

Research Article

Numerov's Method for a Class of Nonlinear Multipoint Boundary Value Problems

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The purpose of this paper is to give a numerical treatment for a class of nonlinear multipoint boundary value problems. The multipoint boundary condition under consideration includes various commonly discussed boundary conditions, such as the three- or four-point boundary condition. The problems are discretized by the fourth-order Numerov's method. The existence and uniqueness of the numerical solution are investigated by the method of upper and lower solutions. The convergence and the fourth-order accuracy of the method are proved. An accelerated monotone iterative algorithm with the quadratic rate of convergence is developed for solving the resulting nonlinear discrete problems. Some applications and numerical results are given to demonstrate the high efficiency of the approach.

1. Introduction

Multipoint boundary value problems arise in various fields of applied science. An often discussed problem is the following nonlinear second-order multipoint boundary value problem:

$$\begin{aligned} -u''(x) &= f(x, u(x)), \quad 0 < x < 1, \\ u(0) &= \sum_{i=1}^p \alpha_i u(\xi_i), \quad u(1) = \sum_{i=1}^p \beta_i u(\eta_i), \end{aligned} \tag{1.1}$$

where $f(x, u)$ is a continuous function of its arguments and for each i , $\alpha_i, \beta_i \in [0, \infty)$ and $\xi_i, \eta_i \in (0, 1)$. An application of this problem appears in the design of a large-size bridge with multipoint supports, where $u(x)$ denotes the displacement of the bridge from the unloaded position (e.g., see [1]). For other applications of problem (1.1), we see [2–4] and the references therein. It is allowed in (1.1) that $\alpha_i = 0$ or $\beta_i = 0$ for some or all i . This implies that the boundary condition in (1.1) includes various commonly discussed multipoint boundary conditions. In particular, the boundary condition in (1.1) is reduced to

$$u(0) = 0, \quad u(1) = \sum_{i=1}^p \beta_i u(\eta_i), \quad (1.1_a)$$

if $\alpha_i = 0$ for all i (see [5–14]), to the form

$$u(0) = \sum_{i=1}^p \alpha_i u(\xi_i), \quad u(1) = 0, \quad (1.1_b)$$

if $\beta_i = 0$ for all i (see [15]), to the four-point boundary condition

$$u(0) = \alpha u(\xi), \quad u(1) = \beta u(\eta), \quad (1.1_c)$$

if $p = 1$ and $\xi_1 = \xi, \eta_1 = \eta$ (see [11, 15–17]), and to the two-point boundary condition

$$u(0) = u(1) = 0, \quad (1.1_d)$$

if $\alpha_i = 0$ and $\beta_i = 0$ for all i . Condition (1.1_c) includes the three-point boundary condition when $\xi = \eta$ (see [16, 17]).

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated in [18, 19] by Il'in and Moiseev. In [20], Gupta studied a three-point boundary value problem for nonlinear second-order ordinary differential equations. Since then, more general nonlinear second-order multipoint boundary value problems in the form (1.1) have been studied. Most of the discussions were concerned with the existence and multiplicity of solutions by using different methods. Applying the fixed point index theorem in cones, the works in [5–14] showed the existence of one or more solutions to the problem (1.1)-(1.1_a), while the works in [15–17] were devoted to the existence of solutions for the three- or four-point boundary value problem (1.1)-(1.1_c). For the problem (1.1) with the more general multipoint boundary conditions, some existence results were obtained in [21, 22] by using the fixed point index theory or the topological degree theory. Based on the method of upper and lower solutions, the authors of [17, 23] obtained some sufficient conditions so that (1.1) or its some special form has at least one solution. Additional works that deal with the existence problem of nonlinear second-order multipoint boundary value problems can be found in [24–29].

On the other hand, there are also some works that are devoted to numerical methods for the solutions of multipoint boundary value problems. The work in [30] made use of the Chebyshev series for approximating solutions of nonlinear first-order multipoint

boundary value problems, and the work in [31] showed how an adaptive finite difference technique can be developed to produce efficient approximations to the solutions of nonlinear multipoint boundary value problems for first-order systems of equations. Another method for computing the solutions of nonlinear first-order multipoint boundary value problems was described in [32], where a multiple shooting technique was developed. Some other works for the computational methods of first-order multipoint boundary value problems can be seen in [33–35]. In [36–38] the authors gave several constructive methods for the solutions of multipoint discrete boundary value problems, including the method of adjoints, the invariant embedding method, and the shooting-type method. In the case of second-order multipoint boundary value problems, there are only a few computational algorithms in the literature. The paper [39] set up a reproducing kernel Hilbert space method for the solution of a second-order three-point boundary value problem. Based upon the shooting technique, a numerical method was developed in [1] for approximating solutions and fold bifurcation solutions of a class of second-order multipoint boundary value problems.

As we know, Numerov's method is one of the well-known difference methods to solve the second-order ordinary differential equation $-u'' = f(x, u)$. Because Numerov's method possesses the fourth-order accuracy and a compact property, it has attracted considerable attention and has been extensively applied in practical computations (cf. [40–51]). Although many theoretical investigations have focused on Numerov's method for two-point boundary conditions such as (1.1_d) (cf. [40, 41, 43, 44, 47–51]), there is relatively little discussion on the analysis of Numerov's method applied to fully multipoint boundary conditions in (1.1). The study presented in this paper is aimed at filling in such a gap by considering Numerov's method for the numerical solution of the multipoint boundary value problem (1.1) with the more general boundary conditions, including the boundary conditions (1.1_a), (1.1_b), and (1.1_c). It is not difficult to give a Numerov's difference approximation to (1.1) in the same manner as that for two-point boundary value problems. However, a lack of explicit information about the boundary value of the solution in the multipoint boundary conditions prevents us from using the standard analysis process of treating two-point boundary value problems, and so we here develop a different approach for the analysis of Numerov's difference approximation to (1.1). Our specific goals are (1) to establish the existence and uniqueness of the numerical solution, (2) to show the convergence of the numerical solution to the analytic solution with the fourth-order accuracy, and (3) to develop an efficient computational algorithm for solving the resulting nonlinear discrete problems. To achieve the above goals, we use the method of upper and lower solutions and its associated monotone iterations. It should be mentioned that the proposed fourth-order Numerov's discretization methodology may be straightforwardly extended to the following nonhomogeneous multipoint boundary condition:

$$u(0) = \sum_{i=1}^p \alpha_i u(\xi_i) + \lambda_1, \quad u(1) = \sum_{i=1}^p \beta_i u(\eta_i) + \lambda_2, \quad (1.2)$$

where λ_1 and λ_2 are two prescribed constants.

The outline of the paper is as follows. In Section 2, we discretize (1.1) into a finite difference system by Numerov's technique. In Section 3, we deal with the existence and

uniqueness of the numerical solution by using the method of upper and lower solutions. The convergence of the numerical solution and the fourth-order accuracy of the method are proved in Section 4. Section 5 is devoted to an accelerated monotone iterative algorithm for solving the resulting nonlinear discrete problem. Using an upper solution and a lower solution as initial iterations, the iterative algorithm yields two sequences that converge monotonically from above and below, respectively, to a unique solution of the resulting nonlinear discrete problem. It is shown that the rate of convergence for the sum of the two produced sequences is quadratic (the error metric is the sum of the infinity norm of the error between the m th-iteration of the upper solution and the true solution with the infinity norm of the error between the m th-iteration of the lower solution and the true solution) and under an additional requirement, the quadratic rate of convergence is attained for one of these two sequences. In Section 6, we give some applications to three model problems and present some numerical results demonstrating the monotone and rapid convergence of the iterative sequences and the fourth-order accuracy of the method. We also compare our method with the standard finite difference method and show its advantages. The final section contains some concluding remarks.

2. Numerov's Method

Let $h = 1/L$ be the mesh size, and let $x_i = ih$ ($0 \leq i \leq L$) be the mesh points in $[0, 1]$. Assume that for all $1 \leq i \leq p$, the points ξ_i and η_i in the boundary condition of (1.1) serve as mesh points. This assumption is always satisfied by a proper choice of mesh size h . For convenience, we use the following notations:

$$S_\alpha[u(\xi)] = \sum_{i=1}^p \alpha_i u(\xi_i), \quad S_\beta[u(\eta)] = \sum_{i=1}^p \beta_i u(\eta_i) \quad (2.1)$$

and introduce the finite difference operators δ_h^2 and \mathcal{D}_h as follows:

$$\begin{aligned} \delta_h^2 u(x_i) &= u(x_{i-1}) - 2u(x_i) + u(x_{i+1}), \quad 1 \leq i \leq L-1, \\ \mathcal{D}_h u(x_i) &= \frac{h^2}{12} (u(x_{i-1}) + 10u(x_i) + u(x_{i+1})), \quad 1 \leq i \leq L-1. \end{aligned} \quad (2.2)$$

Using the following Numerov's formula (cf. [52, 53]):

$$\delta_h^2 u(x_i) = \mathcal{D}_h u''(x_i) + \mathcal{O}(h^6), \quad 1 \leq i \leq L-1, \quad (2.3)$$

we have from (1.1) and (2.1) that

$$\begin{aligned} -\delta_h^2 u(x_i) &= \mathcal{D}_h f(x_i, u(x_i)) + \mathcal{O}(h^6), \quad 1 \leq i \leq L-1, \\ u(0) &= S_\alpha[u(\xi)], \quad u(1) = S_\beta[u(\eta)]. \end{aligned} \quad (2.4)$$

After dropping the $\mathcal{O}(h^6)$ term, we derive a Numerov's difference approximation to (1.1) as follows:

$$\begin{aligned} -\delta_h^2 u_h(x_i) &= \mathcal{D}_h f(x_i, u_h(x_i)), \quad 1 \leq i \leq L-1, \\ u_h(0) &= S_\alpha[u_h(\xi)], \quad u_h(1) = S_\beta[u_h(\eta)], \end{aligned} \quad (2.5)$$

where $u_h(x_i)$ represents the approximation of $u(x_i)$.

For two constants \underline{M} and \overline{M} satisfying $\overline{M} \geq \underline{M} > -\pi^2$, we define

$$h(\underline{M}, \overline{M}) = \begin{cases} \sqrt{\frac{12}{\underline{M}}}, & \underline{M} > -8, \overline{M} > 0, \\ 1, & \underline{M} > -8, \overline{M} \leq 0, \\ \min \left\{ \sqrt{\frac{12}{\underline{M}}}, \sqrt{\frac{12}{\pi^2} \left(1 + \frac{\underline{M}}{\pi^2}\right)} \right\}, & \underline{M} \leq -8, \overline{M} > 0, \\ \sqrt{\frac{12}{\pi^2} \left(1 + \frac{\underline{M}}{\pi^2}\right)}, & \underline{M} \leq -8, \overline{M} \leq 0. \end{cases} \quad (2.6)$$

A fundamental and useful property of the operators δ_h^2 and \mathcal{D}_h is stated below.

Lemma 2.1 (See Lemma 3.1 of [50]). *Let \underline{M} , \overline{M} , and M_i be some constants satisfying*

$$-\pi^2 < \underline{M} \leq M_i \leq \overline{M}, \quad 0 \leq i \leq L. \quad (2.7)$$

If

$$\begin{aligned} -\delta_h^2 u_h(x_i) + \mathcal{D}_h(M_i u_h(x_i)) &\geq 0, \quad 1 \leq i \leq L-1, \\ u_h(0) &\geq 0, \quad u_h(1) \geq 0, \end{aligned} \quad (2.8)$$

and $h < h(\underline{M}, \overline{M})$, then $u_h(x_i) \geq 0$ for all $0 \leq i \leq L$.

The following results are also useful for our forthcoming discussions. Their proofs will be given in the appendix.

Lemma 2.2. *Assume*

$$\sigma \equiv \max \left\{ \sum_{i=1}^p \alpha_i, \sum_{i=1}^p \beta_i \right\} < 1. \quad (2.9)$$

Let \underline{M} , \overline{M} , and M_i be the given constants such that

$$-8(1 - \sigma) < \underline{M} \leq M_i \leq \overline{M}, \quad 0 \leq i \leq L. \quad (2.10)$$

If

$$\begin{aligned} -\delta_h^2 u_h(x_i) + \mathcal{D}_h(M_i u_h(x_i)) &\geq 0, \quad 1 \leq i \leq L-1, \\ u_h(0) &\geq S_\alpha[u_h(\xi)], \quad u_h(1) \geq S_\beta[u_h(\eta)], \end{aligned} \quad (2.11)$$

and $h < h(\underline{M}, \overline{M})$, then $u_h(x_i) \geq 0$ for all $0 \leq i \leq L$.

Lemma 2.3. Let the condition (2.9) be satisfied, and let \underline{M} , \overline{M} , and M_i be the given constants satisfying (2.10). Assume that the functions $u_h(x_i)$ and $g(x_i)$ satisfy

$$\begin{aligned} -\delta_h^2 u_h(x_i) + \mathcal{D}_h(M_i u_h(x_i)) &= g(x_i), \quad 1 \leq i \leq L-1, \\ u_h(0) &= S_\alpha[u_h(\xi)], \quad u_h(1) = S_\beta[u_h(\eta)]. \end{aligned} \quad (2.12)$$

Then when $h < h(\underline{M}, \overline{M})$,

$$\|u_h\|_\infty \leq \frac{\|g\|_\infty}{(8(1-\sigma) + \min(\underline{M}, 0))h^2}, \quad (2.13)$$

where $\|u_h\|_\infty = \max_{1 \leq i \leq L-1} |u_h(x_i)|$ denotes discrete infinity norm for any mesh function $u_h(x_i)$.

Remark 2.4. It is clear from Lemma 2.1 that if $\sigma = 0$ then the condition (2.10) in Lemma 2.2 can be replaced by the weaker condition (2.7). Lemmas 2.1 and 2.2 guarantee that the linear problems based on (2.8) and (2.11) with the inequality relation “ \geq ” replaced by the equality relation “ $=$ ” are well posed.

3. The Existence and Uniqueness of the Solution

To investigate the existence and uniqueness of the solution of (2.5), we use the method of upper and lower solutions. The definition of the upper and lower solutions is given as follows.

Definition 3.1. A function $\tilde{u}_h(x_i)$ is called an upper solution of (2.5) if

$$\begin{aligned} -\delta_h^2 \tilde{u}_h(x_i) &\geq \mathcal{D}_h f(x_i, \tilde{u}_h(x_i)), \quad 1 \leq i \leq L-1, \\ \tilde{u}_h(0) &\geq S_\alpha[\tilde{u}_h(\xi)], \quad \tilde{u}_h(1) \geq S_\beta[\tilde{u}_h(\eta)]. \end{aligned} \quad (3.1)$$

Similarly, a function $\hat{u}_h(x_i)$ is called a lower solution of (2.5) if it satisfies the above inequalities in the reversed order. A pair of upper and lower solutions $\tilde{u}_h(x_i)$ and $\hat{u}_h(x_i)$ are said to be ordered if $\tilde{u}_h(x_i) \geq \hat{u}_h(x_i)$ for all $0 \leq i \leq L$.

It is clear that every solution of (2.5) is an upper solution as well as a lower solution. For a given pair of ordered upper and lower solutions $\tilde{u}_h(x_i)$ and $\hat{u}_h(x_i)$, we set

$$\begin{aligned} \langle \hat{u}_h, \tilde{u}_h \rangle &= \{u_h; \hat{u}_h(x_i) \leq u_h(x_i) \leq \tilde{u}_h(x_i) (0 \leq i \leq L)\}, \\ [\hat{u}_h(x_i), \tilde{u}_h(x_i)] &= \{u_i \in \mathbf{R}; \hat{u}_h(x_i) \leq u_i \leq \tilde{u}_h(x_i)\} \end{aligned} \quad (3.2)$$

and make the following basic hypotheses:

(H₁) For each $0 \leq i \leq L$, there exists a constant M_i such that $M_i > -\pi^2$ and

$$f(x_i, v_i) - f(x_i, v'_i) \geq -M_i(v_i - v'_i) \quad (3.3)$$

whenever $\hat{u}_h(x_i) \leq v'_i \leq v_i \leq \tilde{u}_h(x_i)$;

(H₂) $h < h(\underline{M}, \overline{M})$, where $\overline{M} = \max_{0 \leq i \leq L} M_i$ and $\underline{M} = \min_{0 \leq i \leq L} M_i$.

The existence of the constant M_i in (H₁) is trivial if $f(x_i, u)$ is a C^1 -function of $u \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)]$. In fact, M_i may be taken as any nonnegative constant satisfying

$$M_i \geq \max\{-f_u(x_i, u); u \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)]\}. \quad (3.4)$$

Theorem 3.2. *Let $\tilde{u}_h(x_i)$ and $\hat{u}_h(x_i)$ be a pair of ordered upper and lower solutions of (2.5), and let hypotheses (H₁) and (H₂) be satisfied. Then system (2.5) has a maximal solution $\bar{u}_h(x_i)$ and a minimal solution $\underline{u}_h(x_i)$ in $\langle \hat{u}_h, \tilde{u}_h \rangle$. Here, the maximal property of $\bar{u}_h(x_i)$ means that for any solution $u_h(x_i)$ of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$, one has $u_h(x_i) \leq \bar{u}_h(x_i)$ for all $0 \leq i \leq L$. The minimal property of $\underline{u}_h(x_i)$ is similarly understood.*

Proof. The proof is constructive. Using the initial iterations $\bar{u}_h^{(0)}(x_i) = \tilde{u}_h(x_i)$ and $\underline{u}_h^{(0)}(x_i) = \hat{u}_h(x_i)$ we construct two sequences $\{\bar{u}_h^{(m)}(x_i)\}$ and $\{\underline{u}_h^{(m)}(x_i)\}$, respectively, from the following iterative scheme:

$$\begin{aligned} -\delta_h^2 \underline{u}_h^{(m)}(x_i) + \mathcal{D}_h(M_i \underline{u}_h^{(m)}(x_i)) &= \mathcal{D}_h(M_i \underline{u}_h^{(m-1)}(x_i) + f(x_i, \underline{u}_h^{(m-1)}(x_i))), \quad 1 \leq i \leq L-1, \\ \underline{u}_h^{(m)}(0) &= S_\alpha[u_h^{(m-1)}(\xi)], \quad \underline{u}_h^{(m)}(1) = S_\beta[u_h^{(m-1)}(\eta)], \end{aligned} \quad (3.5)$$

where M_i is the constant in (H₁). By Lemma 2.1, these two sequences are well defined. We shall first prove that for all $m = 0, 1, \dots$,

$$\underline{u}_h^{(m)}(x_i) \leq \underline{u}_h^{(m+1)}(x_i) \leq \bar{u}_h^{(m+1)}(x_i) \leq \bar{u}_h^{(m)}(x_i), \quad 0 \leq i \leq L. \quad (3.6)$$

Let $\bar{w}_h^{(0)}(x_i) = \bar{u}_h^{(0)}(x_i) - \bar{u}_h^{(1)}(x_i)$. Then by (3.1) and (3.5),

$$\begin{aligned} -\delta_h^2 \bar{w}_h^{(0)}(x_i) + \mathcal{D}_h(M_i \bar{w}_h^{(0)}(x_i)) &\geq 0, \quad 1 \leq i \leq L-1, \\ \bar{w}_h^{(0)}(0) &\geq 0, \quad \bar{w}_h^{(0)}(1) \geq 0. \end{aligned} \quad (3.7)$$

It follows from Lemma 2.1 that $\bar{w}_h^{(0)}(x_i) \geq 0$, that is, $\bar{u}_h^{(0)}(x_i) \geq \bar{u}_h^{(1)}(x_i)$ for all $0 \leq i \leq L$. A similar argument using the property of a lower solution gives $\underline{u}_h^{(1)}(x_i) \geq \underline{u}_h^{(0)}(x_i)$ for all $0 \leq i \leq L$. Let $w_h^{(1)}(x_i) = \bar{u}_h^{(1)}(x_i) - \underline{u}_h^{(1)}(x_i)$. We have from (3.3) and (3.5) that

$$\begin{aligned} -\delta_h^2 w_h^{(1)}(x_i) + \mathcal{P}_h \left(M_i w_h^{(1)}(x_i) \right) &\geq 0, \quad 1 \leq i \leq L-1, \\ w_h^{(1)}(0) &\geq 0, \quad w_h^{(1)}(1) \geq 0. \end{aligned} \quad (3.8)$$

Again by Lemma 2.1, $w_h^{(1)}(x_i) \geq 0$, that is, $\bar{u}_h^{(1)}(x_i) \geq \underline{u}_h^{(1)}(x_i)$ for all $0 \leq i \leq L$. This proves (3.6) for $m = 0$. Finally, an induction argument leads to the desired result (3.6) for all $m = 0, 1, \dots$

In view of (3.6), the limits

$$\lim_{m \rightarrow \infty} \bar{u}_h^{(m)}(x_i) = \bar{u}_h(x_i), \quad \lim_{m \rightarrow \infty} \underline{u}_h^{(m)}(x_i) = \underline{u}_h(x_i), \quad 0 \leq i \leq L \quad (3.9)$$

exist and satisfy

$$\underline{u}_h^{(m)}(x_i) \leq \underline{u}_h^{(m+1)}(x_i) \leq \underline{u}_h(x_i) \leq \bar{u}_h(x_i) \leq \bar{u}_h^{(m+1)}(x_i) \leq \bar{u}_h^{(m)}(x_i), \quad 0 \leq i \leq L, \quad m = 0, 1, \dots \quad (3.10)$$

Letting $m \rightarrow \infty$ in (3.5) shows that both $\bar{u}_h(x_i)$ and $\underline{u}_h(x_i)$ are solutions of (2.5).

Now, if $u_h(x_i)$ is a solution of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$, then the pair $u_h(x_i)$ and $\hat{u}_h(x_i)$ are also a pair of ordered upper and lower solutions of (2.5). The above arguments imply that $\underline{u}_h(x_i) \leq u_h(x_i)$ for all $0 \leq i \leq L$. Similarly, we have $u_h(x_i) \leq \bar{u}_h(x_i)$ for all $0 \leq i \leq L$. This shows that $\bar{u}_h(x_i)$ and $\underline{u}_h(x_i)$ are the maximal and the minimal solutions of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$, respectively. The proof is completed. \square

Theorem 3.2 shows that the system (2.5) has a maximal solution $\bar{u}_h(x_i)$ and a minimal solution $\underline{u}_h(x_i)$ in $\langle \hat{u}_h, \tilde{u}_h \rangle$. If $\bar{u}_h(x_i) = \underline{u}_h(x_i)$ for all $0 \leq i \leq L$, then $\bar{u}_h(x_i)$ or $\underline{u}_h(x_i)$ is a unique solution of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$. In general, these two solutions do not coincide. Consider, for example, the case

$$\sum_{i=1}^p \alpha_i = \sum_{i=1}^p \beta_i = 1. \quad (3.11)$$

If there exist two different constants \bar{c} and \underline{c} such that $f(x, \bar{c}) = f(x, \underline{c}) = 0$ for all $x \in (0, 1)$ then both \bar{c} and \underline{c} are solutions of (2.5). Hence to show the uniqueness of a solution it is necessary to impose some additional conditions on α_i , β_i and f . Assume that there exists a constant \underline{M}_u such that

$$f(x_i, v_i) - f(x_i, v'_i) \leq -\underline{M}_u (v_i - v'_i), \quad 0 \leq i \leq L \quad (3.12)$$

whenever $\hat{u}_h(x_i) \leq v'_i \leq v_i \leq \tilde{u}_h(x_i)$. This condition is trivially satisfied if $f(x_i, u)$ is a C^1 -function of $u \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)]$ for all $0 \leq i \leq L$. In fact, \underline{M}_u may be taken as

$$\underline{M}_u = \min_{0 \leq i \leq L} \min \{ -f_u(x_i, u); u \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)] \}. \quad (3.13)$$

The following theorem gives a sufficient condition for the uniqueness of a solution.

Theorem 3.3. *Let the conditions in Theorem 3.2 hold. If, in addition, the conditions (2.9) and (3.12) hold and either*

$$-8(1 - \sigma) < \underline{M}_u \leq 0 \quad \text{or} \quad \underline{M}_u > 0, h < \sqrt{\frac{12}{\underline{M}_u}}, \quad (3.14)$$

then the system (2.5) has a unique solution $u_h^*(x_i)$ in $\langle \hat{u}_h, \tilde{u}_h \rangle$. Moreover, the relation (3.10) holds with $\bar{u}_h(x_i) = \underline{u}_h(x_i) = u_h^*(x_i)$ for all $0 \leq i \leq L$.

Proof. It suffices to show $\bar{u}_h(x_i) = \underline{u}_h(x_i)$ for all $0 \leq i \leq L$, where $\bar{u}_h(x_i)$ and $\underline{u}_h(x_i)$ are the limits in (3.9). Let $w_h(x_i) = \bar{u}_h(x_i) - \underline{u}_h(x_i)$. Then $w_h(x_i) \geq 0$, and by (2.5),

$$\begin{aligned} -\delta_h^2 w_h(x_i) &= \mathcal{D}_h(f(x_i, \bar{u}_h(x_i)) - f(x_i, \underline{u}_h(x_i))), \quad 1 \leq i \leq L-1, \\ w_h(0) &= S_\alpha[w_h(\xi)], \quad w_h(1) = S_\beta[w_h(\eta)]. \end{aligned} \quad (3.15)$$

Therefore, we have from (3.12) that

$$\begin{aligned} -\delta_h^2 w_h(x_i) + \mathcal{D}_h(\underline{M}_u w_h(x_i)) &\leq 0, \quad 1 \leq i \leq L-1, \\ w_h(0) &= S_\alpha[w_h(\xi)], \quad w_h(1) = S_\beta[w_h(\eta)]. \end{aligned} \quad (3.16)$$

By Lemma 2.2, $w_h(x_i) \leq 0$ for all $0 \leq i \leq L$. This proves $\bar{u}_h(x_i) = \underline{u}_h(x_i)$ for all $0 \leq i \leq L$.

To give another sufficient condition, we assume that there exists a nonnegative constant \overline{M}_u^* such that

$$|f(x_i, v_i) - f(x_i, v'_i)| \leq \overline{M}_u^* |v_i - v'_i|, \quad 0 \leq i \leq L \quad (3.17)$$

whenever $\hat{u}_h(x_i) \leq v'_i \leq v_i \leq \tilde{u}_h(x_i)$. If $f(x_i, u)$ is a C^1 -function of $u \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)]$ for all $0 \leq i \leq L$, the above condition is clearly satisfied by

$$\overline{M}_u^* = \max_{0 \leq i \leq L} \max \{ |f_u(x_i, u)|; u \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)] \}. \quad (3.18) \quad \square$$

Theorem 3.4. *Let the conditions in Theorem 3.2 hold. If, in addition, the conditions (2.9) and (3.17) hold and*

$$\overline{M}_u^* < 8(1 - \sigma), \quad (3.19)$$

then the conclusions of Theorem 3.3 are also valid.

Proof. Applying Lemma 2.3 with $M_i = 0$ to (3.15) leads to

$$\|w_h\|_\infty \leq \frac{\|g\|_\infty}{8(1-\sigma)h^2}, \quad (3.20)$$

where $g(x_i) = \mathcal{D}_h(f(x_i, \bar{u}_h(x_i)) - f(x_i, \underline{u}_h(x_i)))$. By (3.17), we obtain $\|g\|_\infty \leq h^2 \bar{M}_u^* \|w_h\|_\infty$. Consequently,

$$\|w_h\|_\infty \leq \frac{\bar{M}_u^* \|w_h\|_\infty}{8(1-\sigma)}. \quad (3.21)$$

This together with (3.19) implies $w_h(x_i) = 0$, that is, $\bar{u}_h(x_i) = \underline{u}_h(x_i)$ for all $0 \leq i \leq L$. \square

It is seen from the proofs of Theorems 3.2–3.4 that the iterative scheme (3.5) not only leads to the existence and uniqueness of the solution of (2.5) but also provides a monotone iterative algorithm for computing the solution. However, the rate of convergence of the iterative scheme (3.5) is only of linear order because it is of Picard type. A more efficient monotone iterative algorithm with the quadratic rate of convergence will be developed in Section 5.

4. Convergence of Numerov's Method

In this section, we deal with the convergence of the numerical solution and show the fourth-order accuracy of Numerov's scheme (2.5). Throughout this section, we assume that the function $f(x, u)$ and the solution $u(x)$ of (1.1) are sufficiently smooth.

Let $u(x_i)$ be the value of the solution of (1.1) at the mesh point x_i , and let $u_h(x_i)$ be the solution of (2.5). We consider the error $e_h(x_i) = u(x_i) - u_h(x_i)$. In fact, we have from (2.4) and (2.5) that

$$\begin{aligned} -\delta_h^2 e_h(x_i) &= \mathcal{D}_h(f(x_i, u(x_i)) - f(x_i, u_h(x_i))) + \mathcal{O}(h^6), \quad 1 \leq i \leq L-1, \\ e_h(0) &= S_\alpha[e_h(\xi)], \quad e_h(1) = S_\beta[e_h(\eta)]. \end{aligned} \quad (4.1)$$

Theorem 4.1. *Let the condition (2.9) hold, and let $[u_{*,i}, u_i^*]$ be an interval in \mathbf{R} such that $u(x_i), u_h(x_i) \in [u_{*,i}, u_i^*]$. Assume that*

$$\max_{0 \leq i \leq L} \max\{f_u(x_i, u); u \in [u_{*,i}, u_i^*]\} < 8(1-\sigma). \quad (4.2)$$

Then for sufficiently small h ,

$$\|u - u_h\|_\infty \leq C^* h^4, \quad (4.3)$$

where C^ is a positive constant independent of h .*

Proof. Applying the mean value theorem to the first equality of (4.1), we have

$$\begin{aligned} -\delta_h^2 e_h(x_i) + \mathcal{D}_h(M_i e_h(x_i)) &= \mathcal{O}(h^6), \quad 1 \leq i \leq L-1, \\ e_h(0) &= S_\alpha[e_h(\xi)], \quad e_h(1) = S_\beta[e_h(\eta)], \end{aligned} \quad (4.4)$$

where $M_i = -f_u(x_i, \theta_i)$ and $\theta_i \in [u_{*,i}, u_i^*]$. Let $\overline{M} = \min_i M_i$ and $\underline{M} = \max_i M_i$. Then by (4.2), $-8(1-\sigma) < \underline{M} \leq M_i \leq \overline{M}$. We, therefore, obtain from Lemma 2.3 that when $h < h(\underline{M}, \overline{M})$,

$$\|e_h\|_\infty \leq \frac{C_1 \|\mathcal{O}(h^6)\|_\infty}{h^2}, \quad (4.5)$$

where C_1 is a positive constant independent of h . Finally, the error estimate (4.3) follows from $\|\mathcal{O}(h^6)\|_\infty \leq C_2 h^6$ for some positive constant independent of h . \square

Theorem 4.1 shows that Numerov's scheme (2.5) possesses the fourth-order accuracy under the conditions of the theorem.

5. An Accelerated Monotone Iterative Algorithm

The iterative scheme (3.5) gives an algorithm for solving the system (2.5). However, as already mentioned in Section 3, its rate of convergence is only of linear order because it is of Picard type. To raise the rate of convergence while maintaining the monotone convergence of the sequence, we propose an accelerated monotone iterative algorithm. An advantage of this algorithm is that its rate of convergence for the sum of the two produced sequences is quadratic (in the sense mentioned in Section 1) with only the usual differentiability requirement on the function $f(\cdot, u)$. If the function $f_u(\cdot, u)$ possesses a monotone property in u , this algorithm is reduced to Newton's method, and one of the two produced sequences converges quadratically.

5.1. Monotone Iterative Algorithm

Let $\tilde{u}_h(x_i)$ and $\hat{u}_h(x_i)$ be a pair of ordered upper and lower solutions of (2.5) and assume that $f(\cdot, u)$ is a C^1 -function of $u \in \langle \hat{u}_h, \tilde{u}_h \rangle$. It follows from Theorems 3.2–3.4 that (2.5) has a unique solution $u_h^*(x_i)$ in $\langle \hat{u}_h, \tilde{u}_h \rangle$ under the conditions of the theorems. To compute this solution, we use the following iterative scheme:

$$\begin{aligned} -\delta_h^2 u_h^{(m)}(x_i) + \mathcal{D}_h(M_i^{(m-1)} u_h^{(m)}(x_i)) \\ = \mathcal{D}_h(M_i^{(m-1)} u_h^{(m-1)}(x_i) + f(x_i, u_h^{(m-1)}(x_i))), \quad 1 \leq i \leq L-1, \\ u_h^{(m)}(0) = S_\alpha[u_h^{(m)}(\xi)], \quad u_h^{(m)}(1) = S_\beta[u_h^{(m)}(\eta)], \end{aligned} \quad (5.1)$$

where $u_h^{(0)}(x_i)$ is either $\tilde{u}_h(x_i)$ or $\hat{u}_h(x_i)$, and for each i ,

$$M_i^{(m)} = \max \left\{ -f_u(x_i, u); u \in [\underline{u}_h^{(m)}(x_i), \overline{u}_h^{(m)}(x_i)] \right\}. \quad (5.2)$$

The functions $\overline{u}_h^{(m)}(x_i)$ and $\underline{u}_h^{(m)}(x_i)$ in the definition of $M_i^{(m)}$ are obtained from (5.1) with $u_h^{(0)}(x_i) = \tilde{u}_h(x_i)$ and $u_h^{(0)}(x_i) = \hat{u}_h(x_i)$, respectively. It is clear from (5.2) that

$$f(x_i, v_i) - f(x_i, v'_i) \geq -M_i^{(m)}(v_i - v'_i), \quad 0 \leq i \leq L, \quad (5.3)$$

whenever $\underline{u}_h^{(m)}(x_i) \leq v'_i \leq v_i \leq \overline{u}_h^{(m)}(x_i)$. Moreover,

$$M_i^{(m)} = \begin{cases} -f_u(x_i, \overline{u}_h^{(m)}(x_i)), & \text{if } f_u(x_i, u) \text{ is monotone nonincreasing in } u \in [\underline{u}_h^{(m)}(x_i), \overline{u}_h^{(m)}(x_i)], \\ -f_u(x_i, \underline{u}_h^{(m)}(x_i)), & \text{if } f_u(x_i, u) \text{ is monotone nondecreasing in } u \in [\underline{u}_h^{(m)}(x_i), \overline{u}_h^{(m)}(x_i)]. \end{cases} \quad (5.4)$$

Hence, if $f_u(x_i, u)$ is monotone nonincreasing/nondecreasing in $u \in [\underline{u}_h^{(m)}(x_i), \overline{u}_h^{(m)}(x_i)]$ for all $0 \leq i \leq L$, then the iterative scheme (5.1) for $\{\overline{u}_h^{(m)}(x_i)\} / \{\underline{u}_h^{(m)}(x_i)\}$ is reduced to Newton's form:

$$\begin{aligned} & -\delta_h^2 \underline{u}_h^{(m)}(x_i) - \mathcal{D}_h \left(f_u(x_i, \underline{u}_h^{(m-1)}(x_i)) \underline{u}_h^{(m)}(x_i) \right) \\ & = -\mathcal{D}_h \left(f_u(x_i, \underline{u}_h^{(m-1)}(x_i)) \underline{u}_h^{(m-1)}(x_i) - f(x_i, \underline{u}_h^{(m-1)}(x_i)) \right), \quad 1 \leq i \leq L-1, \quad (5.5) \\ & \underline{u}_h^{(m)}(0) = S_\alpha \left[\underline{u}_h^{(m)}(\xi) \right], \quad \underline{u}_h^{(m)}(1) = S_\beta \left[\underline{u}_h^{(m)}(\eta) \right]. \end{aligned}$$

To show that the sequences given by (5.1) are well-defined and monotone for an arbitrary C^1 -function $f(\cdot, u)$, we let \underline{M}_u be given by (3.13) and let

$$\overline{M}_u = \max_{0 \leq i \leq L} \max \left\{ -f_u(x_i, u); u \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)] \right\}. \quad (5.6)$$

Lemma 5.1. *Let the condition (2.9) hold, and let $\tilde{u}_h(x_i)$ and $\hat{u}_h(x_i)$ be a pair of ordered upper and lower solutions of (2.5). Assume that $\underline{M}_u > -8(1 - \sigma)$ and $h < h(\underline{M}_u, \overline{M}_u)$. Then the sequences $\{\overline{u}_h^{(m)}(x_i)\}$, $\{\underline{u}_h^{(m)}(x_i)\}$, and $\{M_i^{(m)}\}$ given by (5.1) and (5.2) with $\overline{u}_h^{(0)}(x_i) = \tilde{u}_h(x_i)$ and $\underline{u}_h^{(0)}(x_i) = \hat{u}_h(x_i)$ are all well defined and possess the monotone property*

$$\underline{u}_h^{(m)}(x_i) \leq \underline{u}_h^{(m+1)}(x_i) \leq \overline{u}_h^{(m+1)}(x_i) \leq \overline{u}_h^{(m)}(x_i), \quad 0 \leq i \leq L, \quad m = 0, 1, \dots \quad (5.7)$$

Proof. Since $M_i^{(0)} = \max\{-f_u(x_i, u); u_i \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)]\}$, $-8(1-\sigma) < \underline{M}_u \leq M_i^{(0)} \leq \overline{M}_u$ and $h < h(\underline{M}_u, \overline{M}_u)$, we have from Lemma 2.2 that the first iterations $\overline{u}_h^{(1)}(x_i)$ and $\underline{u}_h^{(1)}(x_i)$ are well defined. Let $\overline{w}_h^{(0)}(x_i) = \overline{u}_h^{(0)}(x_i) - \underline{u}_h^{(1)}(x_i)$. Then, by (3.1) and (5.1),

$$\begin{aligned} -\delta_h^2 \overline{w}_h^{(0)}(x_i) + \rho_h \left(M_i^{(0)} \overline{w}_h^{(0)}(x_i) \right) &\geq 0, \quad 1 \leq i \leq L-1, \\ \overline{w}_h^{(0)}(0) &\geq S_\alpha \left[\overline{w}_h^{(0)}(\xi) \right], \quad \overline{w}_h^{(0)}(1) \geq S_\beta \left[\overline{w}_h^{(0)}(\eta) \right]. \end{aligned} \quad (5.8)$$

We have from Lemma 2.2 that $\overline{w}_h^{(0)}(x_i) \geq 0$, that is, $\overline{u}_h^{(0)}(x_i) \geq \underline{u}_h^{(1)}(x_i)$ for every $0 \leq i \leq L$. Similarly by the property of a lower solution, $\underline{u}_h^{(1)}(x_i) \geq \underline{u}_h^{(0)}(x_i)$ for every $0 \leq i \leq L$. Let $w_h^{(1)}(x_i) = \overline{u}_h^{(1)}(x_i) - \underline{u}_h^{(1)}(x_i)$. Then by (5.1) and (5.3),

$$\begin{aligned} -\delta_h^2 w_h^{(1)}(x_i) + \rho_h \left(M_i^{(0)} w_h^{(1)}(x_i) \right) &\geq 0, \quad 1 \leq i \leq L-1, \\ w_h^{(1)}(0) &= S_\alpha \left[w_h^{(1)}(\xi) \right], \quad w_h^{(1)}(1) = S_\beta \left[w_h^{(1)}(\eta) \right]. \end{aligned} \quad (5.9)$$

It follows from Lemma 2.2 that $w_h^{(1)}(x_i) \geq 0$, that is, $\overline{u}_h^{(1)}(x_i) \geq \underline{u}_h^{(1)}(x_i)$ for every $0 \leq i \leq L$. This proves the monotone property (5.7) for $m = 0$.

Assume, by induction, that there exists some integer $m_0 \geq 0$ such that for all $0 \leq m \leq m_0$, the iterations $\overline{u}_h^{(m)}(x_i)$, $\underline{u}_h^{(m)}(x_i)$, $\underline{u}_h^{(m)}(x_i)$, and $\underline{u}_h^{(m+1)}(x_i)$ are well-defined and satisfy (5.7). Then $M_i^{(m_0+1)}$ is well defined and $-8(1-\sigma) < \underline{M}_u \leq M_i^{(m_0+1)} \leq \overline{M}_u$. Since $h < h(\underline{M}_u, \overline{M}_u)$, we have from Lemma 2.2 that the iterations $\overline{u}_h^{(m_0+2)}(x_i)$ and $\underline{u}_h^{(m_0+2)}(x_i)$ exist uniquely. Let $\overline{w}_h^{(m_0+1)}(x_i) = \overline{u}_h^{(m_0+1)}(x_i) - \underline{u}_h^{(m_0+2)}(x_i)$. Since

$$\begin{aligned} M_i^{(m_0+1)} \overline{w}_h^{(m_0+1)}(x_i) \\ = \left(M_i^{(m_0+1)} - M_i^{(m_0)} \right) \overline{u}_h^{(m_0+1)}(x_i) + M_i^{(m_0)} \overline{u}_h^{(m_0+1)}(x_i) - M_i^{(m_0+1)} \underline{u}_h^{(m_0+2)}(x_i), \end{aligned} \quad (5.10)$$

the iterative scheme (5.1) implies that

$$\begin{aligned} &-\delta_h^2 \overline{w}_h^{(m_0+1)}(x_i) + \rho_h \left(M_i^{(m_0+1)} \overline{w}_h^{(m_0+1)}(x_i) \right) \\ &= \rho_h \left(M_i^{(m_0)} \left(\overline{u}_h^{(m_0)}(x_i) - \underline{u}_h^{(m_0+1)}(x_i) \right) + f \left(x_i, \overline{u}_h^{(m_0)}(x_i) \right) - f \left(x_i, \underline{u}_h^{(m_0+1)}(x_i) \right) \right), \quad 1 \leq i \leq L-1, \\ &\overline{w}_h^{(m_0+1)}(0) = S_\alpha \left[\overline{w}_h^{(m_0+1)}(\xi) \right], \quad \overline{w}_h^{(m_0+1)}(1) = S_\beta \left[\overline{w}_h^{(m_0+1)}(\eta) \right]. \end{aligned} \quad (5.11)$$

Using the relation (5.3) yields

$$\begin{aligned} -\delta_h^2 \bar{w}_h^{(m_0+1)}(x_i) + \mathcal{D}_h \left(M_i^{(m_0+1)} \bar{w}_h^{(m_0+1)}(x_i) \right) &\geq 0, \quad 1 \leq i \leq L-1, \\ \bar{w}_h^{(m_0+1)}(0) &= S_\alpha \left[\bar{w}_h^{(m_0+1)}(\xi) \right], \quad \bar{w}_h^{(m_0+1)}(1) = S_\beta \left[\bar{w}_h^{(m_0+1)}(\eta) \right]. \end{aligned} \quad (5.12)$$

By Lemma 2.2, $\bar{w}_h^{(m_0+1)}(x_i) \geq 0$, that is, $\bar{u}_h^{(m_0+1)}(x_i) \geq \bar{u}_h^{(m_0+2)}(x_i)$ for every $0 \leq i \leq L$. Similarly we have $\underline{u}_h^{(m_0+2)}(x_i) \geq \underline{u}_h^{(m_0+1)}(x_i)$ for every $0 \leq i \leq L$. Let $w_h^{(m_0+2)}(x_i) = \bar{u}_h^{(m_0+2)}(x_i) - \underline{u}_h^{(m_0+2)}(x_i)$. Then by (5.1) and (5.3), $w_h^{(m_0+2)}(x_i)$ satisfies (5.12) with $\bar{w}_h^{(m_0+1)}(x_i)$ replaced by $w_h^{(m_0+2)}(x_i)$. Therefore, by Lemma 2.2, $w_h^{(m_0+2)}(x_i) \geq 0$, that is, $\bar{u}_h^{(m_0+2)}(x_i) \geq \underline{u}_h^{(m_0+2)}(x_i)$ for every $0 \leq i \leq L$. This shows that the monotone property (5.7) is also true for $m = m_0 + 1$. Finally, the conclusion of the lemma follows from the principle of induction. \square

We next show monotone convergence of the sequences $\{\bar{u}_h^{(m)}(x_i)\}$ and $\{\underline{u}_h^{(m)}(x_i)\}$.

Theorem 5.2. *Let the hypothesis in Lemma 5.1 hold. Then the sequences $\{\bar{u}_h^{(m)}(x_i)\}$ and $\{\underline{u}_h^{(m)}(x_i)\}$ given by (5.1) converge monotonically to the unique solution $u_h^*(x_i)$ of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$, respectively. Moreover,*

$$\underline{u}_h^{(m)}(x_i) \leq \underline{u}_h^{(m+1)}(x_i) \leq u_h^*(x_i) \leq \bar{u}_h^{(m+1)}(x_i) \leq \bar{u}_h^{(m)}(x_i), \quad 0 \leq i \leq L, \quad m = 0, 1, \dots \quad (5.13)$$

Proof. It follows from the monotone property (5.7) that the limits

$$\lim_{m \rightarrow \infty} \bar{u}_h^{(m)}(x_i) = \bar{u}_h(x_i), \quad \lim_{m \rightarrow \infty} \underline{u}_h^{(m)}(x_i) = \underline{u}_h(x_i), \quad 0 \leq i \leq L \quad (5.14)$$

exist and they satisfy (3.10). Since the sequence $\{M_i^{(m)}\}$ is monotone nonincreasing and is bounded from below by \underline{M}_u given in (3.13), it converges as $m \rightarrow \infty$. Letting $m \rightarrow \infty$ in (5.1) shows that both $\bar{u}_h(x_i)$ and $\underline{u}_h(x_i)$ are solutions of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$. Since $\underline{M}_u > -8(1 - \sigma)$ and $h < h(\underline{M}_u, \bar{M}_u)$, the condition (3.14) of Theorem 3.3 is satisfied. Thus by Theorem 3.3, $\bar{u}_h(x_i) = \underline{u}_h(x_i) (\equiv u_h^*(x_i))$ and $u_h^*(x_i)$ is the unique solution of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$. The monotone property (5.13) follows from (3.10). \square

When $f_u(x_i, u)$ is monotone nonincreasing/nondecreasing in $u \in [\underline{u}_h^{(m)}(x_i), \bar{u}_h^{(m)}(x_i)]$ for all $0 \leq i \leq L$, the iterative scheme (5.1) for $\{\bar{u}_h^{(m)}(x_i)\} / \{\underline{u}_h^{(m)}(x_i)\}$ is reduced to Newton iteration (5.5). As a consequence of Theorem 5.2, we have the following conclusion.

Corollary 5.3. *Let the hypothesis in Lemma 5.1 be satisfied. If $f_u(x_i, u)$ is monotone nonincreasing in $u \in [\underline{u}_h^{(m)}(x_i), \bar{u}_h^{(m)}(x_i)]$ for all $0 \leq i \leq L$, the sequence $\{\bar{u}_h^{(m)}(x_i)\}$ given by (5.5) with $\bar{u}_h^{(0)}(x_i) = \tilde{u}_h(x_i)$ converges monotonically from above to the unique solution $u_h^*(x_i)$ of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$. Otherwise, if $f_u(x_i, u)$ is monotone nondecreasing in $u \in [\underline{u}_h^{(m)}(x_i), \bar{u}_h^{(m)}(x_i)]$ for all $0 \leq i \leq L$, the sequence $\{\underline{u}_h^{(m)}(x_i)\}$ given by (5.5) with $\underline{u}_h^{(0)}(x_i) = \hat{u}_h(x_i)$ converges monotonically from below to $u_h^*(x_i)$.*

5.2. Rate of Convergence

We now show the quadratic rate of convergence of the sequences given by (5.1). Assume that there exists a nonnegative constant \overline{M}^* such that

$$|f_u(x_i, v_i) - f_u(x_i, v'_i)| \leq \overline{M}^* |v_i - v'_i| \quad \forall v_i, v'_i \in [\hat{u}_h(x_i), \tilde{u}_h(x_i)], \quad 0 \leq i \leq L. \quad (5.15)$$

Clearly, this assumption is satisfied if $f(\cdot, u)$ is a C^2 -function of u .

Theorem 5.4. *Let the hypotheses in Lemma 5.1 and (5.15) hold. Also let $\{\overline{u}_h^{(m)}(x_i)\}$ and $\{\underline{u}_h^{(m)}(x_i)\}$ be the sequences given by (5.1) and let $u_h^*(x_i)$ be the unique solution of (2.5) in $\langle \hat{u}_h, \tilde{u}_h \rangle$. Then there exists a constant ρ , independent of m , such that*

$$\left\| \overline{u}_h^{(m)} - u_h^* \right\|_\infty + \left\| \underline{u}_h^{(m)} - u_h^* \right\|_\infty \leq \rho \left(\left\| \overline{u}_h^{(m-1)} - u_h^* \right\|_\infty + \left\| \underline{u}_h^{(m-1)} - u_h^* \right\|_\infty \right)^2, \quad m = 1, 2, \dots \quad (5.16)$$

Proof. Let $\overline{w}_h^{(m)}(x_i) = \overline{u}_h^{(m)}(x_i) - u_h^*(x_i)$. Subtracting (2.5) from (5.1) gives

$$\begin{aligned} & -\delta_h^2 \overline{w}_h^{(m)}(x_i) + \rho_h \left(M_i^{(m-1)} \overline{w}_h^{(m)}(x_i) \right) \\ &= \rho_h \left(M_i^{(m-1)} \overline{w}_h^{(m-1)}(x_i) + f \left(x_i, \overline{u}_h^{(m-1)}(x_i) \right) - f \left(x_i, u_h^*(x_i) \right) \right), \quad 1 \leq i \leq L-1, \quad (5.17) \\ & \overline{w}_h^{(m)}(0) = S_\alpha \left[\overline{w}_h^{(m)}(\xi) \right], \quad \overline{w}_h^{(m)}(1) = S_\beta \left[\overline{w}_h^{(m)}(\eta) \right]. \end{aligned}$$

By the intermediate value theorem,

$$M_i^{(m-1)} = -f_u \left(x_i, \theta_i^{(m-1)} \right), \quad (5.18)$$

where $\theta_i^{(m-1)} \in [\underline{u}_h^{(m-1)}(x_i), \overline{u}_h^{(m-1)}(x_i)]$, and by the mean value theorem,

$$f \left(x_i, \overline{u}_h^{(m-1)}(x_i) \right) - f \left(x_i, u_h^*(x_i) \right) = f_u \left(x_i, \gamma_i^{(m-1)} \right) \overline{w}_h^{(m-1)}(x_i), \quad (5.19)$$

where $\gamma_i^{(m-1)} \in [u_h^*(x_i), \overline{u}_h^{(m-1)}(x_i)]$. Let

$$g_i^{(m-1)} = \left(f_u \left(x_i, \gamma_i^{(m-1)} \right) - f_u \left(x_i, \theta_i^{(m-1)} \right) \right) \overline{w}_h^{(m-1)}(x_i). \quad (5.20)$$

Then we have from (5.17) that

$$\begin{aligned} & -\delta_h^2 \overline{w}_h^{(m)}(x_i) + \rho_h \left(M_i^{(m-1)} \overline{w}_h^{(m)}(x_i) \right) = \rho_h g_i^{(m-1)}, \quad 1 \leq i \leq L-1, \\ & \overline{w}_h^{(m)}(0) = S_\alpha \left[\overline{w}_h^{(m)}(\xi) \right], \quad \overline{w}_h^{(m)}(1) = S_\beta \left[\overline{w}_h^{(m)}(\eta) \right]. \end{aligned} \quad (5.21)$$

Since $-8(1 - \sigma) < \underline{M}_u \leq M_i^{(m-1)} \leq \overline{M}_u$ and $h < h(\underline{M}_u, \overline{M}_u)$, it follows from Lemma 2.3 that there exists a constant ρ_1 , independent of m , such that

$$\|\overline{w}_h^{(m)}\|_\infty \leq \frac{\rho_1 \|\rho_h g_i^{(m-1)}\|_\infty}{h^2}. \quad (5.22)$$

To estimate $g_i^{(m-1)}$, we observe from (5.15) that

$$|g_i^{(m-1)}| \leq \overline{M}^* |\gamma_i^{(m-1)} - \theta_i^{(m-1)}| \cdot |\overline{w}_h^{(m-1)}(x_i)|. \quad (5.23)$$

Since both $\gamma_i^{(m-1)}$ and $\theta_i^{(m-1)}$ are in $[\underline{u}_h^{(m-1)}(x_i), \overline{u}_h^{(m-1)}(x_i)]$, the above estimate implies that

$$|g_i^{(m-1)}| \leq \overline{M}^* |\overline{u}_h^{(m-1)}(x_i) - \underline{u}_h^{(m-1)}(x_i)| \cdot |\overline{w}_h^{(m-1)}(x_i)|. \quad (5.24)$$

Using this estimate in (5.22), we obtain

$$\|\overline{w}_h^{(m)}\|_\infty \leq \rho_1 \overline{M}^* \|\overline{u}_h^{(m-1)} - \underline{u}_h^{(m-1)}\|_\infty \|\overline{w}_h^{(m-1)}\|_\infty, \quad (5.25)$$

or

$$\|\overline{u}_h^{(m)} - u_h^*\|_\infty \leq \rho_1 \overline{M}^* \|\overline{u}_h^{(m-1)} - \underline{u}_h^{(m-1)}\|_\infty \|\overline{u}_h^{(m-1)} - u_h^*\|_\infty. \quad (5.26)$$

Similarly, we have

$$\|\underline{u}_h^{(m)} - u_h^*\|_\infty \leq \rho_1 \overline{M}^* \|\overline{u}_h^{(m-1)} - \underline{u}_h^{(m-1)}\|_\infty \|\underline{u}_h^{(m-1)} - u_h^*\|_\infty. \quad (5.27)$$

Addition of (5.26) and (5.27) gives

$$\begin{aligned} & \|\overline{u}_h^{(m)} - u_h^*\|_\infty + \|\underline{u}_h^{(m)} - u_h^*\|_\infty \\ & \leq \rho_1 \overline{M}^* \|\overline{u}_h^{(m-1)} - \underline{u}_h^{(m-1)}\|_\infty \left(\|\overline{u}_h^{(m-1)} - u_h^*\|_\infty + \|\underline{u}_h^{(m-1)} - u_h^*\|_\infty \right). \end{aligned} \quad (5.28)$$

Then the estimate (5.16) follows immediately. \square

Theorem 5.4 gives a quadratic convergence for the sum of the sequences $\{\overline{u}_h^{(m)}(x_i)\}$ and $\{\underline{u}_h^{(m)}(x_i)\}$ in the sense of (5.16). If $f_u(x_i, u)$ is monotone nonincreasing/nondecreasing in $u \in [\underline{u}_h^{(m)}(x_i), \overline{u}_h^{(m)}(x_i)]$ for all $0 \leq i \leq L$, the sequence $\{\overline{u}_h^{(m)}(x_i)\} / \{\underline{u}_h^{(m)}(x_i)\}$ has the quadratic convergence. This result is stated as follows.

Theorem 5.5. *Let the conditions in Theorem 5.4 hold. Then there exists a constant ρ , independent of m , such that*

$$\left\| \bar{u}_h^{(m)} - u_h^* \right\|_\infty \leq \rho \left\| \bar{u}_h^{(m-1)} - u_h^* \right\|_\infty^2, \quad m = 1, 2, \dots \quad (5.29)$$

if $f_u(x_i, u)$ is monotone nonincreasing in $u \in [\underline{u}_h^{(m)}(x_i), \bar{u}_h^{(m)}(x_i)]$ for all $0 \leq i \leq L$ and

$$\left\| \underline{u}_h^{(m)} - u_h^* \right\|_\infty \leq \rho \left\| \underline{u}_h^{(m-1)} - u_h^* \right\|_\infty^2, \quad m = 1, 2, \dots \quad (5.30)$$

if $f_u(x_i, u)$ is monotone nondecreasing in $u \in [\underline{u}_h^{(m)}(x_i), \bar{u}_h^{(m)}(x_i)]$ for all $0 \leq i \leq L$.

Proof. Consider the monotone nonincreasing case. In this case, the sequence $\{\bar{u}_h^{(m)}(x_i)\}$ is given by (5.5) with $\bar{u}_h^{(0)}(x_i) = \tilde{u}_h(x_i)$. This implies that $\theta_i^{(m-1)} = \bar{u}_h^{(m-1)}(x_i)$, where $\theta_i^{(m-1)}$ is the intermediate value in (5.18). Since $\gamma_i^{(m-1)}$ in (5.19) is in $[u_h^*(x_i), \bar{u}_h^{(m-1)}(x_i)]$, we see that

$$\left| \gamma_i^{(m-1)} - \theta_i^{(m-1)} \right| \leq \left| \bar{u}_h^{(m-1)}(x_i) - u_h^*(x_i) \right|. \quad (5.31)$$

Thus, (5.24) is now reduced to

$$\left| g_i^{(m-1)} \right| \leq \bar{M}^* \left| \bar{w}_h^{(m-1)}(x_i) \right|^2. \quad (5.32)$$

The argument in the proof of Theorem 5.4 shows that (5.29) holds with $\rho = \rho_1 \bar{M}^*$, where ρ_1 is the constant in (5.22). The proof of (5.30) is similar. \square

6. Applications and Numerical Results

In this section, we give some applications of the results in the previous sections to three model problems. We present some numerical results to demonstrate the monotone and rapid convergence of the sequence from (5.1) and to show the fourth-order accuracy of Numerov's scheme (2.5), as predicted in the analysis.

In order to implement the monotone iterative algorithm (5.1), it is necessary to find a pair of ordered upper and lower solutions of (2.5). The construction of this pair depends mainly on the function $f(\cdot, u)$, and much discussion on the subject can be found in [54] for continuous problems. To demonstrate some techniques for the construction of ordered upper and lower solutions of (2.5), we assume that $f(x, 0) \geq 0$ for all $x \in [0, 1]$ and there exists a nonnegative constant C such that

$$f(x, C) \leq 0, \quad x \in [0, 1]. \quad (6.1)$$

Then $-\delta_h^2 C = 0 \geq \mathcal{D}_h f(x_i, C)$ for all $1 \leq i \leq L-1$. This implies that $\tilde{u}_h(x_i) \equiv C$ and $\hat{u}_h(x_i) \equiv 0$ are a pair of ordered upper and lower solutions of (2.5) if, in addition, the condition (2.9) holds. On the other hand, assume that there exist nonnegative constants a, b with $a < 8(1-\sigma)$ such that

$$f(x, u) \leq au + b \quad \text{for } x \in [0, 1], u \geq 0, \quad (6.2)$$

where $\sigma < 1$ is defined by (2.9). We have from Lemma 2.2 that the solution $\tilde{u}_h(x_i)$ of the linear problem

$$\begin{aligned} -\delta_h^2 \tilde{u}_h(x_i) - a \mathcal{D}_h \tilde{u}_h(x_i) &= h^2 b, \quad 1 \leq i \leq L-1, \\ \tilde{u}_h(0) &= S_\alpha[\tilde{u}_h(\xi)], \quad \tilde{u}_h(1) = S_\beta[\tilde{u}_h(\eta)] \end{aligned} \quad (6.3)$$

exists uniquely and is nonnegative. Clearly by (6.2), this solution is a nonnegative upper solution of (2.5).

As applications of the above construction of upper and lower solutions, we next consider three specific examples. In each of these examples, the analytic solution $u(x)$ of (1.1) is explicitly known, against which we can compare the numerical solution $u_h^*(x_i)$ of the scheme (2.5) to demonstrate the fourth-order accuracy of the scheme. The order of accuracy is calculated by

$$\text{error}_\infty(h) = \|u - u_h^*\|_\infty, \quad \text{order}_\infty(h) = \log_2 \left(\frac{\text{error}_\infty(h)}{\text{error}_\infty(h/2)} \right). \quad (6.4)$$

All computations are carried out by using a MATLAB subroutine on a Pentium 4 computer with 2G memory, and the termination criterion of iterations for (5.1) is given by

$$\left\| \bar{u}_h^{(m)} - \underline{u}_h^{(m)} \right\|_\infty < 10^{-14}. \quad (6.5)$$

Example 6.1. Consider the four-point boundary value problem:

$$\begin{aligned} -u''(x) &= \theta u(x)(1-u(x)) + q(x), \quad 0 < x < 1, \\ u(0) &= \frac{1}{9} u\left(\frac{1}{2}\right), \quad u(1) = \frac{1}{8} u\left(\frac{1}{4}\right), \end{aligned} \quad (6.6)$$

where θ is a positive constant and $q(x)$ is a nonnegative continuous function. Clearly, problem (6.6) is a special case of (1.1) with

$$f(x, u) = \theta u(1-u) + q(x). \quad (6.7)$$

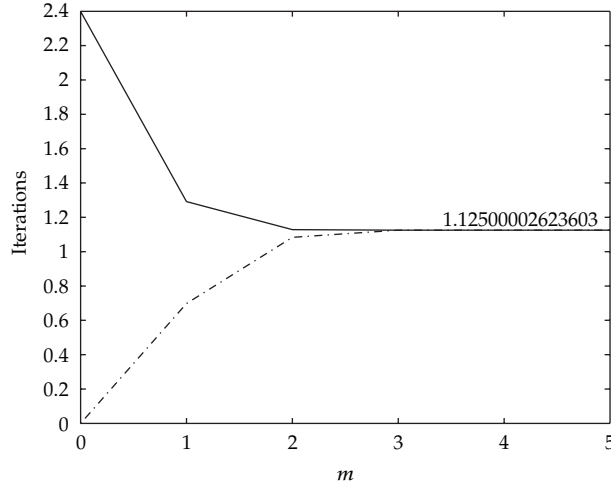


Figure 1: The monotone convergence of $(\{\bar{u}_h^{(m)}(x_i)\}, \{\underline{u}_h^{(m)}(x_i)\})$ at $x_i = 0.5$ for Example 6.1.

To obtain an explicit analytic solution of (6.6), we choose

$$q(x) = 8 + \left(\frac{\pi^2}{2}\right) \sin(2\pi x) - \theta z(x)(1 - z(x)), \quad z(x) = 4x(1 - x) + \frac{1 + \sin(2\pi x)}{8}. \quad (6.8)$$

Then the function $u(x) = z(x)$ is a solution of (6.6). Moreover, $q(x) \geq 0$ in $[0, 1]$ if $\theta \leq 32 - 2\pi^2$.

For problem (6.6), the corresponding Numerov scheme (2.5) is now reduced to

$$\begin{aligned} -\delta_h^2 u_h(x_i) &= \mathcal{P}_h f(x_i, u_h(x_i)), \quad 1 \leq i \leq L-1, \\ u_h(0) &= \frac{1}{9} u_h\left(\frac{1}{2}\right), \quad u_h(1) = \frac{1}{8} u_h\left(\frac{1}{4}\right). \end{aligned} \quad (6.9)$$

To find a pair of ordered upper and lower solutions of (6.9), we observe from (6.7) that $f(x, 0) = q(x) \geq 0$ for all $x \in [0, 1]$, and, therefore, $\hat{u}_h(x_i) \equiv 0$ is a lower solution. Since $q(x) \leq 14$, we have from (6.7) that the condition (6.2) is satisfied for the present function f with $a = \theta$ and $b = 14$. Therefore, the solution $\tilde{u}_h(x_i)$ of (6.3) (corresponding to (6.9)) with $a = \theta$ and $b = 14$ is a nonnegative upper solution if $\theta < 7$. This implies that $\tilde{u}_h(x_i)$ and $\hat{u}_h(x_i) \equiv 0$ are a pair of ordered upper and lower solutions of (6.9).

Let $\theta = \pi/2$. Using $\bar{u}_h^{(0)}(x_i) = \tilde{u}_h(x_i)$ and $\underline{u}_h^{(0)}(x_i) = 0$, we compute the sequences $\{\bar{u}_h^{(m)}(x_i)\}$ and $\{\underline{u}_h^{(m)}(x_i)\}$ from the iterative scheme (5.1) for (6.9) and various values of h . In all the numerical computations, the basic feature of monotone convergence of the sequences was observed. Let $h = 1/32$. In Figure 1, we present some numerical results of these sequences at $x_i = 0.5$, where the solid line denotes the sequence $\{\bar{u}_h^{(m)}(x_i)\}$ and the dashed-dotted line stands for the sequence $\{\underline{u}_h^{(m)}(x_i)\}$. As described in Theorem 5.2, the sequences converge to the same limit as $m \rightarrow \infty$, and their common limit $u_h^*(x_i)$ is the unique solution of (6.9) in $\langle 0, \tilde{u}_h \rangle$. Besides, these sequences converge rapidly (in five iterations). More numerical results

Table 1: Solutions $u_h^*(x_i)$ and $u(x_i)$ of Example 6.1.

x_i	$u_h^*(x_i)$	$u(x_i)$
1/16	0.40721072351325	0.40721042904564
1/8	0.65088888830290	0.65088834764832
3/16	0.84986064600007	0.84985994156391
1/4	1.00000076215892	1
5/16	1.09986064798991	1.09985994156391
3/8	1.15088889437534	1.15088834764832
7/16	1.15721073688765	1.15721042904564
1/2	1.12500002623603	1.12500000000000

Table 2: The accuracy of the numerical solution $u_h^*(x_i)$ of Example 6.1.

h	Scheme (6.9)		SFD scheme	
	$\text{error}_\infty(h)$	$\text{order}_\infty(h)$	$\text{error}_\infty(h)$	$\text{order}_\infty(h)$
1/4	$3.43422969288754e-03$	4.10451040194224	$2.82813777940529e-02$	2.12179230623150
1/8	$1.99640473104834e-04$	4.02647493758726	$6.49796599949681e-03$	2.03066791458431
1/16	$1.22506422453039e-05$	4.00662171813206	$1.59032351665434e-03$	2.00765985597618
1/32	$7.62158923750533e-07$	4.00165559258019	$3.95475554192615e-04$	2.00191427472146
1/64	$4.75802997002006e-08$	4.00040916222332	$9.87377889736241e-05$	2.00047851945960
1/128	$2.97292546136418e-09$	4.00024505190767	$2.46762611546547e-05$	2.00011961094792
1/256	$1.85776283245787e-10$	4.03319325068531	$6.16855384505399e-06$	2.00002977942424
1/512	$1.13469234008790e-11$		$1.54210662950405e-06$	

of $u_h^*(x_i)$ at various x_i are explicitly given in Table 1. We also list the values of the analytic solution $u(x_i)$. Clearly, the numerical solution $u_h^*(x_i)$ meets the analytic solution $u(x_i)$ closely.

To further demonstrate the accuracy of the numerical solution $u_h^*(x_i)$, we list the maximum error $\text{error}_\infty(h)$ and the order $\text{order}_\infty(h)$ in the first three columns of Table 2 for various values of h . The data demonstrate that the numerical solution $u_h^*(x_i)$ has the fourth-order accuracy. This coincides with the analysis very well.

For comparison, we also solve (6.6) by the standard finite difference (SFD) method. This method leads to a difference scheme in the form (6.9) with $\mathcal{P}_h = \mathcal{O}$ (an identical operator). Thus, a similar iterative scheme as (5.1) can be used in actual computations. The corresponding maximum error $\text{error}_\infty(h)$ and the order $\text{order}_\infty(h)$ are listed in the last two columns of Table 2. We see that the standard finite difference method possesses only the second-order accuracy.

Example 6.2. Our second example is for the following five-point boundary value problem:

$$\begin{aligned}
 -u''(x) &= \frac{\theta u(x)}{1+u(x)} + q(x), \quad 0 < x < 1, \\
 u(0) &= \frac{\sqrt{3}}{6}u\left(\frac{1}{4}\right) + \frac{1}{4}u\left(\frac{1}{2}\right), \quad u(1) = \frac{1}{4}u\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{6}u\left(\frac{3}{4}\right),
 \end{aligned} \tag{6.10}$$

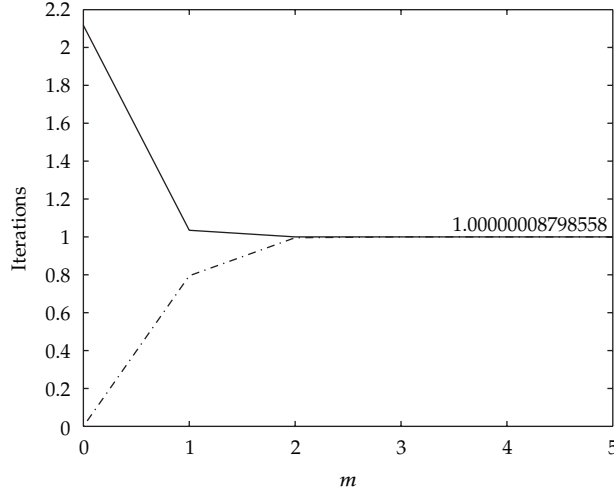


Figure 2: The monotone convergence of $(\{\bar{u}_h^{(m)}(x_i)\}, \{\underline{u}_h^{(m)}(x_i)\})$ at $x_i = 0.5$ for Example 6.2.

where θ is a positive constant and $q(x)$ is a nonnegative continuous function. The corresponding Numerov’s scheme (2.5) for this example is given by

$$\begin{aligned}
 -\delta_h^2 u_h(x_i) &= \rho_h f(x_i, u_h(x_i)), \quad 1 \leq i \leq L-1, \\
 u_h(0) &= \frac{\sqrt{3}}{6} u_h\left(\frac{1}{4}\right) + \frac{1}{4} u_h\left(\frac{1}{2}\right), \quad u_h(1) = \frac{1}{4} u_h\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{6} u_h\left(\frac{3}{4}\right),
 \end{aligned} \tag{6.11}$$

where

$$f(x_i, u_h(x_i)) = \frac{\theta u_h(x_i)}{1 + u_h(x_i)} + q(x_i), \quad 0 \leq i \leq L. \tag{6.12}$$

Let

$$q(x) = \left(\kappa^2 - \frac{\theta}{1 + \sin(\kappa x + \pi/6)} \right) \sin\left(\kappa x + \frac{\pi}{6}\right), \quad \kappa = \frac{2\pi}{3}. \tag{6.13}$$

Then $q(x) \geq 0$ in $[0, 1]$ if $\theta \leq 3\kappa^2/2$, and $u(x) = \sin(\kappa x + \pi/6)$ is a solution of (6.10). Clearly, $\hat{u}_h(x_i) \equiv 0$ is a lower solution of (6.11). On the other hand, the condition (6.2) is satisfied for the present problem with $a = \theta$ and $b = \kappa^2$. Therefore, the solution $\tilde{u}_h(x_i)$ of (6.3) (corresponding to (6.11)) with $a = \theta$ and $b = \kappa^2$ is a nonnegative upper solution of (6.11) if $\theta < 2(9 - 2\sqrt{3})/3$.

Let $\theta = \kappa$. Using $\bar{u}_h^{(0)}(x_i) = \tilde{u}_h(x_i)$ and $\underline{u}_h^{(0)}(x_i) = 0$, we compute the sequences $\{\bar{u}_h^{(m)}(x_i)\}$ and $\{\underline{u}_h^{(m)}(x_i)\}$ from the iterative scheme (5.1) for (6.11). Let $h = 1/32$. Some numerical results of these sequences at $x_i = 0.5$ are plotted in Figure 2, where the solid line denotes the sequence $\{\bar{u}_h^{(m)}(x_i)\}$ and the dashed-dotted line stands for the sequence $\{\underline{u}_h^{(m)}(x_i)\}$. We see that the sequences possess the monotone convergence given in

Table 3: The accuracy of the numerical solution $u_h^*(x_i)$ of Example 6.2.

h	Scheme (6.11)		SFD scheme	
	$\text{error}_\infty(h)$	$\text{order}_\infty(h)$	$\text{error}_\infty(h)$	$\text{order}_\infty(h)$
1/4	3.64212838788403e-04	4.01156400486606	2.66151346177448e-02	2.01341045503011
1/8	2.25815711794031e-05	4.00292832787037	6.59222051766362e-03	2.00338885776990
1/16	1.40848640284297e-06	4.00073477910484	1.64418842865244e-03	2.00084988143279
1/32	8.79855768243232e-08	4.00018946428474	4.10805033531858e-04	2.00021264456991
1/64	5.49837642083162e-09	3.99948214984524	1.02686121950857e-04	2.00005317293630
1/128	3.43771899835588e-10	3.99684542351672	2.56705843380001e-05	2.00001326834657
1/256	2.15327755626049e-11	4.03735960917218	6.41758706221296e-06	2.00000333671353
1/512	1.31139543668724e-12		1.60439305485482e-06	

Theorem 5.2 and converge rapidly (in five iterations) to the unique solution $u_h^*(x_i)$ of (6.11) in $\langle 0, \tilde{u}_h \rangle$. The maximum error $\text{error}_\infty(h)$ and the order $\text{order}_\infty(h)$ of the numerical solution $u_h^*(x_i)$ by the scheme (6.11) and the SFD scheme are presented in Table 3. The numerical results clearly indicate that the proposed scheme (6.11) is more efficient than the SFD scheme.

Example 6.3. Our last example is given by

$$\begin{aligned}
 -u''(x) &= \theta \left(q^4(x) - u^4(x) \right), \quad 0 < x < 1, \\
 u(0) &= \frac{\sqrt{2}}{8} u\left(\frac{1}{8}\right) + \frac{\sqrt{3}}{12} u\left(\frac{1}{4}\right) + \frac{1}{4} u\left(\frac{1}{2}\right), \\
 u(1) &= \frac{\sqrt{3}}{12} u\left(\frac{1}{4}\right) + \frac{1}{4} u\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{12} u\left(\frac{3}{4}\right),
 \end{aligned} \tag{6.14}$$

where θ is a positive constant and $q(x)$ is a continuous function. For this example, the corresponding Numerov scheme (2.5) is reduced to

$$\begin{aligned}
 -\delta_h^2 u_h(x_i) &= \mathcal{D}_h f(x_i, u_h(x_i)), \quad 1 \leq i \leq L-1, \\
 u_h(0) &= \frac{\sqrt{2}}{8} u_h\left(\frac{1}{8}\right) + \frac{\sqrt{3}}{12} u_h\left(\frac{1}{4}\right) + \frac{1}{4} u_h\left(\frac{1}{2}\right), \\
 u_h(1) &= \frac{\sqrt{3}}{12} u_h\left(\frac{1}{4}\right) + \frac{1}{4} u_h\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{12} u_h\left(\frac{3}{4}\right),
 \end{aligned} \tag{6.15}$$

where

$$f(x_i, u_h(x_i)) = \theta \left(q^4(x_i) - u_h^4(x_i) \right), \quad 0 \leq i \leq L. \tag{6.16}$$

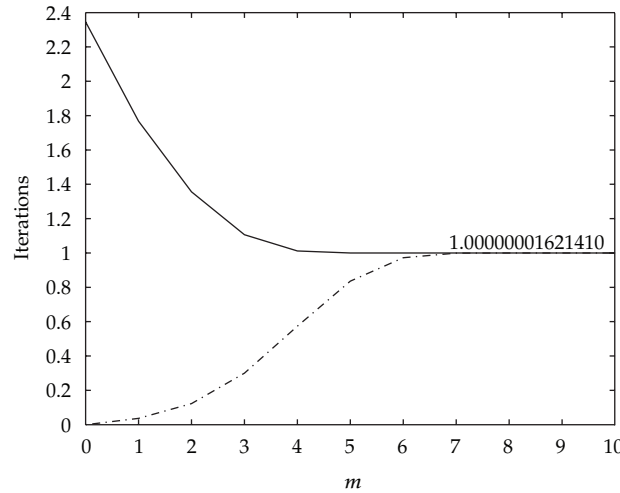


Figure 3: The monotone convergence of $(\{\bar{u}_h^{(m)}(x_i)\}, \{\underline{u}_h^{(m)}(x_i)\})$ at $x_i = 0.5$ for Example 6.3

Table 4: Table 1 The accuracy of the numerical solution $u_h^*(x_i)$ of Example 6.3.

h	Scheme (6.15)		SFD scheme	
	$\text{error}_\infty(h)$	$\text{order}_\infty(h)$	$\text{error}_\infty(h)$	$\text{order}_\infty(h)$
1/8	$4.16044850615194e-06$	4.00262945151917	$1.20005715017424e-03$	1.99011955907584
1/16	$2.59554536974349e-07$	4.00071651350112	$3.02076017246300e-04$	1.99756596363740
1/32	$1.62141038373420e-08$	4.00019299964688	$7.56465233968662e-05$	1.99939372559153
1/64	$1.01324593160257e-09$	4.00017895476307	$1.89195798938613e-05$	1.99984571226086
1/128	$6.33200158972613e-11$	4.00592116975355	$4.73040083492915e-06$	1.99995314107427
1/256	$3.94129173741931e-12$	4.06977174278409	$1.18263862036727e-06$	1.99999043828107
1/512	$2.34701147405758e-13$		$2.95661614635456e-07$	

To accommodate the analytical solution of $u(x) = \sin(\kappa x + \pi/6)$ where $\kappa = 2\pi/3$, we let

$$q(x) = \left(\frac{\kappa^2}{\theta} \sin\left(\kappa x + \frac{\pi}{6}\right) + \sin^4\left(\kappa x + \frac{\pi}{6}\right) \right)^{1/4}. \quad (6.17)$$

As in the previous examples, $\hat{u}_h(x_i) \equiv 0$ is a lower solution of (6.15) and the solution $\tilde{u}_h(x_i)$ of (6.3) (corresponding to (6.15)) with $a = 0$ and $b = \kappa^2 + \theta$ is a nonnegative upper solution.

Let $\theta = \pi^2/2$. We compute the corresponding sequences $\{\bar{u}_h^{(m)}(x_i)\}$ and $\{\underline{u}_h^{(m)}(x_i)\}$ from the iterative scheme (5.1) with the initial iterations $\bar{u}_h^{(0)}(x_i) = \tilde{u}_h(x_i)$ and $\underline{u}_h^{(0)}(x_i) = 0$. Let $h = 1/32$. Figure 3 shows the monotone and rapid convergence of these sequences at $x_i = 0.5$, where the solid line denotes the sequence $\{\bar{u}_h^{(m)}(x_i)\}$ and the dashed-dotted line stands for the sequence $\{\underline{u}_h^{(m)}(x_i)\}$ as before. The data in Table 4 show the maximum error $\text{error}_\infty(h)$ and the order $\text{order}_\infty(h)$ of the numerical solution $u_h^*(x_i)$ by the scheme (6.15) and the SFD scheme for various values of h . The fourth-order accuracy of the numerical solution $u_h^*(x_i)$ by the present Numerov scheme is demonstrated in this table.

7. Conclusions

In this paper, we have given a numerical treatment for a class of nonlinear multipoint boundary value problems by the fourth-order Numerov method. The existence and uniqueness of the numerical solution and the convergence of the method have been discussed. An accelerated monotone iterative algorithm with the quadratic rate of convergence has been developed for solving the resulting nonlinear discrete problem. The proposed Numerov method is more attractive due to its fourth-order accuracy, compared to the standard finite difference method.

In this work, we have generalized the method of upper and lower solutions to nonlinear multipoint boundary value problems. We have also developed a technique for designing and analyzing compact and monotone finite difference schemes with high accuracy. They are very useful for accurate numerical simulations of many other nonlinear problems, such as those related to integrodifferential equations (e.g., [55, 56]) and those in information modeling (e.g., [57–60]).

Appendix

A. Proofs of Lemmas 2.2 and 2.3

Lemmas 2.2 and 2.3 are the special cases of Lemmas 2.2 and 2.3 in [61]. We include their proofs here in order to make the paper self-contained. Define

$$\alpha_i^* = \begin{cases} \alpha_{i'}, & x_i = \xi_{i'} \text{ for some } i', \\ 0, & \text{otherwise,} \end{cases} \quad \beta_i^* = \begin{cases} \beta_{i'}, & x_i = \eta_{i'} \text{ for some } i', \\ 0, & \text{otherwise,} \end{cases} \quad 1 \leq i \leq L-1. \quad (\text{A.1})$$

Let $A = (a_{i,j})$, $B = (b_{i,j})$, and $D = (d_{i,j})$ be the $(L-1)$ th-order matrices with

$$a_{i,j} = 2\delta_{i,j} - \delta_{i,j-1} - \delta_{i,j+1}, \quad b_{i,j} = \frac{5}{6}\delta_{i,j} + \frac{1}{12}\delta_{i,j-1} + \frac{1}{12}\delta_{i,j+1}, \quad d_{i,j} = \delta_{i,1}\alpha_j^* + \delta_{i,L-1}\beta_j^*, \quad (\text{A.2})$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

Lemma A.1. *Let the condition (2.9) be satisfied. Then the inverse $(A - D)^{-1} > 0$, and*

$$\|(A - D)^{-1}\|_{\infty} \leq \frac{1}{8(1 - \sigma)h^2}. \quad (\text{A.3})$$

Proof. It can be checked by Corollary 3.20 on Page 91 of [62] that the inverse $(A - D)^{-1} > 0$. Let $E = (1, 1, \dots, 1)^T \in \mathbf{R}^{L-1}$ and let $S = (A - D)^{-1}E$. Then $\|(A - D)^{-1}\|_{\infty} = \|S\|_{\infty}$. It is known

that the inverse $A^{-1} = (J_{i,j})$ exists, and its elements $J_{i,j}$ are given by

$$J_{i,j} = \begin{cases} \frac{(L-j)i}{L}, & i \leq j, \\ \frac{(L-i)j}{L}, & i > j. \end{cases} \quad (\text{A.4})$$

A simple calculation shows that $\|A^{-1}\|_\infty \leq L^2/8 = 1/(8h^2)$ and $J_{i,1} + J_{i,L-1} = 1$ for each $1 \leq i \leq L-1$. This implies

$$S = A^{-1}E + A^{-1}DS \leq \|A^{-1}\|_\infty E + \sigma \|S\|_\infty E \leq \left(\frac{1}{8h^2}\right)E + \sigma \|S\|_\infty E. \quad (\text{A.5})$$

Thus the estimate (A.3) follows immediately. \square

Proof of Lemma 2.2. Define the following $(L-1)$ th-order matrices or vectors:

$$\begin{aligned} U_h &= (u_h(x_1), u_h(x_2), \dots, u_h(x_{L-1}))^T, \\ M &= \text{diag}(M_1, M_2, \dots, M_{L-1}), \quad M_b = \text{diag}(M_0, 0, \dots, 0, M_L), \\ G_b &= \left(\left(1 - \frac{h^2}{12}M_0\right)u_h(0), 0, \dots, 0, \left(1 - \frac{h^2}{12}M_L\right)u_h(1) \right)^T. \end{aligned} \quad (\text{A.6})$$

Using the matrices A and B defined by (A.2), we have from (2.11) that

$$(A + h^2BM)U_h \geq G_b. \quad (\text{A.7})$$

Since $\underline{M} > -8(1 - \sigma)$ and $h < h(\underline{M}, \overline{M})$, it is easy to check that $1 - (h^2/12)M_i \geq 0 (i = 0, L)$. Thus by the boundary condition in (2.11),

$$G_b \geq DU_h - \frac{h^2}{12}M_bDU_h, \quad (\text{A.8})$$

where D is the $(L-1)$ th-order matrix defined by (A.2). This leads to

$$\left(A - D + h^2BM + \frac{h^2}{12}M_bD \right) U_h \geq 0. \quad (\text{A.9})$$

Let $Q \equiv A - D + h^2BM + (h^2/12)M_bD$. To prove $u_h(x_i) \geq 0$ for all $0 \leq i \leq L$, it suffices to show that the inverse of Q exists and is nonnegative.

Case 1 ($\underline{M} \geq 0$). In this case, the matrix Q satisfies the condition of Corollary 3.20 on Page 91 of [62], and, therefore, its inverse Q^{-1} exists and is positive.

Case 2 ($0 > \underline{M} > -8(1 - \sigma)$). For this case, we define

$$M^+ = \text{diag}(M_1^+, M_2^+, \dots, M_{L-1}^+), \quad M_i^+ = \max\{M_i, 0\}, \quad M^- = M - M^+. \quad (\text{A.10})$$

The matrices M_b^+ and M_b^- can be similarly defined. Let $\bar{Q} = A - D + h^2 BM^+ + (h^2/12)M_b^+ D$. We know from Case 1 that \bar{Q}^{-1} exists and is positive. Since

$$Q = \bar{Q} + h^2 BM^- + \frac{h^2}{12} M_b^- D = \bar{Q} \left(I + h^2 \bar{Q}^{-1} \left(BM^- + \frac{1}{12} M_b^- D \right) \right), \quad (\text{A.11})$$

we need only to prove that the inverse $(I + h^2 \bar{Q}^{-1} (BM^- + (1/12)M_b^- D))^{-1}$ exists and is nonnegative. By Theorem 3 on Page 298 of [63], this is true if

$$\left\| h^2 \bar{Q}^{-1} \left(BM^- + \frac{1}{12} M_b^- D \right) \right\|_{\infty} < 1. \quad (\text{A.12})$$

Since $\bar{Q} \geq A - D$ which implies $0 \leq \bar{Q}^{-1} \leq (A - D)^{-1}$, we have from Lemma A.1 that

$$\left\| \bar{Q}^{-1} \right\|_{\infty} \leq \left\| (A - D)^{-1} \right\|_{\infty} \leq \frac{1}{8(1 - \sigma)h^2}. \quad (\text{A.13})$$

It is clear that $\|B + (1/12)D\|_{\infty} = 1$, $\|M^-\|_{\infty} \leq -\underline{M}$ and $\|M_b^-\|_{\infty} \leq -\underline{M}$. Thus, we have

$$\left\| h^2 \bar{Q}^{-1} \left(BM^- + \frac{1}{12} M_b^- D \right) \right\|_{\infty} \leq \frac{-\underline{M}}{8(1 - \sigma)}. \quad (\text{A.14})$$

The estimate (A.12) follows from $\underline{M} > -8(1 - \sigma)$. □

Proof of Lemma 2.3. Using the same notation as before, the system (2.12) can be written as

$$QU_h = G, \quad (\text{A.15})$$

where $G = (g(x_1), g(x_2), \dots, g(x_{L-1}))^T$.

Case 1 ($\underline{M} \geq 0$). Since the inverse Q^{-1} exists and is positive, we have $0 < Q^{-1} \leq (A - D)^{-1}$. This shows

$$\left\| Q^{-1} \right\|_{\infty} \leq \left\| (A - D)^{-1} \right\|_{\infty} \leq \frac{1}{8(1 - \sigma)h^2}. \quad (\text{A.16})$$

Thus, by (A.15), $\|U_h\|_{\infty} \leq \|G\|_{\infty} / (8(1 - \sigma)h^2)$ which implies the desired estimate (2.13).

Case 2 ($0 > \underline{M} > -8(1 - \sigma)$). It follows from (A.11) that

$$\|Q^{-1}\|_{\infty} \leq \|\overline{Q}^{-1}\|_{\infty} \left\| \left(I + h^2 \overline{Q}^{-1} \left(BM^{-} + \frac{1}{12} M_b^{-} D \right) \right)^{-1} \right\|_{\infty}. \quad (\text{A.17})$$

By (A.13) and (A.14),

$$\|Q^{-1}\|_{\infty} \leq \frac{1}{8(1 - \sigma)h^2} \cdot \frac{8(1 - \sigma)}{8(1 - \sigma) + \underline{M}} = \frac{1}{(8(1 - \sigma) + \underline{M})h^2}. \quad (\text{A.18})$$

This together with (A.15) leads to the estimate (2.13). \square

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