Research Article

Enhancement of the Quality and Robustness in Synchronization of Nonlinear Lur'e Dynamical Networks

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In order to improve the synchronous reliability and dependability of complex dynamical networks, methods need to be proposed to enhance the quality and robustness of the synchronization scheme. The present study focuses on the robust fault detection issue within the synchronization for a class of nonlinear dynamical networks composed by identical Lur'e systems. Sufficient conditions in terms of linear matrix inequalities (LMIs) are established to guarantee global robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization of the network. Under such a synchronization scheme, the error dynamical system is globally asymptotically stable, the effect of external disturbances is suppressed, and at the same time, the network is sensitive to possible faults based on a mixed $\mathcal{H}_-/\mathcal{H}_\infty$ performance. The fault sensitivity $\mathcal{H}_-$ index, moreover, can be optimized via a convex optimization algorithm. The effectiveness and applicability of the analytical results are demonstrated through a network example composed by the Chua's circuit, and it shows that the quality and robustness of synchronization has been greatly enhanced.

1. Introduction

In daily life, many physical systems can be characterized by various complex network models whose nodes are the elements of the network and the edges represent the interactions among them [1]. Treated as typical versions of large-scale systems, the notion of complex dynamical networks has drawn more and more attentions in recent years [2, 3]. One of the interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes, which is a kind of typical collective behaviors and basic motions

in nature [4–9]. Aiming at deriving global synchronization conditions, attempts have been made to consider the synchronization for a special class of networks composed of nonlinear Lur'e systems [10–12]. The main reason is that, in various fields of theory and engineering applications, vast amounts of nonlinear systems can be represented as the Lur'e type, including the Chua's circuit [13], the Goodwin model [14], and the swarm model [15]. Primary methods of dealing with such problems, among others, are developed under the framework of absolute stability theory [16].

In order to improve the synchronous reliability and dependability, methods have been proposed to enhance the quality and robustness of the synchronization scheme. Due to the instability and poor performance that caused by noise or disturbances, it is reasonable to take the noise phenomenon into account during the synchronization process of complex dynamical networks [17, 18]. On the other hand, research in fault diagnosis has been gaining increasing consideration worldwide in the past decades [19–23]. One of the key issues related to fault detection is concerned with its robustness. Large amounts of the relevant jobs have been done for the linear systems in order to examine the robust fault detection (RFD) problem (see [22, 23] and the references therein). In a recent work, we have investigated the robust fault sensitive synchronization of nonlinear Lur'e systems coupled in a master-slave fashion [24]. Similarly, in complex dynamical networks, since it is inevitable for faults to happen within each of the single node, a fault-free synchronization process cannot always be guaranteed. Even though, there is a few work concentrating upon the RFD problem of large-scale nonlinear systems, and hardly there is any previous work that brought the notion "fault" into physical aspects such as synchronization of nonlinear dynamical networks.

Based on these considerations, this present study considers the fault detection and disturbance rejection problem within robust synchronization for a class of dynamical networks. The network model is composed by identical nodes with each node being a perturbed nonlinear Lur'e system, while at the same time, subject to possible faults. The main challenge in evaluating the synchronization scheme is to distinguish failures from other disturbances, and accordingly, the $\mathcal{H}_-/\mathcal{H}_\infty$ paradigm is introduced [25]. For the purpose of description, the robustness objectives during synchronization are considered in virtue of the $\mathcal{H}_\infty$ norm, while the fault sensitivity specifications are expressed by utilizing the formulation of $\mathcal{H}_-$ index. In this manner, the closed-loop error system is asymptotically stable with the $\mathcal{H}_\infty$-norm from the disturbance input to controlled output reduced to a prescribed level, and at the same time, with the $\mathcal{H}_-$ performance index maximized. By transforming the synchronization problem of dynamical networks into absolute stability problem of corresponding error systems as well as applying Lur'e system method in control theory [16], sufficient conditions to the global robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization within nonlinear Lur'e networks are developed in terms of sets of linear matrix inequalities (LMI) [26]. Furthermore, the derived high-dimensional LMI condition is simplified into three groups of lower-dimensional LMIs, which are easier to handle. It should be pointed out that no linearization technique is involved through derivation of all the synchronization criteria.

The rest of the paper is organized as follows. Section 2 proposes the model to be examined in this study, and gives the mathematical formulations of the global robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization problem to be solved. In Section 3, the global robust $\mathcal{H}_\infty$ synchronization scheme of the networks is firstly studied, based on which the criteria on $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization are then proposed in virtue of the LMI technique. Moreover, performance analysis of the network is also discussed in this part. The dynamical network composed by ten identical Chua's circuits is adopted as a numerical example in Section 4, and Section 5 closes the paper.

2. Notations and Preliminaries

The notations used in this study are fairly standard. $\mathbb{R}^{n \times n}$ is the set of $n \times n$ real matrices. For a matrix A, A^T denotes its transpose. He is the Hermit operator with He $A = A + A^T$. If A is a real symmetric negative definite matrix, it is shown by $A < 0$. diag($\cdot$) implies a diagonal or block-diagonal matrix. $A \otimes B$ indicates the Kronecker product of an $n \times m$ matrix A and a $p \times q$ matrix B, that is,

$$
A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix}.
\tag{2.1}
$$

If not explicitly stated, matrices are assumed to have compatible dimensions, and the terms replaced by $*$ of a matrix refer to the terms in a symmetric position that do not need to be written out.

2.1. Basic Knowledge on Lur'e Systems

The basic model of nonlinear Lur'e systems subject to input noise and possible faults considered in this paper is described by

$$
\dot{x} = Ax + B\varphi(y) + B_d d_0 + B_f f_0, \quad y = Cx,
$$
$$
z = Hx + Df,
\tag{2.2}
$$

where $x \in \mathbb{R}^n$ is the state vector and $z \in \mathbb{R}^m$ represents the measurement output vector. $d_0 \in \mathbb{R}^p$ is an unknown input vector (including disturbance, uninterested fault as well as some norm-bounded unstructured model uncertainty) belonging to $L_2[0, +\infty)$, while $f_0 \in \mathbb{R}^q$ denotes the process, sensor, or actuator fault vector to be detected and isolated. Depending on specific situations under consideration, f_0 and d_0 can be modeled as different types of signals. The matrices A, B, C, D, B_d, B_f and H, are known constant matrices with appropriate dimensions. Nonlinearity $\varphi : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m$ is continuous and locally Lipschitz in the first argument with $\varphi(0) = 0, y = (y_1^T, \ldots, y_m^T)^T$ and $\varphi(y) = (\varphi_1^T(y_1), \ldots, \varphi_m^T(y_m))^T$, where the functions $\varphi_l(y_l)$, $l = 1, 2 \ldots, m$ are assumed to satisfy the following inequalities:

$$
0 \le \varphi_l(y_l) y_l \le \theta_l y_l^2, \quad l = 1, 2 \ldots, m,
\tag{2.3}
$$

where $\theta_l \in \mathbb{R}$, $l = 1, 2 \ldots, m$. Denoting $\Theta_0 = \text{diag}(\theta_1, \theta_2, \ldots, \theta_m)$, it is obvious to get

$$
\varphi^T(y)(\varphi(y) - \Theta_0 y) \le 0,
\tag{2.4}
$$

and the nonlinearity $\varphi(y)$ is said to be in the sector $[0, \Theta_0]$ if it satisfies (2.4).

Definition 2.1. Nonlinear system (2.2) is said to be absolutely stable with respect to the sector $[0, \Theta_0]$, if the equilibrium point $x = 0$ is globally asymptotically stable for the nonlinearity $\varphi(y)$ satisfying (2.4).

In order to characterize the influence of the disturbance and fault input, several definitions are introduced.

Definition 2.2. Consider the following transfer function $d_0 \mapsto z$ of system (2.2):

$$K_{zd0}(s) \triangleq H(sI - A)^{-1}B_d. \tag{2.5}$$

Then its $\mathscr{H}_\infty$ norm is defined as $\|K_{zd0}\|_\infty = \sup_{d_0 \in \mathbb{L}_2} \overline{\sigma}[K_{zd0}(j\omega)] = \sup_{d_0 \in \mathbb{L}_2}(\|K_{zd0}d_0\|_2 / \|d_0\|_2)$, where $\overline{\sigma}$ represents the maximal singular value.

Definition 2.3. For system (2.2), the transfer function from the input f_0 to output z is given as:

$$K_{zf0}(s) \triangleq H(sI - A)^{-1}B_f + D, \tag{2.6}$$

whose $\mathscr{H}_-$ index is defined by $\|K_{zf0}(s)\|_-^{[0,\overline{\omega}]} \triangleq \inf_{\omega \in [0, \overline{\omega}]} \underline{\sigma}[K_{zf0}(j\omega)] = \inf_{f_0 \in \mathbb{L}_2}(\|K_{zf0}f_0\|_2 / \|f_0\|_2)$, where $\underline{\sigma}$ stands for the minimum singular value and $\overline{\omega}$ denotes the frequency band $[0, \overline{\omega}]$.

Remark 2.4. The $\mathscr{H}_-$ index defined has been widely adopted to measure the sensitivity of residual to fault in the frequency domain. A system is said to possess a better level of RFD, if the $\mathscr{H}_\infty$ norm of its transfer function from the disturbances to the performance variable is small; meanwhile, the $\mathscr{H}_-$ index of the transfer function from fault to the output variable is large [19]. Various kinds of $\mathscr{H}_- / \mathscr{H}_\infty$ performance criteria have been proposed to determine the RFD issue [21], and the performance is mostly adopted as a trade-off between robustness and sensitivity. In this study, for the sake of simplicity, we will consider the case of maximizing the fault sensitivity $\|K_{rf}(s)\|_-$ with disturbance attenuation $\|K_{rd}(s)\|_\infty$ being a prescribed constant.

2.2. Dynamical Networks Composed of Lur'e Nodes

Consider a class of complex dynamical network model with each node being a general Lur'e system (2.2) shown as follows:

$$\dot{x}_i = Ax_i + B\varphi(y_i) + \sum_{j=1}^{N} g_{ij}\Gamma z_j + B_d d_0 + B_f f_0, \quad y_i = Cx_i,$$

$$z_i = Hx_i + Df_0, \quad i = 1, 2, \ldots, N, \tag{2.7}$$

where $x_i \in \mathbb{R}^n$ and $z_i \in \mathbb{R}^m$ are the state and measurement output of the ith node, respectively. d_0 and f_0 are defined as in system (2.2), which are supposed to be the same with respect to each node. The inner coupling matrix $\Gamma = (\tau_{ij})_{n \times n}$ denotes the coupling pattern between two nodes. $G = (g_{ij})_{N \times N}$ is the outer coupling matrix, standing for the coupling configuration

of the network. If there is a connection between node i and node $j(i \neq j)$, then $g_{ij} = g_{ji} = 1$; otherwise, $g_{ij} = g_{ji} = 0(i \neq j)$. The row sums of G are zero, that is, $\sum_{j=1, j \neq i}^{N} g_{ij} = -g_{ii}, i = 1, 2, \ldots, N$. Let $\varphi_i = (\varphi_{i1}^{T}, \ldots, \varphi_{im}^{T})^{T} \in R^m$ and $\varphi(y_i) = (\varphi_1^{T}(y_{i1}) \cdots \varphi_m^{T}(y_{im}))^{T} \in R^m$ with the following properties:

$$0 \leq \varphi_l(y_{il}) y_{il} \leq \theta_{1l} y_{il}^2, \qquad 0 \leq \varphi_l'(y_{il}) \leq \theta_{1l}, \quad i = 1, 2, \ldots, N, \ l = 1, 2, \ldots, m. \tag{2.8}$$

Denote $\Theta_1 = \text{diag}(\theta_{11}, \ldots, \theta_{1m})$ then the nonlinear function $\varphi(y)$ belongs to the sector $[0, \Theta_1]$.

Lemma 2.5 (Wu [27]). *The eigenvalues of an irreducible matrix $G_0 = (G_{0ij}) \in R^{N \times N}$ with $\sum_{j=1, j \neq i}^{N} G_{0ij} = -G_{0ii}, i = 1, 2, \ldots, N$ satisfy the following.*

(i) *0 is an eigenvalue of G_0 associated with the eigenvector $(1, 1, \ldots, 1)^{T}$.*

(ii) *If $G_{0ij} \geq 0$ for all $1 \leq i, j \leq N, i \neq j$, then the real parts of all eigenvalues of G_0 are less than or equal to 0 and all possible eigenvalues with zero part are 0. In fact, 0 is its eigenvalue of multiplicity 1.*

Assume that the network (2.7) has no isolate clusters; namely, the network is connected. Under this circumstance, the coupling matrix G is symmetric and irreducible; hence it satisfies all the properties given in Lemma 2.5. Besides, suppose that the coupling matrix G has q distinct different eigenvalues $\lambda_1, \ldots, \lambda_q$; then there exists a nonsingular matrix U with $U^T U = I_N$ such that $U^T G U = \Lambda$, where Λ is in the following form:

$$\Lambda = \text{diag}\left(\lambda_1, \underbrace{\lambda_2, \ldots, \lambda_2}_{m_2}, \underbrace{\lambda_3, \ldots, \lambda_3}_{m_3}, \ldots, \underbrace{\lambda_q, \ldots, \lambda_q}_{m_q} \right). \tag{2.9}$$

Here, $\lambda_1 = 0$ is the maximum eigenvalue of multiply 1 and λ_i is the eigenvalue of multiply $m_i, i = 2, 3, \ldots, q$ satisfying $m_2 + \cdots + m_q = N - 1$ and $\lambda_2 > \lambda_3 > \cdots > \lambda_q$.

Definition 2.6. When $d_0 = f_0 = 0$, the dynamical network (2.7) is said to achieve global (asymptotical) synchronization if

$$\lim_{t \to \infty} \|x_i - x_s\|_2 = 0, \quad i = 1, 2, \ldots, N, \tag{2.10}$$

where $\| \cdot \|_2$ means the Euclidean norm. $x_s \in R^n$ is a solution of an isolate node given by

$$\dot{x}_s = A x_s + B \varphi(y_s), \quad y_s = C x_s,$$

$$z_s = H x_s, \tag{2.11}$$

which can be an equilibrium point, a periodic orbit, or even a nonperiodic orbit.

From the properties of the internal coupling matrix G, the following condition holds:

$$\dot{x}_s = Ax_s + B\varphi(Cx_s) + \sum_{j=1}^{N} g_{ij}\Gamma H x_s. \tag{2.12}$$

Define error signals $e_i = x_i - x_s$ and residual signals $r_i = z_i - z_s$ for $i = 1, 2, \ldots, N$. By subtracting (2.12) from (2.7), one arrives at the dynamics of synchronization residual error:

$$\dot{e}_i = Ae_i + B\phi(Ce_i; x_s) + \sum_{j=1}^{N} g_{ij}\Gamma H e_j + B_d d_0 + B_f f_0 + \sum_{j=1}^{N} g_{ij}\Gamma D f_0,$$

$$r_i = He_i + Df_0, \quad i = 1, 2, \ldots, N, \tag{2.13}$$

where $\phi(Ce_i; x_s) = \varphi(Ce_i + Cx_s) - \varphi(Cx_s)$. Let $\phi(\cdot) = (\phi_1^T(\cdot), \ldots, \phi_m^T(\cdot))^T$; it is not difficult to derive from (2.8) that for $c_i^T e \neq 0$, the nonlinear functions $\phi_l(c_l^T e_i; x_s)$, $l = 1, 2, \ldots, m$, satisfy the following sector restrictions:

$$0 \leq \frac{\phi_l(c_l^T e_i; x_s)}{c_l^T e_i} = \frac{\varphi(c_l^T e_i + c_l^T x_s) - \varphi(c_l^T x_s)}{c_l^T e_i} \leq \theta_{1l}, \quad i = 1, 2, \ldots, N, \ l = 1, 2, \ldots, m, \tag{2.14}$$

which leads to

$$\phi_l(c_l^T e_i; x_s)(\phi_l(c_l^T e_i; x_s) - \theta_{1l} c_l^T e_i) \leq 0, \quad i = 1, 2, \ldots, N, \ l = 1, 2, \ldots, m, \tag{2.15}$$

and thus $\phi(Ce_i; x_s)$ also belongs to the sector $[0, \Theta_1]$.

Remark 2.7. Based on the basic knowledge of synchronization, the residual error dynamics must be asymptotically stable in order for the whole process to work. Note that the dynamics of the residual error signal r depends not only on f_0, d_0, and $\phi(y)$ but also on the states of each isolated node x_i. In consequence, this study aims at ensuring the residual error dynamical system to be sensitive to possible faults in the regard of $\mathcal{H}_-$ index, but the error dynamics also remain robustly asymptotically stable to external disturbance in the $\mathcal{H}_\infty$ sense. Under such circumstances, the dynamical network composed of Lur'e nodes is said to achieve global synchronization with a guaranteed $\mathcal{H}_- / \mathcal{H}_\infty$ performance.

Reformulating system (2.13) in virtue of the Kronecker product [28] as

$$\dot{e} = (I_N \otimes A + G \otimes \Gamma H)e + (I_N \otimes B)\Phi((I_N \otimes C)e; X_s) + (I_N \otimes B_d)d + (I_N \otimes B_f + G \otimes \Gamma D)f$$

$$\triangleq \overline{A}e + \overline{B}\Phi(\overline{C}e; X_s) + \overline{B}_d d + \overline{B}_f f,$$

$$r = (I_N \otimes H)e + (I_N \otimes D)f \triangleq \overline{H}e + \overline{D}f,$$

$$\tag{2.16}$$

where

$$e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{pmatrix} \in R^{Nn}, \qquad d = \begin{pmatrix} d_0 \\ d_0 \\ \vdots \\ d_0 \end{pmatrix} \in R^{Np}, \qquad f = \begin{pmatrix} f_0 \\ f_0 \\ \vdots \\ f_0 \end{pmatrix} \in R^{Nq},$$

$$\Phi\left(\overline{C}e; X_s\right) = \begin{pmatrix} \phi(Ce_1; x_s) \\ \phi(Ce_2; x_s) \\ \vdots \\ \phi(Ce_N; x_s) \end{pmatrix} \in R^{Nm}, \quad X_s = \begin{pmatrix} x_s \\ x_s \\ \vdots \\ x_s \end{pmatrix} \in R^{Nn},$$

(2.17)

with $\Theta = I_N \otimes \Theta_1 \in R^{Nm} > 0$ and $\Phi(\overline{C}e; X_s)$ belonging to the sector $[0, \Theta]$. accordingly, the residual error dynamical system (2.16) can be treated as an Nn-dimensional nonlinear Lur'e system, and the $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization of the nonlinear dynamical networks (2.7) can be transformed into the performance analysis and stabilization problem of the corresponding residual error dynamics (2.16).

For system (2.16), the transfer function $d \mapsto r$ is given by $K_{rd}(s) \triangleq \overline{H}(sI - \overline{A})^{-1}\overline{B}_d$, while $K_{rf}(s) \triangleq \overline{H}(sI - \overline{A})^{-1}\overline{B}_f + \overline{D}$ denotes the transfer function $f \mapsto r$. Specifically speaking, the main objective of this present study is to determine under what condition the residual error dynamics (2.16) could be asymptotically stable and, at the same time, satisfy the following conditions:

$$\left\|K_{rf}(s)\right\|_- > \beta, \qquad \left\|K_{rd}(s)\right\|_\infty < \gamma, \tag{2.18}$$

where γ is a prescribed positive constant, and β is a constant to be optimized. By applying the well-known Parseval theorem to the frequency-domain expressions (2.18), where the ratios $\mathcal{H}_\infty$ norm and $\mathcal{H}_-$ index are presented in Definitions 2.2 and 2.3, respectively, we arrive at the equivalent statements as follows:

$$J_1 = \int_0^\infty \left[r(t)^T r(t) - \gamma^2 d^T(t)d(t)\right]dt < 0, \tag{2.19}$$

$$J_2 = \int_0^\infty \left[r(t)^T r(t) - \beta^2 f^T(t)f(t)\right]dt > 0. \tag{2.20}$$

Accordingly, the definition of robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization is derived as follows.

Definition 2.8. The dynamical networks composed of nonlinear Lur'e nodes in (2.7) are said to achieve global robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization with disturbance attenuation γ and fault sensitivity β over the frequency range $[0, \overline{\omega}]$ (where $\overline{\omega}$ could be both finite and infinite), if with zero disturbance and zero fault, the synchronization residual error signal (2.16) is asymptotically stable, while with zero initial condition and given constants $\gamma > 0$, $\beta > 0$, conditions (2.19)-(2.20) hold.

3. Main Results

The intention of this part is to investigate the fault sensitivity as well as disturbance rejection ability of the complex dynamical network (2.7). In order to quantify these two performance indices, one borrows the concept of $\mathcal{H}_-$ index and $\mathcal{H}_\infty$-norm defined in the previous section.

3.1. Global $\mathcal{H}_\infty$ Synchronization of Nonlinear Lur'e Networks

In this subsection, we first consider the case that there is no fault existed in the network by extending previous results on $\mathcal{H}_\infty$ synchronization between two identical Lur'e systems to that of nonlinear Lur'e dynamical networks. Accordingly, the network model is described by

$$\dot{x}_i = Ax_i + B\varphi(y_i) + \sum_{j=1}^{N} G_{ij}\Gamma z_j + B_d d_0, \quad y_i = Cx_i,$$

$$z_i = Hx_i, \quad i = 1, 2, \ldots, N,$$

(3.1)

and the corresponding error dynamics in form of Kronecker product is expressed as

$$\dot{e} = \overline{A}e + \overline{B}\Phi\left(\overline{C}e; X_s\right) + \overline{B}_d d,$$

$$r = \overline{H}e.$$

(3.2)

The disturbance rejection problem within the synchronization of nonlinear dynamical network (3.1) is summarized in the following definition.

Definition 3.1. Given constant scalar $\gamma > 0$, the dynamical network (3.1) is said to achieve global robust $\mathcal{H}_\infty$ synchronization, if system (3.2) is globally asymptotically stable with zero disturbance, and meanwhile, the performance index (2.19) is satisfied with zero initial conditions.

The robust $\mathcal{H}_\infty$ synchronization can be determined in virtue of the following criterion.

Theorem 3.2. *Suppose that $\gamma > 0$ is a prescribed constant. For a given scalar α, if there exist positive-definite matrices $P = P^T > 0$, diagonal matrices $\Delta_1 = \mathrm{diag}(\delta_1, \ldots, \delta_m) > 0$, $\Pi_1 = \mathrm{diag}(\pi_1, \ldots, \pi_m) > 0$, and $\Omega_1 = \mathrm{diag}(\omega_1, \ldots, \omega_m) > 0$, and matrices $\overline{Q}_1$ and $\overline{Q}_2$ such that the following LMI*

$$\Xi_1 = \begin{pmatrix} -\mathrm{He}\left(\overline{Q}_1\overline{A}\right) + \overline{H}^T\overline{H} & \overline{C}^T\Theta\Delta - \overline{Q}_1\overline{B} & P + \overline{Q}_1 - \overline{A}^T\overline{Q}_2^T & \overline{C}^T\Theta\Omega & -\overline{Q}_1\overline{B}_d \\ * & -\mathrm{He}\,\Delta & -\overline{B}^T\overline{Q}_2^T & 0 & 0 \\ * & * & \mathrm{He}\,\overline{Q}_2 & \overline{C}^T\Pi & -\overline{Q}_2\overline{B}_d \\ * & * & * & -\mathrm{He}\,\Omega & 0 \\ * & * & * & * & -\gamma^2 I \end{pmatrix} < 0 \quad (3.3)$$

holds, where $\Delta = I_N \otimes \Delta_1, \Pi = I_N \otimes \Pi_1,$ *and* $\Omega = I_N \otimes \Omega_1,$ *then the dynamical network* (3.1) *achieves global robust* $\mathcal{H}_\infty$ *synchronization with disturbance attenuation* γ.

Proof. See Appendix A. □

Remark 3.3. Theorem 3.2 has provided a sufficient condition for the global robust $\mathcal{H}_\infty$ synchronization of nonlinear Lur'e networks by introducing slack matrices $\overline{Q}_1$ and $\overline{Q}_2$ into LMI (3.3). It is thus expected that Theorem 3.2 will be less conservative than some existing results due to the increasing freedom of these slack variables [29]. With the derived $\mathcal{H}_\infty$ synchronization conditions on Lur'e networks, the fault detection issue will then be examined in the next subsection. However, if the number of nodes is large, condition (3.3) would become a high-dimensional LMI, which is rather tedious to verify. To this end, both of these criteria will be further simplified to the test of three groups of lower-dimensional LMIs.

3.2. Fault Detection within Global $\mathcal{H}_\infty$ Synchronization

The RFD within a synchronization configuration can be treated as a multiple objective design task; that is, the design objective is not only being as sensitive as possible to faults such that early detection of faults is possible, but on the other hand, the sensitivity of possible faults is maximized, also suppressing the effect of disturbances and modeling errors on the synchronization error and subsequently on the residual, in order to prevent the synchronization process from being destroyed. Next theorem gives an LMI formulation for global robust $\mathcal{H}_- / \mathcal{H}_\infty$ synchronization.

Theorem 3.4. *Suppose that* $\gamma > 0$, $\beta > 0$ *are prescribed constant scalars. For a given constant* α, *if there exist a positive-definite matrix* $P = P^T > 0$, *diagonal matrices* $\Delta_1 = \mathrm{diag}(\delta_{11}, \ldots, \delta_{1m}) > 0, \Pi_1 = \mathrm{diag}(\pi_{11}, \ldots, \pi_{1m}) > 0,$ *and* $\Omega_1 = \mathrm{diag}(\omega_{11}, \ldots, \omega_{1m}) > 0$, *and matrices* $\overline{Q}_1$ *and* $\overline{Q}_2$ *such that LMIs* (3.3) *as well as*

$$
\Xi_2 = \begin{pmatrix}
-\mathrm{He}(\overline{Q}_1\overline{A}) - \overline{H}^T\overline{H} & \overline{C}^T\Theta\Delta - Q_1\overline{B} & \overline{P} + \overline{Q}_1 - \overline{A}^T\overline{Q}_2^T & \overline{C}^T\Theta\Omega & -\overline{H}\,\overline{D} - \overline{Q}_1\overline{B}_f \\
* & -\mathrm{He}\,\Delta & -\overline{B}^T\overline{Q}_2^T & 0 & 0 \\
* & * & \mathrm{He}\,\overline{Q}_2 & \overline{C}^T\Pi & -\overline{Q}_2\overline{B}_f \\
* & * & * & -\mathrm{He}\,\Omega & 0 \\
* & * & * & * & \beta^2 I - \overline{D}^T\overline{D}
\end{pmatrix} < 0
$$

(3.4)

hold, then the dynamical network in (2.7) *achieves global robust* $\mathcal{H}_- / \mathcal{H}_\infty$ *synchronization with disturbance attenuation* γ *and fault sensitivity* β.

Proof. See Appendix B. □

Theorem 3.5. *Suppose that $\alpha, \beta > 0$ and $\gamma > 0$ are given scalars. If there exist matrices $W_i > 0, V_i$, and diagonal matrices $\Delta_1 > 0, \Pi_1 > 0,$ and $\Omega_1 > 0$ such that the following conditions for $i = 1, 2$ and q hold:*

$$\begin{pmatrix} \Upsilon_{11} + H^T H & C^T \Theta_1 \Delta_1 - V_i B & \Upsilon_{31} & C^T \Theta_1 \Omega_1 & -V_i B_d \\ * & -\text{He}\,\Delta_1 & -\alpha B^T V_i^T & 0 & 0 \\ * & * & \text{He}\,\alpha V_i & C^T \Pi_1 & -\alpha V_i B_d \\ * & * & * & -\text{He}\,\Omega_1 & 0 \\ * & * & * & * & -\gamma^2 I \end{pmatrix} < 0, \qquad (3.5)$$

$$\begin{pmatrix} \Upsilon_{11} - H^T H & C^T \Theta_1 \Delta_1 - V_i B & \Upsilon_{31} & C^T \Theta_1 \Omega_1 & -HD - V_i B_f - \lambda_i V_i \Gamma D \\ * & -\text{He}\,\Delta_1 & -\alpha B^T V_i^T & 0 & 0 \\ * & * & \text{He}\,\alpha V_i & C^T \Pi_1 & -\alpha V_i B_f - \lambda_i \alpha V_i \Gamma D \\ * & * & * & -\text{He}\,\Omega_1 & 0 \\ * & * & * & * & \beta^2 I - D^T D \end{pmatrix} < 0, \qquad (3.6)$$

where $\Upsilon_{11} = -\text{He}(V_i A + \lambda_i V_i \Gamma H)$ and $\Upsilon_{31} = W_i + V_i - \alpha A^T V_i^T - \lambda_i \alpha H^T \Gamma^T V_i^T$, then the conditions given in Theorem 3.4 are ensured.

Proof. See Appendix C.   □

Corollary 3.6. *For a constant α, let $\beta > 0$ and $\gamma > 0$ be prescribed constant scalars. The dynamical network (2.7) is said to achieve global robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization with disturbance attenuation γ and fault sensitivity β, if there exist matrices $W_i > 0$, $V_i T_i$, and diagonal matrices $\Delta_1 > 0, \Omega_1 >, \Pi_1 > 0$ such that the LMI conditions (3.5)-(3.6) hold for $i = 1, 2$ and q (corresponding to the largest, second largest, and smallest eigenvalues, resp.).*

Remark 3.7. If the number of nodes N is large, the $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization criterion of the dynamical network would become a group of LMIs with rather high dimensions. In order to tackle this problem, the synchronization of the $nN \times nN$-dimensional network has been disposed in a lower n-dimensional space through verifying three groups of n-dimensional LMIs in Corollary 3.6, and the derived conditions are quite convenient to use.

As an immediate consequence, we arrive at the simplified criterion for global robust $\mathcal{H}_\infty$ synchronization of nonlinear dynamical network (3.1) summarized as in the following corollary.

Corollary 3.8. *For a constant α, let $\beta > 0$ and $\gamma > 0$ be prescribed constant scalars. If there exist matrices $W_i > 0$, $V_i T_i$, and diagonal matrices $\Delta_1 > 0, \Omega_1 >, \Pi_1 > 0$ such that the LMIs (3.5) for $i = 1, 2$ and q are feasible, then the dynamical network (2.7) is said to achieve global robust $\mathcal{H}_\infty$ synchronization.*

3.3. $\mathcal{H}_-/\mathcal{H}_\infty$ Performance Analysis

It comes from Corollary 3.6 that the $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization within a dynamical network can be cast into that of three sets of independent systems whose dimensions are the same as that of each isolate node. Namely, if the following systems

$$\dot{e}_{\lambda i} = (A + \lambda_i \Gamma H)e_{\lambda i} + B\phi(Ce_{\lambda i}) + (B_f + \lambda_i \Gamma D)f_0 + B_d d_0,$$

$$r_{\lambda i} = He_{\lambda i} + Df_0,$$

(3.7)

satisfy (3.5)-(3.6) for $i = 1, 2$ and q, then the conditions given in Definition 2.8 will be guaranteed. Suppose the transfer function of system (3.7) from $d_0 \mapsto r_{\lambda i}$ and $f_0 \mapsto r_{\lambda i}$ for $i = 1, 2, \dots, N$ as K_{rd_i} and K_{rf_i}, respectively. Then denote

$$K_{rd\lambda} = \mathrm{diag}(K_{rd1}, \dots, K_{rdN}),$$

$$K_{rf\lambda} = \mathrm{diag}(K_{rf1}, \dots, K_{rfN}),$$

(3.8)

where $K_{rd\lambda}$ and $K_{rf\lambda}$ are in the following form:

$$K_{rd\lambda} = (I_N \otimes H)(sI - I_N \otimes A - \Lambda \otimes \Gamma H)^{-1}(I_N \otimes B_d),$$

$$K_{rf\lambda} = (I_N \otimes H)(sI - I_N \otimes A - \Lambda \otimes \Gamma H)^{-1}(I_N \otimes B_f - \Lambda \otimes \Gamma D) + (I_N \otimes D).$$

(3.9)

On the other hand, consider the following system:

$$\dot{e}_\lambda = (I_N \otimes A + \Lambda \otimes \Gamma H)e_\lambda + (I_N \otimes B)\Phi((I_N \otimes C)e; X_s) + (I_N \otimes B_d)d + (I_N \otimes B_f + \Lambda \otimes \Gamma D)f$$

$$r_\lambda = (I_N \otimes H)e_\lambda + (I_N \otimes D)f,$$

(3.10)

where $e_\lambda = (e_{\lambda 1}^T, \dots, e_{\lambda N}^T)^T$ and $r_\lambda = (r_{\lambda 1}^T, \dots, r_{\lambda N}^T)^T$. It can be found that the transfer functions from $d \mapsto r_\lambda$ and $f \mapsto r_\lambda$ of system (3.10) are just those defined in (3.8). Moreover, by carrying out unitary transformation, $K_{rd\lambda}$ is similar to K_{rd}, and so do $K_{rf\lambda}$ and K_{rf}. Recall the definition of the $\mathcal{H}_\infty$ norm and $\mathcal{H}_-$ index previously stated in Definitions 2.2 and 2.3; then we arrive at the following relationships between the $\mathcal{H}_\infty$ norms of K_{rd} and K_{rdi} as well as the $\mathcal{H}_-$ indexes of K_{rf} and K_{rfi} for $i = 1, 2, \dots, N$:

$$\|K_{rd}\|_\infty = \|K_{rd\lambda}\|_\infty = \max_{i=1,\dots,N} \|K_{rdi}\|_\infty,$$

$$\|K_{rf}\|_- = \|K_{rf\lambda}\|_- = \min_{i=1,\dots,N} \|K_{rfi}\|_-.$$

(3.11)

Conditions (3.11) show that the $\mathcal{H}_\infty$ norm of the transfer function from $d \mapsto r$ in (2.16) equals to the maximum of those of the N systems (3.7), whilst the corresponding $\mathcal{H}_-$ index is the minimum value within those of (3.7). Accordingly, the RFD of the network (2.7) can be cast into those of (3.7); thus we have the following corollary.

Corollary 3.9. *For a given scalar $\gamma > 0$, the performance indexes of the dynamical network (2.7) satisfy $\|K_{rd}\|_\infty < \gamma$, and $\|K_{rf}\|_- > \beta$, if $\max_{i=1,\ldots,N}\|K_{rdi}\|_\infty < \gamma$ and $\min_{i=1,\ldots,N}\|K_{rfi}\|_- > \beta$ hold in the decoupled systems (3.7) for $i = 1, 2, \ldots, N$.*

The following corollary presents a method of deriving the maximum value of fault sensitivity and, at the same time, suppresses the external disturbance to a prescribed level for the global robust $\mathcal{H}_- / \mathcal{H}_\infty$ synchronization of network (2.7).

Corollary 3.10. *The nonlinear dynamical networks (2.7) are said to achieve global synchronization with guaranteed $\mathcal{H}_\infty$ performance γ and the maximum fault detection sensitivity $\beta_0 = \sqrt{\rho}$, where ρ is the global minimum of the following generalized eigenvalue minimization problem with respect to matrices $W_i > 0, V_i$ for $i = \{1, 2, q\}$ as well as diagonal matrices $\Delta_1 > 0, \Pi_1 > 0$, and $\Omega_1 > 0$:*

$$\min -\rho$$

$$\begin{pmatrix} \Upsilon_{11} - H^T H & C^T \Theta_1 \Delta_1 - V_i B & \Upsilon_{13} & -HD - V_i B_f \\ * & -\mathrm{He}\,\Delta_1 & \Pi_1 C - \alpha B^T V_i^T & 0 \\ * & * & \mathrm{He}\,\alpha V_i & -\alpha V_i B_f \\ * & * & * & \rho I - D^T D \end{pmatrix} < 0, \tag{3.12}$$

as well as the LMI condition (3.5) holds. Here, Υ_{11} and Υ_{13} are described in Theorem 3.5 with constant scalars α and $\gamma > 0$ prescribed.

4. Numerical Examples

A lower-dimensional dynamical network model is concerned in this part so as to demonstrate the applicability and effectiveness of the approaches proposed in the previous sections. Throughout our numerical simulations, each node of the network is supposed to be a concrete Chua's circuit, which is frequently observed in various fields of theory and engineering applications [30].

In the first stage, it will be shown that how the results derived in Section 3.1 can be used to guarantee the global robust $\mathcal{H}_\infty$ synchronization of the dynamical network (2.7). Let us take a group of ten dimensionless state equations of Chua's oscillators, for example, where one of the node system is shown as system S_a in Figure 1, $a = 1, 2, \ldots, 10$:

$$\begin{pmatrix} \dot{v}_{a1} \\ \dot{v}_{a2} \\ \dot{i}_{a3} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{C_1}\left(\dfrac{v_{a2} - v_{a1}}{R} - g(v_{a1}) \right) + \displaystyle\sum_{j=1}^{10} \dfrac{G_{aj}}{R_1 C_1} H v_{j1} \\ \dfrac{1}{C_2}\left(\dfrac{v_{a1} - v_{a2}}{R} + i_{a3} + i_{ad} \right) \\ -\dfrac{1}{L}(v_{a2} + R_0 i_{a3}) \end{pmatrix}. \tag{4.1}$$

Here, R_0 and R are linear resistors. The voltages across the capacitors C_1 and C_2 are denoted by v_{a1} and v_{a2}, i_{a3} is the current through the inductances L, and i_{ad} is an external disturbance
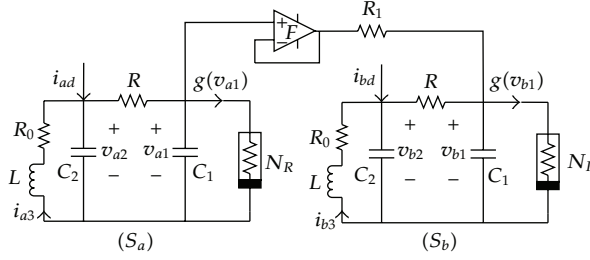
Figure 1: Coupling of any of the two identical Chua's circuits with disturbance signal, $a = 1, 2, \ldots, N$, and $b = 1, 2, \ldots, N$.

current that system $\mathcal{S}_a$ subjects to. The nonlinear characteristic $g(v_1)$ represents the current through the nonlinear resistor N_R, which is a piecewise-linear function expressed as

$$g(v_{a1}) = M_1 v_{a1} + \frac{1}{2}(M_0 - M_1)[|v_{a1} + 1| - |v_{a1} - 1|], \tag{4.2}$$

and it satisfies $\min\{M_0, M_1\} \le g'(v_1) \le \min\{M_0, M_1\}$.

Suppose that each node of the dynamical network developed by (2.7) is a circuit in the form of (4.1). The possible coupling between two arbitrary Chua's circuits, as shown in Figure 1, indicates that there is a connection from S_b to S_a but none from S_a to S_b, where the element F plays the role of unicommunication. Depending on different values of the controller gain, the resistor R_1 can be adjusted. It is straightforward to reformulate system (4.1) into the Lur'e form as

$$\dot{x}_i = Ax_i + B\varphi(Cx_i) + \sum_{j=1}^{10} g_{ij}\Gamma H x_j + B_d d_i, \tag{4.3}$$

where

$$x = \begin{pmatrix} v_1 \\ v_2 \\ i_3 \end{pmatrix}, \quad A = \begin{pmatrix} -p(M_0 + 1) & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & -s \end{pmatrix}, \quad B = \begin{pmatrix} -p(M_1 - M_0) \\ 0 \\ 0 \end{pmatrix},$$

$$\tag{4.4}$$

$$C = (1 \;\; 0 \;\; 0), \quad B_d = \begin{pmatrix} 0 \\ \dfrac{1}{C_2} \\ 0 \end{pmatrix}, \quad d_i = i_{ad},$$

and the nonlinear function $\varphi(Cx) = (1/2)(|x_1 + 1| - |x_1 - 1|)$ satisfies the sector condition on $[0, 1]$. Furthermore, suppose the output equation to be as
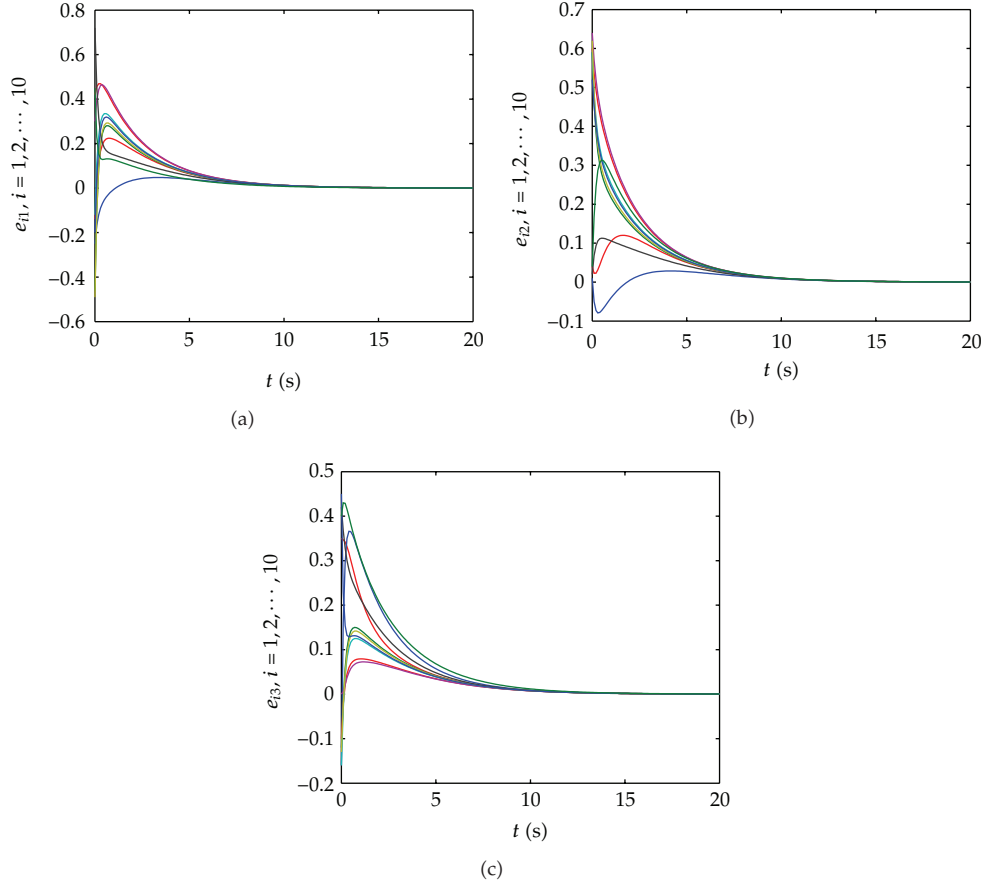
$$z_i = Hx_i, \tag{4.5}$$

(a)

(b)

(c)

Figure 2: The time responses of synchronization errors of the nominal dynamical network.

with parameter matrix $H = (1\ 0\ 0)$. Choose system parameters as $R = C_2 = 1$, $p = 1/RC_1 = 5.5$, $q = 1/L = 7.3$, $s = R_0/L = 4$, $M_0 = -1/7$, $M_1 = 2/7$. In the following, $R_1 = 0.3\,\Omega$ is taken. The network topology is assumed as star-like with ten nodes; thus G has the eigenvalues as follows:

$$\lambda_1 = 0, \qquad \lambda_2 = \cdots = \lambda_9 = -1, \qquad \lambda_{10} = -10. \tag{4.6}$$

Picking $\alpha = 3$, and prescribing disturbance attenuation $\gamma = 0.9$, we arrive at the feasible solutions given in Appendix B by solving the LMIs (3.5), which, according to Corollary 3.8, implies that the dynamical network composed of Chua's circuits has achieved the global robust $\mathcal{H}_\infty$ synchronization.

Simulation results also confirm the effectiveness of the design. Figure 2 depicts the time response of synchronization error of the nominal dynamical network without disturbance signal $d(t)$, and it shows that the synchronization error converges to zero exponentially. Herein, initial values are taken arbitrarily.
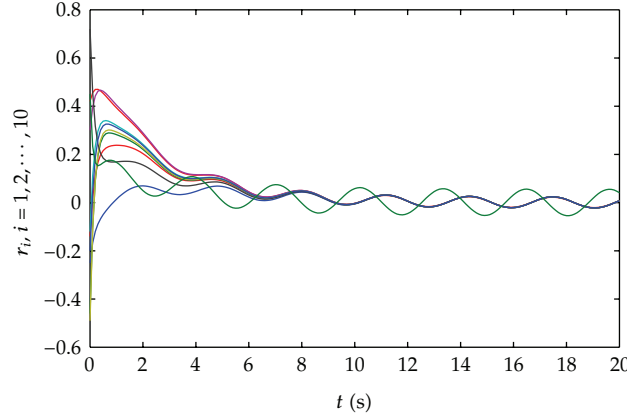
Figure 3: The fault-free residual responses r_i with $d_i(t) = 0.5 \sin(2t)$, $i = 1, 2, \ldots, 10$.

To observe the $\mathcal{H}_\infty$ performance with disturbance attenuation, assume the unknown input noise disturbance d_i to be as

$$d_i(t) = 0.5 \sin(2t), \quad t \geq 0, \ i = 1, 2, \ldots, 10. \tag{4.7}$$

Accordingly, the time response of the output residual error of Lur'e dynamical network with the above disturbance signals and zero initial conditions are shown in Figure 3.

In what follows, let us consider the global robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization of the dynamical network (4.4) in the presence of fault signal f. For the purpose of illustration, the process fault is supposed to be a faulty current flowing in the same direction as i_{m3} along with the leftmost branch of each of the circuits, which will be simulated as two different types. Accordingly, it leads to

$$\dot{x}_i = Ax_i + B\varphi(Cx_i) + \sum_{j=1}^{10} g_{ij} \Gamma H x_j + B_d d_i + B_f f,$$

$$z_i = H x_i + D f, \quad i = 1, 2, \ldots, 10 \tag{4.8}$$

with $B_f = (0 \ 1/C_2 \ -R_0/L)^T$ and $D = 1$.

Remaining $\gamma = 0.9$ and picking the fault sensitivity $\beta = 0.6$, we arrive at solution of the LMI (3.5)-(3.6) with $\alpha = 3$ presented in Appendix C, which on its turn ensures that the network (4.4) has achieved global robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization in the presence of possible faults and external disturbances.

As for the corresponding simulation results, first let the process fault be a pulse of unit amplitude occurred from $5s$ to $10s$ (and is zero otherwise). The generated residual signals $r_i(t)$, $i = 1, 2, \ldots, 10$ are depicted in Figure 4(a), from which one observes that the effect of the disturbance input $d_i(t)$ on the residual error signal $r_i(t)$, $i = 1, 2, \ldots, 10$ has been greatly reduced, and the residuals have rather large amplitudes so that the synchronization process
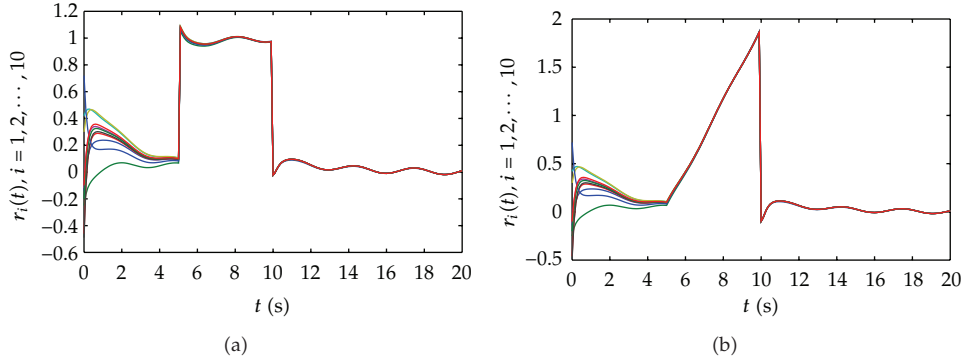
(a)   (b)

Figure 4: The residual responses $r_i(t)$ with $d_i(t) = 0.5\sin(2t)$ and (a) $f(t) = f_1(t)$; (b) $f(t) = f_2(t)$, $i = 1, 2, \ldots, 10$.

remains sensitive to fault. With the same disturbance $d_i(t)$, then redo the simulation with a different:

$$f_2(t) = \begin{cases} 0.4(t-5), & 10 \leq t < 20, \\ 0, & \text{elsewhere,} \end{cases} \tag{4.9}$$

and the results are plotted in Figure 4(b).

By solving the generalized eigenvalue problem corresponding to the minimization problem given in Corollary 3.6, we get estimates of the maximum values of fault sensitivity as $\beta_{1m} = 0.7961$, $\beta_{2m} = \cdots = \beta_{9m} = 0.8548$, and $\beta_{10m} = 0.9524.$, which also guarantees in terms of Corollary 3.6 that Lur'e dynamical networks achieve $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization with $\beta < \beta_{0m}$, where $\beta_{0m} = \min_{i=1,\ldots,10}\{\beta_{im}\} = 0.7961$.

5. Conclusion and Future Work

Aiming at enhancing the reliability and robustness of synchronization, the global robust $\mathcal{H}_-/\mathcal{H}_\infty$ synchronization scheme has been introduced into a class of nonlinear dynamical networks in the existence of possible faults and external disturbances. The criterion on synchronization was developed in virtue of the LMI technique such that each of the node systems of the network is robustly synchronized as well as sensitive to faults according to a mixed $\mathcal{H}_-/\mathcal{H}_\infty$ performance. Since both of the external disturbance and system fault are, respectively, considered, such synchronization scheme proposed here may be more practical than the synchronization in the previous literature. Moreover, the fault sensitivity H_- index could be optimized via a convex optimization algorithm. In order to demonstrate the effectiveness and applicability of the derived results, a low-dimensional dynamical network with each node being a Chua's circuit has been adopted as an example.

As for future work, it will be interesting to study the synchronization of complex networks with different disturbance from various sources. Also, it is possible to extend the present results to stochastic complex networks [31–34].

Appendices

A. Proof of Theorem 3.2

Proof. First, we will show the global asymptotical stability of the residual error dynamics (3.2) with $d = 0$ (no fault is taken into account here), and accordingly (3.2) is represented as

$$\dot{e} = \overline{A}e + \overline{B}\Phi\left(\overline{C}e; X_s\right). \tag{A.1}$$

Under the given conditions, the performance indexes J_1 in condition (2.19) are then proven to be satisfied.

Choose a Lyapunov functional candidate in the form of

$$V = e^T P e + 2\sum_{i=1}^{N}\sum_{j=1}^{m}\pi_{ij}\int_{0}^{c_{ij}^T e_i}\varphi_j(\alpha)d\alpha, \tag{A.2}$$

where $P > 0$ and $\Pi = I_N \otimes \Pi_1$ with $\Pi_1 = \text{diag}(\pi_{11}, \ldots, \pi_{1m}) > 0$ need to be determined. By calculating the time derivative of V along with the trajectory of the residual error dynamics (3.2), it yields

$$\dot{V} = 2e^T P\dot{e} + 2\sum_{i=1}^{N}\sum_{j=1}^{m}\pi_{ij}\varphi_j\left(c_{ij}^T e_i\right)c_{ij}^T\dot{e}_i = 2e^T P\dot{e} + 2\Psi^T\left(\overline{C}e\right)\Pi\overline{C}\dot{e}, \tag{A.3}$$

where $\Psi(\overline{C}e) = (\varphi^T(Ce_1), \ldots, \varphi^T(Ce_N))$. Then consider the sector restrictions that nonlinearities $\Phi(\overline{C}e; X_s)$ and $\Psi(\overline{C}e)$ satisfy, namely, for any diagonal matrices $\Delta_1 = \text{diag}(\delta_{11}, \ldots, \delta_{1m}) > 0$ and $\Omega_1 = \text{diag}(\omega_{11}, \ldots, \omega_{1m}) > 0$:

$$2\sum_{i=1}^{N}\sum_{j=1}^{m}\delta_{ij}\phi_j\left(c_{ij}^T e_i; x_s\right)\left(\phi_j\left(c_{ij}^T e_i; x_s\right) - \theta_j c_{ij}^T e_i\right)$$

$$= 2\Phi^T\left(\overline{C}e; X_s\right)\Delta\Phi\left(\overline{C}e; X_s\right) - 2\Phi^T\left(\overline{C}e; X_s\right)\Delta\Theta\overline{C}e \leq 0,$$

$$2\sum_{i=1}^{N}\sum_{j=1}^{m}\omega_{ij}\varphi_j\left(c_{ij}^T e_i\right)\left(\varphi_j\left(c_{ij}^T e_i\right) - \theta_j c_{ij}^T e_i\right) \tag{A.4}$$

$$= 2\Psi^T\left(\overline{C}e\right)\Omega\Psi\left(\overline{C}e\right) - 2\Psi^T\left(\overline{C}e\right)\Omega\Theta\overline{C}e \leq 0$$

with $\Omega = I_N \otimes \Omega_1\Delta = I_N \otimes \Delta_1$. The results are obtained with the assumption that each subsystem has the same diagonal matrices Π_1, Ω_1, and Δ_1 which does not affect the feasibility of inequality (2.18). For the sake of simplicity, denote $\Phi \triangleq \Phi(\overline{C}e; X_s)$ and $\Psi \triangleq \Psi(\overline{C}e)$ in the following contexts. Moreover, it is known from (A.1) that there exist free-weighting matrices $\overline{Q}_1$ and $\overline{Q}_2$ with appropriate dimensions such that

$$e^T\overline{Q}_1\left(\dot{e} - \overline{A}e - \overline{B}\Phi\right) = \dot{e}^T\overline{Q}_2\left(\dot{e} - \overline{A}e - \overline{B}\Phi\right) = 0. \tag{A.5}$$

Incorporating formulations (A.4)-(A.5) into equality (A.3) derives

$$
\begin{aligned}
\dot{V} &\leq 2e^T P\dot{e} + 2\Psi^T \Pi \overline{C}\dot{e} - 2\Phi^T \Delta \Phi + 2\Phi^T \Delta \Theta \overline{C}e - 2\Psi^T \Omega \Psi + 2\Psi^T \Omega \Theta \overline{C}e \\
&\quad + 2e^T \overline{Q}_1\left(\dot{e} - \overline{A}e - \overline{B}\Phi\right) + 2\dot{e}^T \overline{Q}_2\left(\dot{e} - \overline{A}e - \overline{B}\Phi\right) \\
&= \eta^T \Xi \eta,
\end{aligned}
\tag{A.6}
$$

where

$$
\eta = \begin{pmatrix} e \\ \Phi \\ \dot{e} \\ \Psi \end{pmatrix}, \qquad
\Xi = \begin{pmatrix}
\mathrm{He}\,\overline{Q}_1\overline{A} & \overline{C}^T \Theta \Delta - \overline{Q}_1 \overline{B} & P + \overline{Q}_1 - \overline{A}^T \overline{Q}_2^T & \overline{C}^T \Theta \Omega \\
* & -\mathrm{He}\,\Delta & -\overline{B}^T \overline{Q}_2^T & 0 \\
* & * & \mathrm{He}\,\overline{Q}_2 & \overline{C}^T \Pi \\
* & * & * & -\mathrm{He}\,\Omega
\end{pmatrix},
\tag{A.7}
$$

and it follows that $\Xi < 0$ is guaranteed by the upper left block of LMI (3.3); hence the synchronization residual error dynamics (3.2) is globally asymptotically stable.

   In the following, we will show that the restriction on performance index J_1 given in (2.19) is satisfied under zero initial conditions for all nonzero $d \in L_2[0, \infty)$. In this case, the error dynamics (3.2) is expressed by

$$
\dot{e} = \overline{A}e + \overline{B}\Phi\left(\overline{C}e; X_s\right) + \overline{B}_d d, \quad r = \overline{H}e.
\tag{A.8}
$$

Based on (A.6) and (A.8), it is not difficult to derive

$$
r^T r - \gamma^2 d^T d + \dot{V} \leq \eta_1^T \Xi_1 \eta_1,
\tag{A.9}
$$

where $\eta_1 = [e^T \ \Phi^T \ \dot{e}^T \ \Psi^T \ d^T]^T$, and Ξ_1 is described in condition (3.3) with $\Xi_1 < 0$. It further implies that for any $d \neq 0$, $r(t)^T r(t) - \gamma^2 d^T(t)d(t) + \dot{V}(t) < 0$. Under zero initial condition, the Lyapunov function V defined in (A.2) satisfies $V(0) = 0$ and $V(t) \geq 0$ for $t > 0$, hence

$$
\begin{aligned}
J_1 &\leq \int_0^\infty \left[r(t)^T r(t) - \gamma^2 d^T(t)d(t)\right] dt + V(t)|_{t\to\infty} - V(0) \\
&= \int_0^\infty \left[r(t)^T r(t) - \gamma^2 d^T(t)d(t) + \dot{V}(t)\right] < 0,
\end{aligned}
\tag{A.10}
$$

and (2.19) is satisfied, which completes the proof.   □

B. Proof of Theorem 3.4

Proof. On the basis of Theorem 3.2, it is known that if there exist solutions to LMI (3.3), the network achieves global synchronization and robust to input disturbances. As for the

condition of fault detection within the $\mathcal{H}_\infty$ synchronization, namely, the synchronization process should be sensitive to possible input faults, it then comes to an extra verification of condition (2.20) under zero initial conditions for all nonzero $f \in \mathbb{L}_2[0, \infty]$. In this situation, the error dynamics is given by

$$\dot{e} = \overline{A}e + \overline{B}\Phi\left(\overline{C}e; X_s\right) + \overline{B}_f f, \quad r = \overline{H}e + \overline{D}f. \tag{B.1}$$

Following the same line of the proof of $J_1 < 0$ in Theorem 3.2, we know that if

$$-J_2 \leq \int_0^\infty \left[\beta^2 f^T(t) f(t) - r(t)^T r(t)\right] dt + V(t)|_{t \to \infty} - V(0)$$

$$= \int_0^\infty \left[\beta^2 f^T(t) f(t) - r(t)^T r(t) + \dot{V}(t)\right] < 0 \tag{B.2}$$

holds, then the constraint (2.20) will be satisfied where V is defined in (A.2), and further, the inequality condition (B.2) is guaranteed by

$$\beta^2 f^T f - r^T r + \dot{V} \leq \eta_2^T \Xi_2 \eta_2^T < 0, \tag{B.3}$$

where $\eta_2 = [e^T\ \Phi^T\ \dot{e}^T\ \Psi^T\ f^T]^T$ with $\Xi_2 < 0$ given in (3.4). Thus the performance index $J_2 > 0$ is satisfied, and the proof is completed. $\qquad\square$

C. Proof of Theorem 3.5

Proof. To facilitate the design of the coupling matrix Γ, we designate $\overline{Q}_1 = \overline{S}$ and $\overline{Q}_2 = \alpha\overline{S}$, respectively, where α is a constant scalar; also it can be seen from (3.3) that $\alpha(\overline{S} + \overline{S}^T) < 0$, and thus $\overline{S}$ is nonsingular.

Recall that there exists a unitary matrix U such that $U^T G U = \Lambda$ with Λ defined in (2.9). Pre- and postmultiplying to both sides of LMIs (3.3) by $\overline{U} = \text{diag}(U^T \otimes I_N, U^T \otimes I_m, U^T \otimes I_m, U^T \otimes I_m)$ and $\overline{U}^T$ yields

$$\begin{pmatrix} -\text{He}\left(\overline{VA}_\Lambda\right) + \overline{H}^T\overline{H} & \overline{C}^T\Theta\Delta - \overline{V}\overline{B} & \overline{W} + \overline{V} - \alpha\overline{A}_\lambda^T\overline{V}^T & \overline{C}^T\Theta\Omega & -\overline{V}\overline{B}_d \\ * & -\text{He}\,\Delta & -\alpha\overline{B}^T\overline{V}^T & 0 & 0 \\ * & * & \text{He}\,\alpha\overline{V} & \overline{C}^T\Pi & -\alpha\overline{V}\overline{B}_d \\ * & * & * & -\text{He}\,\Omega & 0 \\ * & * & * & * & -\gamma^2 I \end{pmatrix} < 0, \tag{C.1}$$

where $\overline{V} = (U^T \otimes I_N)\overline{S}(U \otimes I_N)$, $\overline{W} = (U^T \otimes I_N)\overline{P}(U \otimes I_N)$, and $\overline{A}_\Lambda = I \otimes A + \Lambda \otimes \Gamma H$. It implies from (C.1) that all the matrices appearing in this LMI are diagonal except for matrices $\overline{V}, \overline{W}$. To this end, suppose that there exist matrices V_i and W_i such that for $i = 1, 2, \ldots, N$, the N LMIs (3.5) hold; then there must exist diagonal matrices $\overline{V} = \text{diag}(V_1, V_2, \ldots, V_N)$

and $\overline{W} = \text{diag}(W_1, W_2, \ldots, W_N)$ as solutions to condition (C.1), and accordingly (3.3) holds. In a similar pattern, the feasibility of LMI (3.6) means that condition (3.4) holds.

Moreover, since the coupling matrix G has q distinct different eigenvalues as (2.9), it is evident to find that the number of LMI groups to be examined in (3.5)-(3.6) can be reduced from N to q. On the other hand, it is noted that due to the convex property of LMI [26], each of the rest $q - 3$ groups of LMIs for $i = 3, \ldots, q - 1$ can be written as a linear combination of the two groups of LMIs corresponding to the second-maximum λ_2 and the minimum eigenvalue λ_q. In this situation, the synchronization condition only requires the feasibility of three groups LMIs (3.5)-(3.6) with $i = 1, 2$ and q; thus it completes the proof.   □

D. Solution of LMIs (3.5) for $i = 1, 2$ and q

One has

$$W_1 = \begin{pmatrix} 14.4037 & -10.8380 & 0.4396 \\ -10.8380 & 30.5854 & 3.0167 \\ 0.4396 & 3.0167 & 7.0477 \end{pmatrix}, \qquad V_1 = \begin{pmatrix} -1.6269 & -2.0020 & -0.2305 \\ -1.0598 & -6.5768 & -0.7952 \\ 0.0584 & 0.3076 & -0.2281 \end{pmatrix},$$

$$\Pi_1 = 1.9512, \qquad \Delta_1 = 7.6035, \qquad \Lambda_1 = 2.5538,$$

$$W_2 = \cdots = W_9 = \begin{pmatrix} 11.0778 & -5.0711 & 1.8522 \\ -5.0711 & 17.0729 & 3.4321 \\ 1.8522 & 3.4321 & 9.6299 \end{pmatrix},$$

$$V_2 = \cdots = V_9 = \begin{pmatrix} -0.9218 & -1.0373 & -0.1830 \\ -0.3643 & -3.0024 & -0.4393 \\ -0.0293 & 0.3875 & -0.3124 \end{pmatrix},$$

$$\Pi_2 = \cdots = \Pi_9 = 1.8828, \qquad \Delta_2 = \cdots = \Delta_9 = 5.0301, \qquad \Lambda_2 = \cdots = \Lambda_9 = 2.9445$$

$$W_{10} = \begin{pmatrix} 7.1144 & -1.1548 & 0.5316 \\ -1.1548 & 5.1002 & 1.5201 \\ 0.5316 & 1.5201 & 4.5129 \end{pmatrix}, \qquad V_{10} = \begin{pmatrix} -0.1892 & -0.1842 & -0.0411 \\ -0.0345 & -0.7973 & -0.1416 \\ 0.0078 & 0.2423 & -0.1422 \end{pmatrix},$$

$$\Pi_{10} = 1.0396, \qquad \Delta_{10} = 1.9588, \qquad \Lambda_{10} = 2.0045.$$

(D.1)

E. Solution of LMIs (3.5)-(3.6) for $i = 1, 2$ and q

One has

$$W_1 = \begin{pmatrix} 19.4357 & -10.2421 & -0.2580 \\ -10.2421 & 50.9203 & 4.7464 \\ -0.2580 & 4.7464 & 0.5780 \end{pmatrix}, \qquad V_1 = \begin{pmatrix} -2.5375 & -4.4290 & -0.4708 \\ -2.4266 & -15.4850 & -1.8584 \\ -0.3124 & -1.6514 & -0.2009 \end{pmatrix},$$

$$\Pi_1 = 2.1700 \times 10^{-6}, \qquad \Delta_1 = 12.8948, \qquad \Lambda_1 = 1.0216 \times 10^{-12},$$

$$W_2 = \cdots = W_9 = \begin{pmatrix} 31.7702 & -9.3411 & 0.1724 \\ -9.3411 & 56.7357 & 5.3885 \\ 0.1724 & 5.3885 & 0.6592 \end{pmatrix},$$

$$V_2 = \cdots = V_9 = \begin{pmatrix} -3.0576 & -6.0476 & -0.6670 \\ -1.8406 & -15.4908 & -1.8934 \\ -0.2677 & -1.7028 & -0.2106 \end{pmatrix},$$

$$\Pi_2 = \cdots = \Pi_9 = 1.5037 \times 10^{-6}, \qquad \Delta_2 = \cdots = \Delta_9 = 20.8139,$$

$$\Lambda_2 = \cdots = \Lambda_9 = 2.0313 \times 10^{-12} w$$

$$W_{10} = \begin{pmatrix} 275.7624 & -44.0445 & 1.2220 \\ -44.0445 & 59.4247 & 5.0031 \\ 1.2220 & 5.0031 & 0.6368 \end{pmatrix}, \qquad V_{10} = \begin{pmatrix} -7.0320 & -9.6985 & -1.1098 \\ 0.4957 & -10.1060 & -1.2957 \\ -0.0845 & -1.1706 & -0.1522 \end{pmatrix},$$

$$\Pi_{10} = 3.6238 \times 10^{-6}, \qquad \Delta_{10} = 114.4167, \qquad \Lambda_{10} = 3.0198 \times 10^{-12}.$$

(E.1)

Acknowledgments

References

[1] R. Albert and A. Barabási, "Statistical mechanics of complex networks," *Reviews of Modern Physics*, vol. 74, no. 1, pp. 47–97, 2002.

[2] X. F. Wang and G. Chen, "Complex networks: small-world, scale-free and beyond," *IEEE Circuits and Systems Magazine*, vol. 3, no. 1, pp. 6–20, 2003.

[3] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, "Complex networks: structure and dynamics," *Physics Reports*, vol. 424, no. 4-5, pp. 175–308, 2006.

[4] X. F. Wang and G. Chen, "Synchronization in small-world dynamical networks," *International Journal of Bifurcation and Chaos*, vol. 12, no. 1, pp. 187–192, 2002.

[5] M. Barahona and L. M. Pecora, "Synchronization in small-world systems," *Physical Review Letters*, vol. 89, no. 5, Article ID 054101, pp. 1–4, 2002.

[6] S. Strogatz, *Sync: The Emerging Science of Spontaneous Order*, Hyperion Books, New York, NY, USA, 2003.

[7] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Transactions on Automatic Control*, vol. 50, no. 6, pp. 841–846, 2005.

[8] S. Xu and Y. Yang, "Global asymptotical stability and generalized synchronization of phase synchronous dynamical networks," *Nonlinear Dynamics*, vol. 59, no. 3, pp. 485–496, 2010.

[9] C. W. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems*, World Scientific, 2006.

[10] X. Liu, J. Wang, and L. Huang, "Global synchronization for a class of dynamical complex networks," *Physica A*, vol. 386, no. 1, pp. 543–556, 2007.

[11] X. Liu, J. Wang, and L. Huang, "Stabilization of a class of dynamical complex networks based on decentralized control," *Physica A*, vol. 383, no. 2, pp. 733–744, 2007.

[12] S. Xu and Y. Yang, "Synchronization for a class of complex dynamical networks with time-delay," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 8, pp. 3230–3238, 2009.

[13] L. O. Chua, "Chuas circuit: an overview ten years later," *Journal of Circuits, Systems, and Computers*, vol. 4, pp. 117–159, 1994.

[14] P. Ruoff, M. Vinsjevik, C. Monnerjahn, and L. Rensing, "The goodwin model: simulating the effect of light pulses on the circadian sporulation rhythm of Neurospora crassa," *Journal of Theoretical Biology*, vol. 209, no. 1, pp. 29–42, 2001.

[15] V. Gazi and K. M. Passino, "Stability analysis of swarms," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 692–697, 2003.

[16] M. Vidyasagar, *Nonlinear Systems Analysis*, Prentice-Hall, Upper Saddle River, NJ, USA, 1993.

[17] B. Shen, Z. Wang, and X. Liu, "Bounded H_∞ synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon," *IEEE Transactions on Neural Networks*, vol. 22, no. 1, pp. 145–157, 2011.

[18] Y. Tang, Z. Wang, W. K. Wong, J. Kurths, and J. Fang, "Multiobjective synchronization of coupled systems," *Chaos*, vol. 21, no. 2, Article ID 025114, 2011.

[19] J. Chen and R. J. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Kluwer Academic Publishers, Boston, Mass, USA, 1999.

[20] P. M. Frank, "Enhancement of robustness in observer-based fault detection," *International Journal of Control*, vol. 59, no. 4, pp. 955–981, 1994.

[21] S. X. Ding, *Model-Based Fault Diagnosis Techniques, Design Schemes, Algorithms, and Tools*, Springer, Berlin, Germany, 2008.

[22] J. L. Wang, G.-H. Yang, and J. Liu, "An LMI approach to H_∞ index and mixed H_- / H_∞ fault detection observer design," *Automatica*, vol. 43, no. 9, pp. 1656–1665, 2007.

[23] X. Li and K. Zhou, "A time domain approach to robust fault detection of linear time-varying systems," *Automatica*, vol. 45, no. 1, pp. 94–102, 2009.

[24] S. -Y. Xu, Y. Yang, X. Liu, Y. Tang, and H. -D. Sun, "Robust fault-sensitive synchronization of a class of nonlinear systems," *Chinese Physics B*, vol. 20, no. 2, Article ID 020509, 2011.

[25] M. Hou and R. J. Patton, "An LMI approach to H_- / H_∞ fault detection observers," in *Proceedings of the UKACC International Conference on Control*, pp. 305–310, 1996.

[26] S. Boyd, L. ELGhaoui, E. Feron, and V. Balakrishnam, *Linear Matrix Inequalities in Systems and Control*, SIMA, Philadelphia, Pa, USA, 1994.

[27] C. W. Wu, *Synchronization in Coupled Chaotic Circuits and Systems*, World Scientific, Singapore, 2002.

[28] C. W. Wu and L. O. Chua, "Application of Kronecker products to the analysis of systems with uniform linear coupling," *IEEE Transactions on Circuits and Systems I*, vol. 42, no. 10, pp. 775–778, 1995.

[29] M. Wu, Y. He, J. H. She, and G. P. Liu, "Delay-dependent criteria for robust stability of time-varying delay systems," *Automatica*, vol. 40, no. 8, pp. 1435–1439, 2004.

[30] *Chua's Circuit: A Paradigm for Chaos*, vol. 1, World Scientific, Singapore, 1993.

[31] Z. Wang, Y. Wang, and Y. Liu, "Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 1, Article ID 5342442, pp. 11–25, 2010.

[32] J. Liang, Z. Wang, and X. Liu, "State estimation for coupled uncertain stochastic networks with missing measurements and time-varying delays: the discrete-time case," *IEEE Transactions on Neural Networks*, vol. 20, no. 5, pp. 781–793, 2009.

[33] Y. Liu, Z. Wang, J. Liang, and X. Liu, "Synchronization and state estimation for discrete-time complex networks with distributed delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 38, no. 5, pp. 1314–1325, 2008.

[34] J. Liang, Z. Wang, Y. Liu, and X. Liu, "Global synchronization control of general delayed discrete-time networks with stochastic coupling and disturbances," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 38, no. 4, pp. 1073–1083, 2008.