). \tag{3.10}$$

The following lemma comes from Xu and Li [11]. For the completeness, we provide its proof here.

Lemma 3.1. *Let x be a nondegenerated random variable, and let ξ be a positive random variable under the information at time t, then $E_t(x^2\xi)E_t(\xi) > (E_t(x\xi))^2$.*

Proof. Since ξ is a positive random variable, we can define a new probability measure Q as

$$dQ \triangleq \frac{\xi}{E_t(\xi)}dP, \tag{3.11}$$

where P is the original measure. Since x is a nondegenerated random variable, we have, under measure Q,

$$\mathrm{Var}_t^Q(x) = E_t^Q\left(x^2\right) - \left(E_t^Q(x)\right)^2 > 0. \tag{3.12}$$

Transforming the above inequality to under measure P, we obtain

$$E_t\left(x^2\frac{\xi}{E_t(\xi)}\right) - \left(E_t\left(x\frac{\xi}{E_t(\xi)}\right)\right)^2 > 0. \tag{3.13}$$

Multiplying both sides by $(E_t(\xi))^2$ in the above inequality produces

$$E_t\left(x^2\xi\right)E_t(\xi) > (E_t(x\xi))^2. \tag{3.14}$$

This completes the proof. □

Theorem 3.2. *For $t = 0, 1, \ldots, T-1$, $\omega_t > 0$, $\theta_t \geq 0$, and $\Xi_t \geq 0$.*

Proof. We use induction. For $t = T-1$, since the return of the risky asset at period T, R_{T-1}, is a nondegenerated random variable, then

$$\mathrm{Var}_{T-1}(R_{T-1}) = E_{T-1}\left(R_{T-1}^2\right) - \left(E_{T-1}(R_{T-1})\right)^2 > 0, \tag{3.15}$$

$E_{T-1}(R_{T-1}^2) > 0$. So

$$\begin{aligned} 0 \leq \theta_{T-1} &= \frac{E_{T-1}^2(p_T R_{T-1})}{E_{T-1}(p_T R_{T-1}^2)} = p_T \frac{E_{T-1}^2(R_{T-1})}{E_{T-1}(R_{T-1}^2)} < p_T, \\ \Xi_{T-1} &= E_{T-1}\left(\frac{\lambda^2}{4\omega}\theta_{T-1}\right) \geq 0. \end{aligned} \tag{3.16}$$

Therefore,

$$\omega_{T-1} = p_{T-1} + \left(r_{T-1}^0\right)^2 \left(p_T - \frac{E_{T-1}^2(p_T R_{T-1})}{E_{T-1}(p_T R_{T-1}^2)}\right) > 0. \tag{3.17}$$

Suppose that $\omega_s > 0$, $\theta_s \geq 0$, and $\Xi_s \geq 0$ hold true for $s = t+1, \ldots, T-2, T-1$, then for period t,

$$\theta_t = \frac{E_t^2(\lambda_{t+1} R_t)}{E_t(\omega_{t+1} R_t^2)} \geq 0. \tag{3.18}$$

By Lemma 3.1, we can easily see that

$$E_t(\omega_{t+1}) > \frac{E_t^2(\omega_{t+1} R_t)}{E_t(\omega_{t+1} R_t^2)}. \tag{3.19}$$

Hence, we obtain

$$\begin{aligned} \Xi_t &= E_t\left(\sum_{s=t}^{T-1} \frac{\lambda^2}{4\omega}\theta_s\right) = \frac{\lambda^2}{4\omega}\theta_t + E_t(\Xi_{t+1}) \geq 0, \\ \omega_t &= p_t + \left(r_t^0\right)^2 \left[E_t(\omega_{t+1}) - \frac{E_t^2(\omega_{t+1} R_t)}{E_t(\omega_{t+1} R_t^2)}\right] > 0. \end{aligned} \tag{3.20}$$

By induction, it shows that for $t = 0, 1, \ldots, T-1$, $\omega_t > 0$, $\theta_t \geq 0$, and $\Xi_t \geq 0$. □

The analytical optimal strategy and the value function of problem $A(\lambda, \omega)$ can be derived by using dynamic programming approach, which are summarized in the following theorem.

Theorem 3.3. *The optimal strategy and the value functions of problem $A(\lambda,\omega)$ are, respectively, given by*

$$u_t^* = \frac{\lambda}{2\omega}\frac{E_t(\lambda_{t+1}R_t)}{E_t(\omega_{t+1}R_t^2)} - \frac{E_t(\omega_{t+1}R_t)}{E_t(\omega_{t+1}R_t^2)} r_t^0 x_t, \qquad t = 0,1,\ldots,T-1, \tag{3.21}$$

$$f_t^*(x_t) = -\omega\omega_t x_t^2 + \lambda\lambda_t x_t + \Xi_t, \qquad t = 0,1,\ldots,T-1, \tag{3.22}$$

where Ξ_t, ω_t, and λ_t are given as defined in (3.7)–(3.9).

Proof. We will show that the above recursive formulas hold true by induction starting with the boundary condition $f_T(x_T) = (\lambda x_T - \omega x_T^2)p_T$. Note that for $t = T-1$,

$$\begin{aligned} f_{T-1}^*(x_{T-1}) &= \max_{u_{T-1}} f_{T-1}(x_{T-1}) \\ &= \max_{u_{T-1}} E_{T-1}\left[\left(\lambda x_{T-1} - \omega x_{T-1}^2\right)p_{T-1} + f_T^*(x_T)\right] \\ &= \max_{u_{T-1}} E_{T-1}\left[\left(\lambda x_{T-1} - \omega x_{T-1}^2\right)p_{T-1} + \left(\lambda x_T - \omega x_T^2\right)p_T\right] \\ &= \max_{u_{T-1}} \left(\lambda x_{T-1} - \omega x_{T-1}^2\right)p_{T-1} + \lambda\left[r_{T-1}^0 p_T x_{T-1} + E_{T-1}(p_T R_{T-1})u_{T-1}\right] \\ &\quad - \omega\left[\left(r_{T-1}^0\right)^2 p_T x_{T-1}^2 + 2r_{T-1}^0 E_{T-1}(p_T R_{T-1})u_{T-1}x_{T-1} + E_{T-1}\left(p_T R_{T-1}^2\right)u_{T-1}^2\right]. \end{aligned} \tag{3.23}$$

Since $E_{T-1}(R_{T-1}^2) > 0$ by assumption, the function $f_{T-1}(x_{T-1})$ is a concave function of u_{T-1}. The first-order condition gives

$$\lambda E_{T-1}(p_T R_{T-1}) - 2\omega\left[r_{T-1}^0 E_{T-1}(p_T R_{T-1})x_{T-1} + E_{T-1}\left(p_T R_{T-1}^2\right)u_{T-1}\right] = 0, \tag{3.24}$$

which yields the optimal solution u_{T-1}^* as

$$u_{T-1}^* = \frac{\lambda}{2\omega}\frac{E_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R_{T-1}^2)} - \frac{E_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R_{T-1}^2)} r_{T-1}^0 x_{T-1}. \tag{3.25}$$

Substituting u^*_{T-1} back into $f_{T-1}(x_{T-1})$, it follows that

$$\begin{aligned}
f^*_{T-1}(x_{T-1}) &= \left(\lambda x_{T-1} - \omega x^2_{T-1}\right)p_{T-1} \\
&\quad + \lambda\left[r^0_{T-1}p_T x_{T-1} + E_{T-1}(p_T R_{T-1})\left(\frac{\lambda}{2\omega}\frac{E_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})} - \frac{E_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})}r^0_{T-1}x_{T-1}\right)\right] \\
&\quad - \omega\left[\left(r^0_{T-1}\right)^2 p_T x^2_{T-1} + 2r^0_{T-1}E_{T-1}(p_T R_{T-1})\right. \\
&\qquad \times\left(\frac{\lambda}{2\omega}\frac{E_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})} - \frac{E_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})}r^0_{T-1}x_{T-1}\right)x_{T-1} \\
&\qquad \left.+E_{T-1}\left(p_T R^2_{T-1}\right)\left(\frac{\lambda}{2\omega}\frac{E_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})} - \frac{E_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})}r^0_{T-1}x_{T-1}\right)^2\right] \\
&= -\omega\left[p_{T-1} + \left(r^0_{T-1}\right)^2\left(p_T - \frac{E^2_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})}\right)\right]x^2_{T-1} \\
&\quad + \lambda\left[p_{T-1} + r^0_{T-1}\left(p_T - \frac{E^2_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})}\right)\right]x_{T-1} + \frac{\lambda^2}{4\omega}\frac{E^2_{T-1}(p_T R_{T-1})}{E_{T-1}(p_T R^2_{T-1})} \\
&= -\omega\omega_{T-1}x^2_{T-1} + \lambda\lambda_{T-1}x_{T-1} + \frac{\lambda^2}{4\omega}\theta_{T-1} \\
&= -\omega\omega_{T-1}x^2_{T-1} + \lambda\lambda_{T-1}x_{T-1} + \Xi_{T-1}.
\end{aligned} \tag{3.26}$$

Hence, the conclusion holds true for $t = T-1$.

Now we assume that the conclusion holds true for time $t+1$, in other words,

$$\begin{aligned}
f^*_{t+1}(x_{t+1}) &= -\omega\omega_{t+1}x^2_{t+1} + \lambda\lambda_{t+1}x_{t+1} + \Xi_{t+1}, \\
u^*_{t+1} &= \frac{\lambda}{2\omega}\frac{E_{t+1}(\lambda_{t+2}R_{t+1})}{E_{t+1}(\omega_{t+2}R^2_{t+1})} - \frac{E_{t+1}(\omega_{t+2}R_{t+1})}{E_{t+1}(\omega_{t+2}R^2_{t+1})}r^0_{t+1}x_{t+1},
\end{aligned} \tag{3.27}$$

then the optimization problem at time t for given state x_t is

$$\begin{aligned}
f^*_t(x_t) &= \max_{u_t} f_t(x_t) \\
&= \max_{u_t} E_t\left[\left(\lambda x_t - \omega x^2_t\right)p_t + f^*_{t+1}(x_{t+1})\right] \\
&= \max_{u_t} E_t\left[\left(\lambda x_t - \omega x^2_t\right)p_t - \omega\omega_{t+1}x^2_{t+1} + \lambda\lambda_{t+1}x_{t+1} + \Xi_{t+1}\right]
\end{aligned}$$

$$= \max_{u_t} \left\{ \left(\lambda x_t - \omega x_t^2\right) p_t - \omega \left[\left(r_t^0\right)^2 x_t^2 E_t(\omega_{t+1}) + 2 r_t^0 E_t(\omega_{t+1} R_t) u_t x_t + E_t\left(\omega_{t+1} R_t^2\right) u_t^2 \right] \right.$$
$$\left. + \lambda \left(r_t^0 x_t E_t(\lambda_{t+1}) + E_t(\lambda_{t+1} R_t) u_t \right) + E_t(\Xi_{t+1}) \right\} \tag{3.28}$$

Noting that $E_t(\omega_{t+1} R_t^2) > 0$ by Theorem 3.2, the function $f_t(x_t)$ is also a concave function of u_t. The first-order condition yields

$$u_t^* = \frac{\lambda}{2\omega} \frac{E_t(\lambda_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} - \frac{E_t(\omega_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} r_t^0 x_t. \tag{3.29}$$

Therefore, for the above given u_t^*,

$$\begin{aligned} f_t^*(x_t) &= \left(\lambda x_t - \omega x_t^2\right) p_t \\ &\quad + \lambda \left[r_t^0 x_t E_t(\lambda_{t+1}) + E_t(\lambda_{t+1} R_t) \left(\frac{\lambda}{2\omega} \frac{E_t(\lambda_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} - \frac{E_t(\omega_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} r_t^0 x_t \right) \right] \\ &\quad - \omega \left[\left(r_t^0\right)^2 x_t^2 E_t(\omega_{t+1}) + 2 r_t^0 E_t(\omega_{t+1} R_t) \left(\frac{\lambda}{2\omega} \frac{E_t(\lambda_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} - \frac{E_t(\omega_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} r_t^0 x_t \right) x_t \right. \\ &\quad \left. + E_t\left(\omega_{t+1} R_t^2\right) \left(\frac{\lambda}{2\omega} \frac{E_t(\lambda_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} - \frac{E_t(\omega_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} r_t^0 x_t \right)^2 \right] + E_t(\Xi_{t+1}) \\ &= -\omega \left[p_t + \left(r_t^0\right)^2 \left(E_t(\omega_{t+1}) - \frac{E_t^2(\omega_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} \right) \right] x_t^2 \\ &\quad + \lambda \left[p_t + r_t^0 \left(E_t(\lambda_{t+1}) - \frac{E_t(\omega_{t+1} R_t) E_t(\lambda_{t+1} R_t)}{E_t\left(\omega_{t+1} R_t^2\right)} \right) \right] x_t + \frac{\lambda^2}{4\omega} \theta_t + E_t(\Xi_{t+1}) \\ &= -\omega \omega_t x_t^2 + \lambda \lambda_t x_t + \Xi_t. \end{aligned} \tag{3.30}$$

Hence, the conclusion is true for t. By induction, the theorem is true. □

4. Optimal Strategy and the Efficient Frontier of the Original Problem $P(\omega)$

If we insert the optimal strategy given in Theorem 3.3 into the dynamic process of wealth, x_T and x_T^2 can be expressed as

$$\begin{aligned} x_T &= r_{T-1}^0 x_{T-1} + R_{T-1} u_{T-1}^* \\ &= \left[1 - \frac{R_{T-1} E_{T-1}(\omega_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)} \right] r_{T-1}^0 x_{T-1} + \frac{\lambda}{2\omega} \frac{R_{T-1} E_{T-1}(\lambda_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)}, \end{aligned}$$

$$x_T^2 = \left[1 - \frac{2R_{T-1}E_{T-1}(\omega_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)} + \frac{R_{T-1}^2 E_{T-1}^2(\omega_T R_{T-1})}{E_{T-1}^2\left(\omega_T R_{T-1}^2\right)}\right]\left(r_{T-1}^0\right)^2 x_{T-1}^2$$
$$+ \frac{\lambda}{\omega}\left[\frac{R_{T-1}E_{T-1}(\lambda_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)} - \frac{R_{T-1}^2 E_{T-1}(\omega_T R_{T-1})E_{T-1}(\lambda_T R_{T-1})}{E_{T-1}^2\left(\omega_T R_{T-1}^2\right)}\right] r_{T-1}^0 x_{T-1}$$
$$+ \frac{\lambda^2}{4\omega^2}\frac{R_{T-1}^2 E_{T-1}^2(\lambda_T R_{T-1})}{E_{T-1}^2\left(\omega_T R_{T-1}^2\right)}, \tag{4.1}$$

$$\lambda_T x_T = \left[\lambda_T - \frac{\lambda_T R_{T-1}E_{T-1}(\omega_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)}\right] r_{T-1}^0 x_{T-1} + \frac{\lambda}{2\omega}\frac{\lambda_T R_{T-1}E_{T-1}(\lambda_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)},$$
$$\omega_T x_T^2 = \left[\omega_T - \frac{2\omega_T R_{T-1}E_{T-1}(\omega_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)} + \frac{\omega_T R_{T-1}^2 E_{T-1}^2(\omega_T R_{T-1})}{E_{T-1}^2\left(\omega_T R_{T-1}^2\right)}\right]\left(r_{T-1}^0\right)^2 x_{T-1}^2$$
$$+ \frac{\lambda}{\omega}\left[\frac{\omega_T R_{T-1}E_{T-1}(\lambda_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)} - \frac{\omega_T R_{T-1}^2 E_{T-1}(\omega_T R_{T-1})E_{T-1}(\lambda_T R_{T-1})}{E_{T-1}^2\left(\omega_T R_{T-1}^2\right)}\right] r_{T-1}^0 x_{T-1} \tag{4.2}$$
$$+ \frac{\lambda^2}{4\omega^2}\frac{\omega_T R_{T-1}^2 E_{T-1}^2(\lambda_T R_{T-1})}{E_{T-1}^2\left(\omega_T R_{T-1}^2\right)}.$$

Taking expectations on both sides of (4.2) based on the information available at time $T-1$, we conclude that

$$E_{T-1}(\lambda_T x_T) = \lambda_{T-1}x_{T-1} - p_{T-1}x_{T-1} + \frac{\lambda}{2\omega}\theta_{T-1}, \tag{4.3}$$

$$E_{T-1}\left(\omega_T x_T^2\right) = \omega_{T-1}x_{T-1}^2 - p_{T-1}x_{T-1}^2 + \frac{\lambda^2}{4\omega^2}\theta_{T-1}, \tag{4.4}$$

where

$$\theta_{T-1} = \frac{E_{T-1}^2(\lambda_T R_{T-1})}{E_{T-1}\left(\omega_T R_{T-1}^2\right)}. \tag{4.5}$$

The above equations are recursive equations, and by taking expectations on both sides of (4.3) and (4.4) at time $T-2,\ldots,1,0$ repeatedly, we obtain

$$E_0(x_{T\wedge\tau}) = \sum_{t=1}^{T} p_t E_0(x_t) = \lambda_0 x_0 + \frac{\lambda}{2\omega}\sum_{t=1}^{T} E_0(\theta_{t-1}) = \lambda_0 x_0 + \frac{\lambda}{2\omega}\Theta, \tag{4.6}$$

$$E_0\left(x_{T\wedge\tau}^2\right) = \sum_{t=1}^{T} p_t E_0\left(x_t^2\right) = \omega_0 x_0^2 + \frac{\lambda^2}{4\omega^2}\sum_{t=1}^{T} E_0(\theta_{t-1}) = \omega_0 x_0^2 + \frac{\lambda^2}{4\omega^2}\Theta, \tag{4.7}$$

where

$$\theta_t = \frac{E_t^2(\lambda_{t+1}R_t)}{E_t\left(\omega_{t+1}R_t^2\right)}, \qquad \Theta = \sum_{t=1}^{T} E_0(\theta_{t-1}). \tag{4.8}$$

With the results (4.6) and (4.7), the variance of the terminal wealth $x_{T\wedge\tau}$ under the optimal strategy (3.21) can be written as

$$\begin{aligned}
\mathrm{Var}_0(x_{T\wedge\tau}) &= E_0\left(x_{T\wedge\tau}^2\right) - [E_0(x_{T\wedge\tau})]^2 \\
&= E_0\left(\sum_{t=1}^{T} p_t x_t^2\right) - \left[E_0\left(\sum_{t=1}^{T} p_t x_t\right)\right]^2 \\
&= \omega_0 x_0^2 + \frac{\lambda^2}{4\omega^2}\Theta - \left[\lambda_0 x_0 + \frac{\lambda}{2\omega}\Theta\right]^2 \\
&= \frac{\lambda^2}{4\omega^2}\left(\Theta - \Theta^2\right) - \frac{\lambda}{\omega}\lambda_0\Theta x_0 + \left(\omega_0 - \lambda_0^2\right)x_0^2.
\end{aligned} \tag{4.9}$$

Lemma 4.1. $0 < \Theta < 1$, $\omega_0 - \lambda_0^2/(1-\Theta) > 0$.

Proof. First of all, we claim that $\mathrm{Var}_0(x_{T\wedge\tau}) > 0$, since it measures the risk of investor at the time of exiting market, and the risky asset cannot degenerate into the riskless asset. Especially, when $x_0 = 0$, $\mathrm{Var}_0(x_{T\wedge\tau})$ can be reduced to

$$\mathrm{Var}_0(x_{T\wedge\tau}) = \frac{\lambda^2}{4\omega^2}\left(\Theta - \Theta^2\right) > 0, \tag{4.10}$$

and it is easy to show that $0 < \Theta < 1$.

The expression of $\mathrm{Var}_0(x_{T\wedge\tau})$ can be further converted into

$$\mathrm{Var}_0(x_{T\wedge\tau}) = \left(\Theta - \Theta^2\right)\left[\frac{\lambda}{2\omega} - \frac{\lambda_0}{1-\Theta}x_0\right]^2 + \left(\omega_0 - \frac{\lambda_0^2}{1-\Theta}\right)x_0^2 > 0. \tag{4.11}$$

Since we know that $0 < \Theta < 1$, the above inequality implies $\omega_0 - \lambda_0^2/(1-\Theta) > 0$, and we finish the proof of Lemma 4.1. □

According to Theorem 2.2, a necessary condition for the optimal solution of auxiliary problem $A(\lambda^*, \omega)$ to attain the optimality of problem $P(\omega)$ at the same time is

$$\lambda^* = 1 + 2\omega E_0(x_{T\wedge\tau})\,|_{u*} = 1 + 2\omega\left(\lambda_0 x_0 + \frac{\lambda^*}{2\omega}\Theta\right). \tag{4.12}$$

We can easily obtain

$$\lambda^* = \frac{1 + 2\omega\lambda_0 x_0}{1-\Theta}. \tag{4.13}$$

Finally, substituting (4.13) back into (3.21) yields the analytically optimal strategy of the original problem $P(\omega)$, which is summarized in the following theorem.

Theorem 4.2. *For the mean-variance problem $P(\omega)$, the optimal strategy is given by*

$$u_t^* = \frac{1 + 2\omega\lambda_0 x_0}{2\omega(1-\Theta)} \frac{E_t(\lambda_{t+1}R_t)}{E_t\left(\omega_{t+1}R_t^2\right)} - \frac{E_t(\omega_{t+1}R_t)}{E_t\left(\omega_{t+1}R_t^2\right)} r_t^0 x_t, \quad t = 0, 1, \dots, T-1, \tag{4.14}$$

where $\Theta, \lambda_0, \omega_{t+1}$, and λ_{t+1} are given as defined.

Referring to (4.6),

$$\frac{\lambda^*}{2\omega} = \frac{E_0(x_{T\wedge\tau}) - \lambda_0 x_0}{\Theta}. \tag{4.15}$$

Substituting (4.15) back into (4.11), the relationship between $\mathrm{Var}_0(x_{T\wedge\tau})$ and $E_0(x_{T\wedge\tau})$ can be shown as follows:

$$\mathrm{Var}_0(x_{T\wedge\tau}) = \frac{(1-\Theta)}{\Theta}\left[E_0(x_{T\wedge\tau}) - \frac{\lambda_0 x_0}{1-\Theta}\right]^2 + \left(\omega_0 - \frac{\lambda_0^2}{1-\Theta}\right)x_0^2. \tag{4.16}$$

Therefore, the efficient frontier of the original problem $P(\omega)$ is given by (4.16) for

$$E_0(x_{T\wedge\tau}) \in \left[\frac{\lambda_0 x_0}{1-\Theta}, +\infty\right). \tag{4.17}$$

From the efficient frontier (4.16) of the optimal dynamic mean-variance portfolio selection problem with an uncertain exit time, when returns are serially correlated, we can obtain the trade-off between the return and the risk when investor exits from market. Since all of the parameters Θ, λ_0, and ω_0 are functions of p_t and R_t for $t = 0, 1, \dots, T-1$, both the exiting time and the correlations of the risky asset returns have impacts on the optimal strategy and the efficient frontier, and this is quite different from the cases with deterministic terminal time, and the risky asset returns at different periods are independent.

Remark 4.3. In Xu and Li [11], a multi-period portfolio selection problem with serial correlation and a certain exit time is studied. If $p_1 = p_2 = \cdots = p_{T-1} = 0$, $p_T = 1$, and $r_t^0 = r$, $\; t = 0, 1, \dots, T-1$ in our model, our result is exactly the same as the one of Xu and Li [11]. So we generalize the model and results of Xu and Li [11] to the case with an uncertain investment horizon.

5. Extension to the Situation with Multiple Risky Assets

The results in the previous sections can be extended to the general situation with multiple risky assets. Suppose that there are n risky assets and one riskless asset with period-$t+1$ returns r_t^i $(i = 1, 2, \dots, n)$ and r_t^0, respectively. Define $e_t^i = r_t^i - r_t^0$, $e_t = (e_t^1, e_t^2, \dots, e_t^n)'$ and

$U_t = (u_t^1, u_t^2, \ldots, u_t^n)'$ for $i = 1, 2, \ldots, n$ and $t = 0, 1, \ldots, T-1$, where u_t^i is the amount invested in the ith risky asset at time t. In this case, the wealth dynamics is described by

$$x_{t+1} = r_t^0 x_t + e_t' U_t, \quad t = 0, 1, \ldots, T-1. \tag{5.1}$$

Accordingly, the multi-period mean-variance portfolio selection problem with an uncertain exit time and serial correlations can be formulated as

$$\widehat{P}(\omega) \begin{cases} \max\limits_{U} & E_0(x_{T\wedge\tau}) - \omega \mathrm{Var}_0(x_{T\wedge\tau}) \\ \text{s.t.} & x_{t+1} = r_t^0 x_t + e_t' U_t, \quad t = 0, 1, \ldots, T-1, \end{cases} \tag{5.2}$$

where $\omega \geq 0$ is a pregiven parameter, representing the degree of the investor's risk aversion.

With the same method as in the previous section, we can show the following theorem.

Theorem 5.1. *For problem $\widehat{P}(\omega)$, the optimal investment strategy is given by*

$$U_t^* = \frac{1 + 2\omega\tilde{\lambda}_0 x_0}{2\omega\left(1 - \tilde{\Theta}\right)} E_t^{-1}(\tilde{\omega}_{t+1} e_t e_t') E_t\left(\tilde{\lambda}_{t+1} e_t\right) - E_t^{-1}(\tilde{\omega}_{t+1} e_t e_t') E_t(\tilde{\omega}_{t+1} e_t) r_t^0 x_t, \tag{5.3}$$

for $t = 0, 1, \ldots, T-1$, and the efficient frontier is given by

$$Var_0(x_{T\wedge\tau}) = \frac{\left(1 - \tilde{\Theta}\right)}{\tilde{\Theta}} \left[E_0(x_{T\wedge\tau}) - \frac{\tilde{\lambda}_0 x_0}{1 - \tilde{\Theta}}\right]^2 + \left(\tilde{\omega}_0 - \frac{\tilde{\lambda}_0^2}{1 - \tilde{\Theta}}\right) x_0^2, \tag{5.4}$$

where

$$\begin{aligned} &\tilde{\theta}_t = E_t\left(\tilde{\lambda}_{t+1} e_t'\right) E_t^{-1}(\tilde{\omega}_{t+1} e_t e_t') E_t\left(\tilde{\lambda}_{t+1} e_t\right), \qquad \tilde{\Theta} = \sum_{t=1}^{T} E_0\left(\tilde{\theta}_{t-1}\right), \\ &\tilde{\omega}_t = p_t + \left(r_t^0\right)^2 \left[E_t(\tilde{\omega}_{t+1}) - E_t(\tilde{\omega}_{t+1} e_t') E_t^{-1}(\tilde{\omega}_{t+1} e_t e_t') E_t(\tilde{\omega}_{t+1} e_t)\right], \qquad \tilde{\omega}_T = p_T, \\ &\tilde{\lambda}_t = p_t + r_t^0 \left[E_t\left(\tilde{\lambda}_{t+1}\right) - E_t\left(\tilde{\lambda}_{t+1} e_t'\right) E_t^{-1}(\tilde{\omega}_{t+1} e_t e_t') E_t(\tilde{\omega}_{t+1} e_t)\right], \qquad \tilde{\lambda}_T = p_T, \end{aligned} \tag{5.5}$$

for $t = 0, 1, \ldots, T-1$.

Remark 5.2. When the returns rates of the n risky assets are statistically independent, our results reduce to the results of Guo and Hu [3]. That is, we extend the model and results of Guo and Hu [3] to the case with serially correlate returns.

6. Numerical Example

In the previous sections, we derive the optimal strategies and the mean-variance efficient frontiers of two optimal portfolio selection problems with serial correlations and uncertain

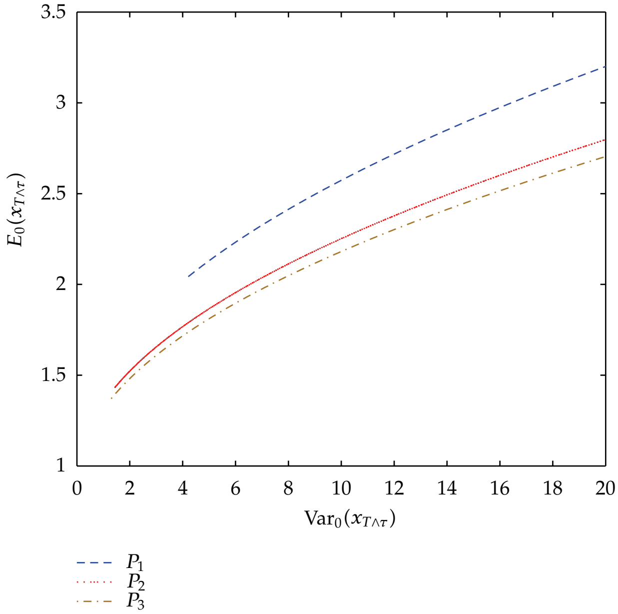

Figure 1: Efficient frontiers with different probability distributions of exit time.

exit time. In this section, we provide a numerical example to demonstrate the impacts of the uncertainty of exit time and the serial correlations of returns on the efficient frontier. In the example, we only consider one risky asset for simplicity and assume that its return rate is subject to AR(1) progress

$$r_t = \mu + \rho(r_{t-1} - \mu) + \sqrt{1-\rho^2}\sigma y_t, \tag{6.1}$$

where μ is the unconditional expectation of r_t, $\rho \in (-1, 1)$ is the first-order autocorrelation coefficient, σ is the unconditional standard deviation of r_t, y_t is a random variable with standard normal distribution, and y_t is independent of y_s $(s < t)$.

To examine the impact of the uncertainty of exit time on the efficient frontier clearly, we compare the efficient frontiers under three different probability distributions of uncertain exit time $t = T \wedge \tau$: $P_1 = (p_0, p_1, p_2, p_3) = (0, 0.09, 0.2, 0.71)$, $P_2 = (p_0, p_1, p_2, p_3) = (0, 0, 0.5, 0.5)$, and $P_3 = (p_0, p_1, p_2, p_3) = (0, 0.2, 0.1, 0.7)$. The remaining parameters are set as $\mu = 0.03$, $b = 0.5$, $\sigma = 0.02$, $x_0 = 1$, and $T = 3$. Figure 1 implies when the investor exits from market later, the investor enjoys more expected wealth returns at the same level of risk than the one terminates the investment earlier.

Furthermore, to test the impact of the correlation coefficient on the efficient frontier, we compare the efficient frontiers under three different settings of correlation coefficient ρ: $\rho_1 = 0.1$, $\rho_2 = 0.4$, and $\rho_3 = 0.7$. The other parameters are given as $\mu = 0.02$, $\sigma = 0.02$, $x_0 = 1$, and $T = 3$. It is obvious from Figure 2 that the investor who takes the risky asset with larger correlation coefficient will suffer less risk than the one with less correlate risky asset to achieve the same expected wealth return.

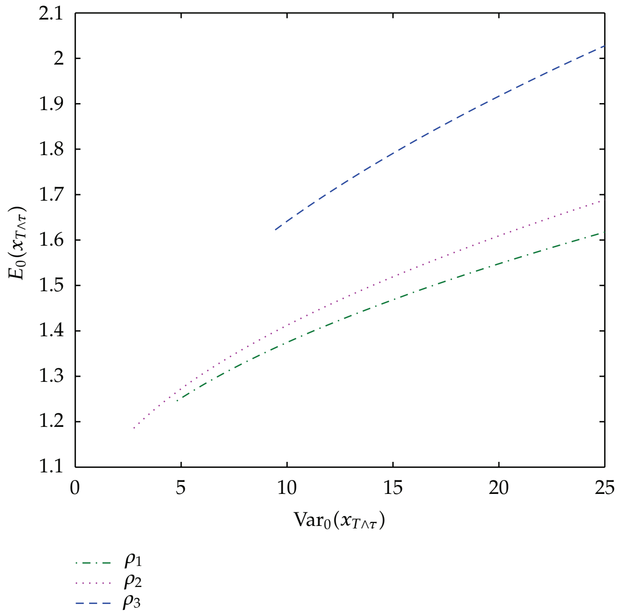

Figure 2: Efficient frontiers with different values of correlation coefficient.

7. Conclusion

In this paper, we consider an optimal portfolio selection problem under multi-period setting and mean-variance framework for an investor, who does not know with certainty when she/he will exit the market in which the capital returns are serially correlated. The problem is much more complicated than the case with deterministic exit time and/or with serially noncorrelate assets returns. By applying the dynamic programming approach and the embedding technique of Li and Ng [2], both the optimal strategy and the efficient frontier of the problem are derived explicitly. Our results include, as special cases, the ones of Li and Ng [2], Guo and Hu [3], and Xu and Li [11]. In addition, a numerical example with AR (1) return process is presented. It shows that both the serial correlations of assets returns and the uncertainty of exit time have significant impacts on the optimal strategy and the efficient frontier.

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