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Research Article

Robust H_∞ Filtering for General Nonlinear Stochastic State-Delayed Systems

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This paper studies the robust H_∞ filtering problem of nonlinear stochastic systems with time delay appearing in state equation, measurement, and controlled output, where the state is governed by a stochastic Itô-type equation. Based on a nonlinear stochastic bounded real lemma and an exponential estimate formula, an exponential (asymptotic) mean square H_∞ filtering design of nonlinear stochastic time-delay systems is presented via solving a Hamilton-Jacobi inequality. As one corollary, for linear stochastic time-delay systems, a Luenberger-type filter is obtained by solving a linear matrix inequality. Two simulation examples are finally given to show the effectiveness of our results.

1. Introduction

Robust H_∞ filtering has been studied extensively for more than two decades, which is very useful in signal processing and engineering applications; see [1–7] and the references therein. Compared with classical Kalman filter, one does not need to know the exact statistic information about the external disturbance in the H_∞ filtering design. H_∞ filtering requires one to design a filter such that the $\mathcal{L}_2$-gain from the external disturbance to the estimation error is below a prescribed level $\gamma > 0$. Stochastic Itô modelling has become more and more important in both theory and practical applications such as in mathematical finance and population models [8]. In recent years, the study of stochastic H_∞ filtering for the systems governed by stochastic Itô-type equations has attracted a great deal of attention, for example, [2, 5, 9]. References [2, 5] presented approaches to linear stochastic delay-free and time delayed H_∞ filtering design via linear matrix inequalities (LMIs), respectively. Reference [9] first solved the nonlinear stochastic delay-free H_∞ filtering problem by means of a stochastic bounded real lemma derived in [10]. References [11, 12], respectively, solve the H_∞ filtering and control of nonlinear stochastic time delayed systems, where time delay only happens in the state equation.

It is well known that time delay phenomena are often encountered in many engineering applications such as network control and communication, and a study of time delay systems has been a popular research topic for a long time [13]. Stochastic time delay systems are ideal models in mathematical finance and population growth theory [8]. Recently, [14, 15] investigated the Kalman filter problem of linear stochastic time delay systems. Reference [5] presented an approach to stochastic H_∞ filtering design for linear uncertain time delay systems via LMIs. Reference [11] first studied the H_∞ design issue for a class of nonlinear stochastic time delayed systems under a stronger assumption (assumption 2.1 of [11]), for which only the state equation contains a time delay. Because, in practice, time delay often exists not only in a state equation but also in a measurement equation and a controlled output, it is necessary to study such a nonlinear stochastic H_∞ filtering design.

To our best knowledge, few works on H_∞ filtering have been reported for general nonlinear stochastic time delay systems. The aim of this paper is to study the robust H_∞ filtering design for nonlinear stochastic state-delayed systems, where the time delay appears in the state equation, measurement equation, and controlled output. Similar to Proposition 1 of [9], a nonlinear stochastic bounded real lemma for time delay systems is obtained, and then an exponential estimate formula is also presented. Finally, based on our developed nonlinear stochastic bounded real lemma and exponential estimate formula, we present a sufficient condition for exponential and asymptotic mean square H_∞ filtering synthesis of nonlinear stochastic time delay systems via solving a constrained Hamilton-Jacobi inequality (HJI), respectively. Compared with the delay-free H_∞ filtering [9], the current HJI depends on more variables due to the appearance of time delays. A key procedure to derive an exponential mean square H_∞ filtering is to develop an exponential estimate formula (Lemma 2.3), which is very useful in its own right. In particular, in the case of linear time-invariant-delayed systems, if a quadratic Lyapunov function is chosen, then the HJI reduces to an LMI, which may be easily solved by the existing Matlab control toolbox [16].

For convenience, we adopt the following traditional notations: A': transpose of the matrix A; $A \geq 0$ ($A > 0$): A is a positive semidefinite (positive definite) matrix; I: identity matrix. $\|x\|$: Euclidean 2-norm of n-dimensional real vector x; $\mathcal{L}^2_{\mathcal{F}}(\mathcal{R}_+, \mathcal{R}^l)$: the space of nonanticipative stochastic processes $y(t)$ with respect to filtration $\mathcal{F}_t$ satisfying $\|y(t)\|^2_{L_2} := E\int_0^\infty \|y(t)\|^2 dt < \infty$; $C^{2,1}(U,T)$: class of functions $V(x,t)$ twice continuously differentiable with respect to $x \in U$ and once continuously differentiable with respect to $t \in T$ except possibly at $x = 0$; $V_t(x,t) := (\partial V(x,t))/\partial t$; $V_x(x,t) := (\partial V(x,t)/\partial x_i)_{n\times 1}$; $V_{xx}(x,t) := (\partial^2 V(x,t)/\partial x_i \partial x_j)_{n\times n}$; $C([-\tau,0],\mathcal{R}^n)$: a vector space of all continuous $\mathcal{R}^n$-valued functions defined on $[-\tau,0]$.

2. Preliminaries

Consider the following nonlinear stochastic time delay system:

$$
\begin{aligned}
dx(t) = &\big(f(x(t), x(t-\tau), t) + g(x(t), x(t-\tau), t)v(t)\big)dt \\
&+ (h(x(t), x(t-\tau), t) + s(x(t), x(t-\tau), t)v(t))dW(t),
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= l(x(t), x(t-\tau), t) + k(x(t), x(t-\tau), t)v(t),\\
z(t) &= m(x(t), x(t-\tau), t),\\
x(t) &= \phi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathcal{R}^n),
\end{aligned}
\tag{2.1}
$$

where $x(t) \in \mathcal{R}^n$ is called the system state, $y(t) \in \mathcal{R}^r$ is the measurement, $W(\cdot)$ is a standard one-dimensional Wiener process defined on a complete filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{R}_+}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in\mathcal{R}_+}$ satisfying usual conditions, $z(t) \in \mathcal{R}^m$ is the state combination to be estimated, $v \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}_+, \mathcal{R}^{n_v})$ stands for the exogenous disturbance signal, which is a square integrable, $\mathcal{F}_t$-adapted stochastic process, and $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathcal{R}^n)$ denotes all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0], R^n)$-valued random variable $\xi(s)$ with $s \in [-\tau, 0]$. We assume that $f, h : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}_+ \mapsto \mathcal{R}^n$, $g, s : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}_+ \mapsto \mathcal{R}^{n\times n_v}$, $l : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}_+ \mapsto \mathcal{R}^r$, $k : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}_+ \mapsto \mathcal{R}^{r\times n_v}$, and $m : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}_+ \mapsto \mathcal{R}^{n_z}$ satisfy the local Lipschitz condition and the linear growth condition, which guarantee that the system (2.1) admits a unique strong solution; see [8]. In addition, suppose that $f(0,0,t) = h(0,0,t) = l(0,0,t) \equiv 0$, so $x \equiv 0$ is an equilibrium point of (2.1).

Since this paper deals with the infinite horizon stochastic H_∞ filtering problem, it is inevitably related to stochastic stability. Hence, we first present the following definition.

Definition 2.1. The nonlinear stochastic time delayed system

$$
\begin{aligned}
dx(t) &= f(x(t), x(t-\tau), t)dt + h(x(t), x(t-\tau), t)dW(t),\\
x(t) &= \phi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathcal{R}^n),
\end{aligned}
\tag{2.2}
$$

is called exponentially mean square stable if there are positive constants A and α such that

$$
E\|x(t)\|^2 \le A\|\phi\|^2 e^{-\alpha t},
\tag{2.3}
$$

where $\|\phi\|^2 = E\max_{-\tau\le t\le 0}\|\phi(t)\|^2$.

Associated with (2.1) and $V : \mathcal{R}^n \times \mathcal{R}_+ \mapsto \mathcal{R}_+$, we define an operator $\mathcal{L}_1 V : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}_+ \mapsto \mathcal{R}$ as follows:

$$
\begin{aligned}
\mathcal{L}_1 V(x, y, t) = {} & V_t(x, t) + V'_x(x, t)\left[f(x, y, t) + g(x, y, t)v(t)\right]\\
& + \frac{1}{2}\left[h(x, y, t) + s(x, y, t)v(t)\right]' V_{xx}(x, t)\left[h(x, y, t) + s(x, y, t)v(t)\right].
\end{aligned}
\tag{2.4}
$$

The following lemma is a generalized version of Proposition 1 in [9], which may be viewed as a nonlinear stochastic bounded real lemma for time delayed systems.

Lemma 2.2. *Consider the following input-output system:*

$$\begin{aligned} dx(t) &= \big(f(x(t), x(t-\tau), t) + g(x(t), x(t-\tau), t)v(t)\big)dt \\ &\quad + (h(x(t), x(t-\tau), t) + s(x(t), x(t-\tau), t)v(t))dW(t), \\ z(t) &= m(x(t), x(t-\tau), t), \quad x(t) = \phi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathcal{R}^n). \end{aligned} \tag{2.5}$$

If there exists a positive definite Lyapunov function $V(x,t) \in C^{2,1}(\mathcal{R}^n, \mathcal{R}_+)$ solving the following HJI:

$$\begin{aligned} \Gamma(x, y, t) \ &:= V_t(x,t) + V'_x(x,t)f(x,y,t) \\ &\quad + \frac{1}{2}(V'_x(x,t)g(x,y,t) + h'(x,y,t)V_{xx}(x,t)s(x,y,t)) \\ &\quad \times\big(\gamma^2 I - s'(x,y,t)V_{xx}(x,t)s(x,y,t)\big)^{-1} \\ &\quad \times\big(g'(x,y,t)V_x(x,t) + s'(x,y,t)V_{xx}(x,t)h(x,y,t)\big) \\ &\quad + \frac{1}{2}\|z(t)\|^2 + \frac{1}{2}h'(x,y,t)V_{xx}(x,t)h(x,y,t) < 0 \\ \gamma^2 I - s'(x,y,t)&V_{xx}(x,t)s(x,y,t) > 0, \quad \forall (x,y,t) \in \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}_+, \\ &V(0,0) = 0 \end{aligned} \tag{2.6}$$

for some $\gamma > 0$, then the following inequality:

$$\|z(t)\|^2_{L_2} \le \gamma^2 \|v(t)\|^2_{L_2}, \quad \forall v \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_v}),\ v \ne 0, \tag{2.7}$$

holds with initial state $x(s) = 0$, a.s., for all, $s \in [-\tau, 0]$.

Proof. See Appendix A. □

Lemma 2.3. *Consider the unforced system*

$$\begin{aligned} dx(t) &= f(x(t), x(t-\tau), t)dt + h(x(t), x(t-\tau), t)dW(t), \\ x(t) &= \phi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathcal{R}^n). \end{aligned} \tag{2.8}$$

If there exists a positive definite Lyapunov function $V(x,t) \in C^{2,1}(R^n, [-\tau, \infty))$, $c_1, c_2, c_3, c_4 > 0$ with $c_1c_3 > c_2c_4$ satisfying the following conditions:

(i) $c_1\|x\|^2 \le V(x,t) \le c_2\|x\|^2$, *for all* $(x,t) \in \mathcal{R}^n \times [-\tau, \infty)$,

(ii) $\mathcal{L}_1 V(x,y,t)|_{v=0} \le -c_3\|x\|^2 + c_4\|y\|^2$, *for all* $t > 0$,

then

$$E\|x(t)\|^2 \le \begin{cases} \dfrac{(c_4c_2/c_1)\tau + c_2}{c_1}\|\phi\|^2 e^{-(c_3/c_2)t}, & 0 \le t \le \tau, \\ \dfrac{(c_4c_2/c_1)\tau + c_2}{c_1}\|\phi\|^2 e^{-((c_3/c_2)-(c_4/c_1))t}, & t > \tau, \end{cases} \tag{2.9}$$

that is, (2.8) *is exponentially mean square stable.*

Proof. See Appendix B. □

In what follows, we construct the following filtering equation for the estimation of $z(t)$:

$$\begin{aligned} d\hat{x}(t) &= \hat{f}(\hat{x}(t), \hat{x}(t-\tau), t)dt + \hat{G}(\hat{x}(t), \hat{x}(t-\tau), t)y(t)\, dt \\ \hat{z}(t) &= \hat{m}(\hat{x}(t), \hat{x}(t-\tau), t), \quad \hat{x}(0) = 0, \end{aligned} \tag{2.10}$$

where $\hat{f}$, $\hat{G}$, and $\hat{m}$ that are to be determined are matrices of appropriate dimensions. One may find that (2.10) is more general, which includes the following Luenberger-type filtering as a special form:

$$\begin{aligned} d\hat{x}(t) &= f(\hat{x}(t), \hat{x}(t-\tau), t)dt + G(\hat{x}(t), \hat{x}(t-\tau), t)\big(y(t) - l(\hat{x}(t), \hat{x}(t-\tau), t)\big)dt, \\ \hat{z}(t) &= m(\hat{x}(t), \hat{x}(t-\tau), t), \qquad \hat{x}(0) = 0. \end{aligned} \tag{2.11}$$

Set $\eta(t) = [x'(t)\hat{x}'(t)]'$, and let

$$\tilde{z}(t) = z(t) - \hat{z}(t) = m(x(t), x(t-\tau), t) - \hat{m}(\hat{x}(t), \hat{x}(t-\tau), t) \tag{2.12}$$

denote the estimation error; then we get the following augmented system:

$$\begin{aligned} d\eta(t) &= \big(f_e(\eta(t)) + g_e(\eta(t))v(t)\big)dt + \big(h_e(\eta(t)) + s_e(\eta(t))v(t)\big)dW(t), \\ \tilde{z}(t) &= z(t) - \hat{z}(t) = m(x(t), x(t-\tau), t) - \hat{m}(\hat{x}(t), \hat{x}(t-\tau), t), \\ \eta(t) &= \begin{bmatrix} \phi(t) \\ 0 \end{bmatrix}, \quad \phi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathcal{R}^n), \quad \forall t \in [-\tau, 0], \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} f_e(\eta(t)) &= \begin{bmatrix} f(x(t), x(t-\tau), t) \\ \hat{f}(\hat{x}(t), \hat{x}(t-\tau), t) + \hat{G}(\hat{x}(t), \hat{x}(t-\tau), t)l(x(t), x(t-\tau), t) \end{bmatrix}, \\ g_e(\eta(t)) &= \begin{bmatrix} g(x(t), x(t-\tau), t) \\ \hat{G}(\hat{x}(t), \hat{x}(t-\tau), t)k(x(t), x(t-\tau), t) \end{bmatrix}, \\ h_e(\eta(t)) &= \begin{bmatrix} h(x(t), x(t-\tau), t) \\ 0 \end{bmatrix}, \qquad s_e(\eta(t)) = \begin{bmatrix} s(x(t), x(t-\tau), t) \\ 0 \end{bmatrix}. \end{aligned} \tag{2.14}$$

In Section 3, we let $\mathcal{L}_\eta$ denote the infinitesimal operator of system (2.13). According to different requirements for internal stability, we are in a position to define various of H_∞ filters as follows.

Definition 2.4 (exponential mean square H_∞ filtering). Find the matrices $\hat{f}$, $\hat{G}$, and $\hat{m}$ in (2.10), such that

(i) the equilibrium point $\eta \equiv 0$ of the augmented system (2.13) is exponentially mean square stable in the case $v = 0$,

(ii) for a given disturbance attenuation level $\gamma > 0$, the following H_∞ performance holds for $x(t) \equiv 0$ on $t \in [-\tau, 0]$:

$$\|\tilde{z}\|_{L_2}^2 \leq \gamma^2 \|v\|_{L_2}^2, \quad \forall v \in \mathcal{L}_{\mathcal{F}}^2(\mathcal{R}^+, \mathcal{R}^{n_v}),\ v \neq 0. \tag{2.15}$$

Definition 2.5 (asymptotic mean square H_∞ filtering). If in (i) of Definition 2.4 the equilibrium point $\eta \equiv 0$ of the augmented system (2.13) is asymptotically mean square stable, that is,

$$\lim_{t \to \infty} E\|\eta(t)\|^2 = 0 \tag{2.16}$$

and (2.15) holds, then (2.10) is called an asymptotic mean square H_∞ filter.

3. Main Results

Our first main result is about exponential mean square H_∞ filter.

Theorem 3.1. *Suppose that there exists a positive Lyapunov function $V(\eta, t) = V(x, \hat{x}, t) \in C^{2,1}(\mathcal{R}^{2n} \times [-\tau, \infty))$, $c_1, c_2, c_3, c_4 > 0$ with $c_1 c_3 > c_2 c_4$, such that*

$$\begin{aligned} &c_1\left(\|x\|^2 + \|\hat{x}\|^2\right) \leq V(x, \hat{x}, t) \leq c_2\left(\|x\|^2 + \|\hat{x}\|^2\right), \quad \forall (x, \hat{x}, t) \in \mathcal{R}^{2n} \times [-\tau, \infty), \\ &-\frac{1}{2}\|m(x, y, t) - \hat{m}(\hat{x}, \hat{y}, t)\|^2 \leq -c_3\left(\|x\|^2 + \|\hat{x}\|^2\right) + c_4\left(\|y\|^2 + \|\hat{y}\|^2\right), \quad \forall t > 0. \end{aligned} \tag{3.1}$$

For given disturbance attenuation level $\gamma > 0$, if $V(\eta, t)$ solves the following HJI:

$$\begin{aligned} \Gamma(x, y, \hat{x}, \hat{y}) \ :=\ & V_t + V_x' f(x, y, t) + V_{\hat{x}}'\left(\hat{f}(\hat{x}, \hat{y}, t) + \widehat{G}(\hat{x}, \hat{y}, t) l(x, y, t)\right) \\ & + \frac{1}{2}\Theta'(x, \hat{x}, y, \hat{y}, t)\left(\gamma^2 I - s'(x, y, t) V_{xx} s(x, y, t)\right)^{-1}\Theta(x, \hat{x}, y, \hat{y}, t) \\ & + \frac{1}{2}\|m(x, y, t) - \hat{m}(\hat{x}, \hat{y}, t)\|^2 + \frac{1}{2}h'(x, y, t) V_{xx} h(x, y, t) < 0, \end{aligned} \tag{3.2}$$

$$\gamma^2 I - s'(x, y, t) V_{xx} s(x, y, t) > 0, \quad \forall (x, y, \hat{x}, \hat{y}, t) \in \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}_+,$$

$$V(0, 0) = 0$$

for some matrices $\hat{f}, \widehat{G}$, and $\hat{m}$ of suitable dimensions, then the exponential mean square H_∞ filtering is obtained by (2.10), *where*

$$\Theta'(x, \hat{x}, y, \hat{y}, t) = V_x' g(x, y, t) + V_{\hat{x}}' \widehat{G}(\hat{x}, \hat{y}, t) k(x, y, t) + h'(x, y, t) V_{xx} s(x, y, t). \tag{3.3}$$

Proof of Theorem 3.1. In Lemma 2.2, we substitute $V(x,\hat{x},t)$, $\tilde{z}=m(x,y,t)-\hat{m}(\hat{x},\hat{y},t)$,

$$f_e=\begin{bmatrix} f(x,y,t) \\ \hat{f}(\hat{x},\hat{y},t)+\hat{G}(\hat{x},\hat{y},t)l(x,y,t)\end{bmatrix},\qquad g_e=\begin{bmatrix} g(x,y,t) \\ \hat{G}(\hat{x},\hat{y},t)k(x,y,t)\end{bmatrix},$$
$$h_e=\begin{bmatrix} h(x,y,t) \\ 0\end{bmatrix},\qquad s_e=\begin{bmatrix} s(x,y,t) \\ 0\end{bmatrix}, \tag{3.4}$$

for $V(x,t), z, f, g, h$, and s, respectively; then, by a series of simple computations, (2.15) is obtained.

Next, we show the augmented system (2.13) to be exponential mean square stable for $v\equiv 0$. Set

$$\mathcal{L}_\eta^{v=0}V(x,\hat{x},t):=V_t+V_\eta' f_e+\frac{1}{2}h_e' V_{\eta\eta}h_e. \tag{3.5}$$

By (3.2),

$$\begin{aligned}\mathcal{L}_\eta^{v=0}V(x,\hat{x},t) &< -\frac{1}{2}\left\|m(x,y,t)-\hat{m}(\hat{x},\hat{y},t)\right\|^2 \\ &\quad -\frac{1}{2}\Theta'(x,\hat{x},y,\hat{y},t)\left(\gamma^2 I-s'(x,y,t)V_{xx}s(x,y,t)\right)^{-1}\Theta(x,\hat{x},y,\hat{y},t) \\ &\le -\frac{1}{2}\left\|m(x,y,t)-\hat{m}(\hat{x},\hat{y},t)\right\|^2 \\ &\le -c_3\left(\|x\|^2+\|\hat{x}\|^2\right)+c_4\left(\|y\|^2+\|\hat{y}\|^2\right).\end{aligned} \tag{3.6}$$

Applying Lemma 2.3, we know that (2.13) is internally stable in exponential mean square sense. The proof of Theorem 3.1 is ended. □

Inequality (3.2) is a constrained HJI, which is not easily tested in practice. However, if in (2.1), $s\equiv 0$, that is, only the state depends on noise, then the constraint condition $\gamma^2 I - s'(x,y,t)V_{xx}s(x,y,t)>0$ holds automatically, and HJI (3.2) becomes an unconstrained one.

The following theorem is about asymptotic mean square H_∞ filter, which is weaker than the exponential mean square H_∞ filter.

Theorem 3.2. *Assume that $V(\eta,t)\in C^{2,1}(\mathcal{R}^{2n},\mathcal{R}_+)$ has an infinitesimal upper limit, that is,*

$$\lim_{\|\eta\|\to\infty}\inf_{t>0}V(\eta,t)=\infty. \tag{3.7}$$

Additionally, one assume that $V(\eta,t)>c\|\eta\|^2$ for some $c>0$. If $V(\eta,t)$ solves HJI (3.2), *then* (2.10) *is an asymptotic mean square H_∞ filter.*

Proof. Obviously, it only needs to show that (2.13) is asymptotically mean square stable while $v=0$. From (3.6), $\mathcal{L}_\eta^{v=0}V(x,\hat{x},t)<0$, so (2.13) is globally asymptotically stable in probability 1 according to the result of [17].

By Itô's formula and the property of stochastic integration, we have

$$\begin{aligned}EV(\eta(t),t) &= EV(\eta(0),0) + E\int_0^t \mathcal{L}_\eta V(\eta(s),s)|_{v=0}ds + E\int_0^t h_e'(\eta(s),s)V_\eta(\eta(s),s)dW(s)\\ &= EV(\eta(0),0) + E\int_0^t \mathcal{L}_\eta V(\eta(s),s)|_{v=0}\, ds\\ &\leq EV(\eta(0),0) - \frac{1}{2}E\int_0^t \|m(x(s),x(s-\tau),s) - \widehat{m}(\widehat{x}(s),\widehat{x}(s-\tau),s)\|^2 ds\\ &\leq EV(\eta(0),0) < \infty.\end{aligned} \tag{3.8}$$

Set $\widetilde{\mathcal{F}}_t = \mathcal{F}_t \cup \sigma(y(s), 0 \leq s \leq t)$; then (3.8) yields

$$E\Big[V(\eta(t),t) \mid \widetilde{\mathcal{F}}_s\Big] \leq V(\eta(s),s) \quad \text{a.s.}, \tag{3.9}$$

which says that $\{V(\eta(t),t), \widetilde{\mathcal{F}}_t, 0 \leq s \leq t\}$ is a nonnegative supermartingale with respect to $\{\widetilde{\mathcal{F}}_t\}_{t\geq 0}$. By Doob's convergence theorem [18] and the fact that $\lim_{t\to\infty}\eta(t) = 0$ a.s., it immediately yields $V(\eta(\infty),\infty) = \lim_{t\to\infty}V(\eta(t),t) = 0$ a.s. Moreover, $\lim_{t\to\infty}EV(\eta(t),t) = EV(\eta(\infty),\infty) = EV(0,\infty) = 0$. Because $V(\eta,t) \geq c\|\eta\|^2$ for some $c > 0$, it follows that $\lim_{t\to\infty}E\|\eta(t)\|^2 = 0$. This theorem is proved. □

As one application of Theorem 3.2, we concentrate our attention on linear stochastic time delay H_∞ filtering design. Consider the following linear time-invariant stochastic time delay system:

$$\begin{aligned}dx(t) &= (A_0x(t) + A_1x(t-\tau) + Bv(t))dt + (C_0x(t) + C_1x(t-\tau) + Dv(t))dW(t),\\ y(t) &= l_0x(t) + l_1x(t-\tau) + Kv(t),\\ z(t) &= m_0x(t) + m_1x(t-\tau),\\ x(t) &= \phi(t) \in C^b_{\mathcal{F}_0}([-\tau,0];\mathcal{R}^n),\end{aligned} \tag{3.10}$$

where, in (3.10), all coefficient matrices are assumed to be constant. Consider the following Luenberger-type filtering equation:

$$\begin{aligned}d\widehat{x}(t) &= A_0\widehat{x}(t) + A_1\widehat{x}(t-\tau)dt + G\big(y(t) - l_0\widehat{x}(t) - l_1\widehat{x}(t-\tau)\big)dt,\\ \widehat{z}(t) &= m_0\widehat{x}(t) + m_1\widehat{x}(t-\tau), \qquad \widehat{x}(0) = 0,\end{aligned} \tag{3.11}$$

with G a constant matrix to be determined later. In this case,

$$\widehat{f}(\widehat{x}(t),\widehat{x}(t-\tau),t) = A_0\widehat{x}(t) + A_1\widehat{x}(t-\tau) - G(l_0\widehat{x}(t) + l_1\widehat{x}(t-\tau)), \qquad \widehat{G} = G. \tag{3.12}$$

Set

$$V(x,\hat{x},t) = x'(t)Px(t) + \int_{t-\tau}^{t} x'(\theta)P_1x(\theta)d\theta + \hat{x}'(t)Q\hat{x}(t) + \int_{t-\tau}^{t} \hat{x}'(\theta)Q_1\hat{x}(\theta)d\theta, \tag{3.13}$$

where $P > 0$, $P_1 > 0$, $Q > 0$, and $Q_1 > 0$ are to be determined. Then by a series of computations, we have from HJI (3.2) that

$$\begin{aligned}
V_t &= x'(t)P_1x(t) - x'(t-\tau)P_1x(t-\tau) + \hat{x}'(t)Q_1\hat{x}(t) - \hat{x}'(t-\tau)Q_1\hat{x}(t-\tau),\\
V_x'f(x,y,t) &= \begin{bmatrix} x' & y' & \hat{x}' & \hat{y}' \end{bmatrix} \begin{bmatrix} PA_0 + A_0'P & \star & 0 & 0 \\ A_1'P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \hat{x} \\ \hat{y} \end{bmatrix},\\
V_{\hat{x}}'Gl(x,y,t) &= \begin{bmatrix} x' & y' & \hat{x}' & \hat{y}' \end{bmatrix} \begin{bmatrix} 0 & 0 & \star & 0 \\ 0 & 0 & \star & 0 \\ QGl_0 & QGl_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \hat{x} \\ \hat{y} \end{bmatrix},\\
V_{\hat{x}}'\hat{f}(\hat{x},\hat{y},t) &= \begin{bmatrix} x' & y' & \hat{x}' & \hat{y}' \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q(A_0 - Gl_0) + (A_0 - l_0'G')Q & \star \\ 0 & 0 & (A_1 - Gl_1)'Q & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \hat{x} \\ \hat{y} \end{bmatrix},\\
\frac{1}{2}h'(x,y,t)V_{xx}'h(x,y,t) &= \begin{bmatrix} x' & y' & \hat{x}' & \hat{y}' \end{bmatrix} \begin{bmatrix} C_0'PC_0 & \star & 0 & 0 \\ C_1'PC_0 & C_1'PC_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \hat{x} \\ \hat{y} \end{bmatrix},\\
\frac{1}{2}\left\| m(x,y,t) - m(\hat{x},\hat{y},t) \right\|^2 &= \begin{bmatrix} x' & y' & \hat{x}' & \hat{y}' \end{bmatrix} \frac{1}{2} \begin{bmatrix} m_0'm_0 & \star & \star & \star \\ m_1'm_0 & m_1'm_1 & \star & \star \\ -m_0'm_0 & -m_0'm_1 & m_0'm_0 & \star \\ -m_1'm_0 & -m_1'm_1 & m_1'm_0 & m_1'm_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ \hat{x} \\ \hat{y} \end{bmatrix},\\
&\frac{1}{2}\Theta'(x,\hat{x},y,\hat{y},t)\left(\gamma^2 I - s'(x,y,t)V_{xx}s(x,y,t)\right)^{-1}\Theta(x,\hat{x},y,\hat{y},t)\\
&= \begin{bmatrix} x' & y' & \hat{x}' & \hat{y}' \end{bmatrix} \begin{bmatrix} C_0'PD + 2PB \\ C_1'PD \\ 2QGK \\ 0 \end{bmatrix} \frac{1}{2}\left[\gamma^2 I - 2D'PD\right]^{-1} \begin{bmatrix} C_0'PD + 2PB \\ C_1'PD \\ 2QGK \\ 0 \end{bmatrix}' \begin{bmatrix} x \\ y \\ \hat{x} \\ \hat{y} \end{bmatrix},
\end{aligned} \tag{3.14}$$

where $\star$ is derived by symmetry. Hence, HJI (3.2) is equivalent to

$$\begin{bmatrix} A_{11} & \star & \star & \star \\ A_{21} & A_{22} & \star & \star \\ QGl_0 - \frac{1}{2}m_0'm_0 & QGl_1 - \frac{1}{2}m_0'm_1 & A_{33} & \star \\ -\frac{1}{2}m_1'm_0 & -\frac{1}{2}m_1'm_1 & (A_1 - Gl_1)'Q + \frac{1}{2}m_1'm_0 & \frac{1}{2}m_1'm_1 - Q_1 \end{bmatrix}$$
$$+ \begin{bmatrix} C_0'PD + 2PB \\ C_1'PD \\ 2QGK \\ 0 \end{bmatrix} \frac{1}{2}\left[\gamma^2 I - 2D'PD\right]^{-1} \begin{bmatrix} C_0'PD + 2PB \\ C_1'PD \\ 2QGK \\ 0 \end{bmatrix}' < 0, \tag{3.15}$$
$$\gamma^2 I - 2D'PD > 0$$

with

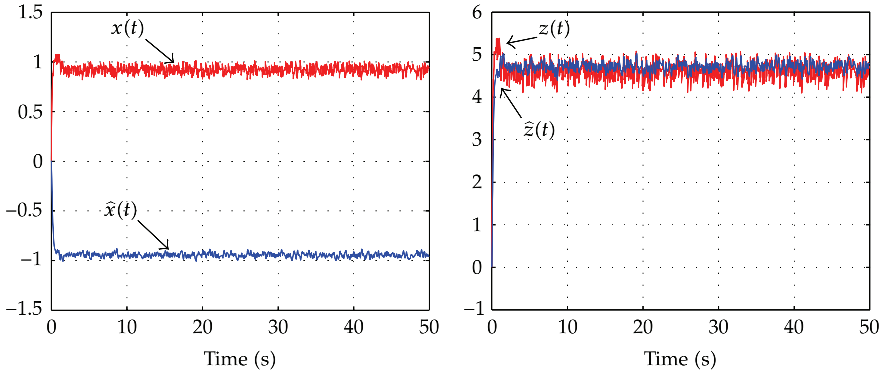
$$\begin{gathered} A_{11} = PA_0 + A_0'P + C_0'PC_0 + P_1 + \frac{1}{2}m_0'm_0, \\ A_{21} = A_1'P + C_1'PC_0 + \frac{1}{2}m_1'm_0, \qquad A_{22} = -P_1 + C_1'PC_1 + \frac{1}{2}m_1'm_1, \\ A_{33} = Q(A_0 - Gl_0) + (A_0 - Gl_0)'Q + Q_1 + \frac{1}{2}m_0'm_0. \end{gathered} \tag{3.16}$$

By Schur's complement, (3.15) are equivalent to

$$\begin{bmatrix} A_{11} & \star & \star & \star & \star \\ A_{21} & A_{22} & \star & \star & \star \\ G_1 l_0 - \frac{1}{2}m_0'm_0 & G_1 l_1 - \frac{1}{2}m_0'm_1 & A_{33} & \star & \star \\ -\frac{1}{2}m_1'm_0 & -\frac{1}{2}m_1'm_1 & A_1 Q - l_1'G_1' + \frac{1}{2}m_1'm_0 & \frac{1}{2}m_1'm_1 - Q_1 & 0 \\ 2B'P + D'PC_0 & D'PC_1 & 2K'G_1' & 0 & -2\gamma^2 I + 4D'PD \end{bmatrix} < 0 \tag{3.17}$$

with $QG = G_1$. Obviously, (3.17) is an LMI on P, P_1, Q, Q_1, G_1. By Theorem 3.2, we immediately obtain the following corollary.

Corollary 3.3. *If* (3.17) *is feasible with solutions $P > 0, P_1 > 0, Q > 0, Q_1 > 0$, and G_1, then* (3.11) *is an asymptotic mean square H_∞ filter with the filtering gain $G = Q^{-1}G_1$.*



(a) The trajectories of x and $\widehat{x}$ (b) The trajectories of z and $\widehat{z}$

Figure 1: Simulation results for Example 4.1.

4. Illustrative Examples

Below, we give two examples to illustrate the validity of our developed theory in the above section.

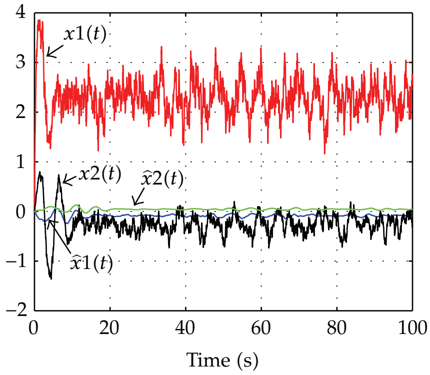
Example 4.1 (one-dimensional exponential mean square H_∞ filtering). Suppose that a stochastic signal z is generated by the following nonlinear stochastic system driven by a standard Wiener process and corrupted by a stochastic external disturbance v, where the power of v is 0.05. We construct an H_∞ filter to estimate z from the measurement signal y:

$$\begin{gathered} dx(t) = \left[\left(-10x(t) - x(t)x^2(t-\tau)\right) + x(t-\tau)v(t)\right]dt + x(t)dW(t) \\ x(t) = \phi(t) \in C^b_{\mathcal{F}_0}([-\tau,0];\mathcal{R}), \\ y(t) = -\frac{25}{2}x(t) - 2x(t)x(t-\tau) + v(t), \\ z(t) = 5x(t). \end{gathered} \tag{4.1}$$

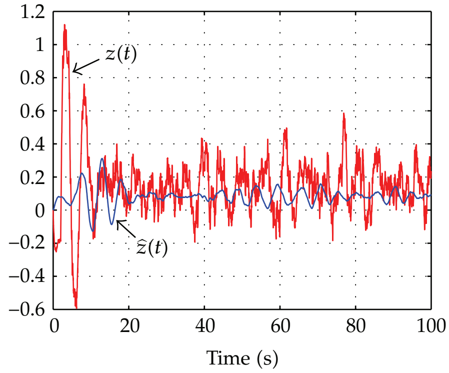
For given disturbance attenuation level $\gamma = 1$, according to Theorem 3.1, in order to determine the filtering parameters $\widehat{f}, \widehat{G}$, and $\widehat{m}$, we must solve HJI (3.2). Set $V(x,\widehat{x}) = x^2 + \widehat{x}^2$, $\widehat{m} = -5\widehat{x}$; then (3.1) hold obviously. In addition, we can easily test that $\Gamma(x,y,\widehat{x},\widehat{y}) = -6.5x^2 - 13.5\widehat{x}^2 < 0$ when we take $\widehat{f} = -14\widehat{x}$, $\widehat{G} = 1$, $\widehat{m} = 5\widehat{x}$. So the exponential mean square H_∞ filter is given as

$$d\widehat{x}(t) = -14\widehat{x}(t)dt + y(t)dt, \qquad \widehat{z}(t) = -5\widehat{x}(t). \tag{4.2}$$

Because there may be more than one triple $(\widehat{f}, \widehat{G}, \widehat{m})$ solving HJI (3.2), H_∞ filtering is in general not unique. The simulation result can be seen in Figures 1(a) and 1(b).



(a) The trajectories of x and $\hat{x}$



(b) The trajectories of z and $\hat{z}$

Figure 2: Simulation results for Example 4.2.

Example 4.2 (linear mean square H_∞ filtering). In (3.10), we take the power of v to be 0.01, and

$$A_0 = \begin{bmatrix} -2.6 & -0.2 \\ 0.4 & -1.8 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1.8 & 0.2 \\ -0.7 & -0.9 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.7 \\ 0.94 \end{bmatrix},$$
$$C_0 = \begin{bmatrix} -0.8 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} -0.3 & 0.4 \\ 0.21 & -1.05 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \tag{4.3}$$
$$l_0 = \begin{bmatrix} 1.3 & 0.8 \end{bmatrix}, \qquad l_1 = \begin{bmatrix} 1.2 & 3 \end{bmatrix}, \qquad K = 0.5,$$
$$m_0 = \begin{bmatrix} -0.11 & 0.3 \end{bmatrix}, \qquad m_1 = \begin{bmatrix} 0.28 & 0.63 \end{bmatrix}.$$

Obviously, substituting the above data into (3.17) with $\gamma = 2$ and solving LMI (3.17), we have

$$P = \begin{bmatrix} 1.6095 & -0.0293 \\ -0.0293 & 0.7909 \end{bmatrix} > 0, \qquad P_1 = \begin{bmatrix} 3.8622 & -0.5054 \\ -0.5054 & 1.6277 \end{bmatrix} > 0,$$
$$Q = \begin{bmatrix} 1.0009 & 0.0275 \\ 0.0275 & 1.3260 \end{bmatrix} > 0, \qquad Q_1 = \begin{bmatrix} 3.6487 & 0.1333 \\ 0.1333 & 3.6199 \end{bmatrix} > 0, \tag{4.4}$$
$$G_1 = \begin{bmatrix} -0.0772 \\ 0.0235 \end{bmatrix}, \qquad G = Q^{-1}G_1 = \begin{bmatrix} -0.0777 \\ 0.0194 \end{bmatrix}.$$

The simulation result can be found in Figures 2(a) and 2(b).

5. Conclusions

This paper presents an approach to the design of H_∞ filtering for general nonlinear stochastic time delay systems via solving HJI (3.2). Although it is difficult to solve the general HJI (3.2), under some special cases such as linear time delay systems, HJI (3.2) reduces to LMIs, which can be easily solved. How to solve HJI (3.2) is a very valuable research topic, which deserves further study. In addition, in order to avoid solving HJI (3.2), a possible scheme is to adopt a fuzzy linearized method for the original system (2.1) as done in [19].

Appendices

A. Proof of Lemma 2.2

As done in [9], applying the completing squares technique and considering (2.6), it is easy to obtain

$$\mathcal{L}_1 V(x,y,t) \le \frac{1}{2}\gamma^2 v'(t)v(t) - \frac{1}{2}z'(t)z(t). \tag{A.1}$$

In addition, by Itô's formula, for any $T > 0$, we have

$$\begin{aligned} EV(x(T),T) &= EV(x(0),0) + E\int_0^T dV(x(s),s) \\ &= EV(x(0),0) + E\int_0^T \mathcal{L}V(x(t),t)dt \\ &= EV(x(0),0) + E\int_0^T \mathcal{L}_1 V(x(t),x(t-\tau,t))dt \\ &\le EV(x(0),0) + \frac{1}{2}E\int_0^T \left(\gamma^2\|v(t)\|^2 - \|z(t)\|^2\right)dt, \end{aligned} \tag{A.2}$$

where, in (A.2), $\mathcal{L}$ is the so-called infinitesimal operator of (2.5), which is defined by

$$\begin{aligned} \mathcal{L}V(x(t),t) = V_t(x(t),t) &+ V_x'(x(t),t)\left[f(x(t),x(t-\tau),t) + g(x(t),x(t-\tau),t)v(t)\right] \\ &+ \frac{1}{2}[h(x(t),x(t-\tau),t) + s(x(t),x(t-\tau),t)v(t)]'V_{xx}(x(t),t) \\ &\cdot [h(x(t),x(t-\tau),t) + s(x(t),x(t-\tau),t)v(t)]. \end{aligned} \tag{A.3}$$

In view of V being positive and $V(0,0) = 0$, it follows that for the zero initial condition $x(s) \equiv 0$, for all $s \in [-\tau,0]$,

$$E\int_0^T \|z(t)\|^2 dt \le E\int_0^T \|v(t)\|^2 dt, \tag{A.4}$$

which proves Lemma 2.2.

B. Proof of Lemma 2.3

By (A.2), we know that, for any $t > 0$,

$$EV(x(t),t) - EV(x(0),0) = \int_0^t E\mathcal{L}_1 V(x(s),x(s-\tau),s)|_{v=0} ds. \tag{B.1}$$

By given conditions (i) and (ii), (B.1) yields

$$\begin{aligned} EV(x(t),t) - EV(x(0),0) &\leq -c_3 \int_0^t E\|x(s)\|^2 ds + c_4 \int_0^t E\|x(s-\tau)\|^2 ds \\ &\leq -\frac{c_3}{c_2} \int_0^t EV(x(s),s) ds + \frac{c_4}{c_1} \int_0^t EV(x(s-\tau),s-\tau) ds. \end{aligned} \tag{B.2}$$

When $0 \leq t \leq \tau$, we have

$$EV(x(t),t) \leq \left(\frac{c_4 c_2}{c_1}\tau + c_2\right)\|\phi\|^2 - \frac{c_3}{c_2}\int_0^t EV(x(s),s) ds. \tag{B.3}$$

Applying Gronwall's inequality, it follows that $EV(x(t),t) \leq ((c_4 c_2/c_1)\tau + c_2)\|\phi\|^2 e^{-(c_3/c_2)t}$. Again, using condition (i),

$$E\|x(t)\|^2 \leq \frac{((c_4 c_2/c_1)\tau + c_2)}{c_1}\|\phi\|^2 e^{-(c_3/c_2)t}. \tag{B.4}$$

When $t > \tau > 0$, letting $\mu = s - \tau$, (B.2) yields

$$\begin{aligned} EV(x(t),t) &\leq c_2\|\phi\|^2 - \frac{c_3}{c_2}\int_0^t EV(x(s),s) ds + \frac{c_4}{c_1}\int_{-\tau}^{t-\tau} EV(x(\mu),\mu) dt\mu \\ &\leq c_2\|\phi\|^2 - \frac{c_3}{c_2}\int_0^t EV(x(s),s) ds \\ &\quad + \frac{c_4}{c_1}\int_{-\tau}^0 EV(x(\mu),\mu) d\mu + \frac{c_4}{c_1}\int_0^t EV(x(\mu),\mu) d\mu \\ &= \left(\frac{c_4 c_2}{c_1}\tau + c_2\right)\|\phi\|^2 - \left(\frac{c_3}{c_2} - \frac{c_4}{c_1}\right)\int_0^t EV(x(s),s) ds. \end{aligned} \tag{B.5}$$

Repeating the same procedure as above, we have

$$E\|x(t)\|^2 \leq \frac{(c_4 c_2/c_1)\tau + c_2}{c_1}\|\phi\|^2 e^{-((c_3/c_2)-(c_4/c_1))t}. \tag{B.6}$$

Lemma 2.3 is hence proved.

Acknowledgments

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