Research Article
Fractional Calculus and Shannon Wavelet

Carlo Cattani

Department of Mathematics, University of Salerno, Via Ponte Don Melillo, 84084 Fisciano, Italy

Correspondence should be addressed to Carlo Cattani, ccattani@unisa.it

An explicit analytical formula for the any order fractional derivative of Shannon wavelet is given as wavelet series based on connection coefficients. So that for any $L_2(\mathbb{R})$ function, reconstructed by Shannon wavelets, we can easily define its fractional derivative. The approximation error is explicitly computed, and the wavelet series is compared with Grünwald fractional derivative by focusing on the many advantages of the wavelet method, in terms of rate of convergence.

1. Introduction

Shannon wavelet theory [1, 2] is based on a family of orthogonal functions having many interesting properties. They enjoy the many advantages of wavelets [3, 4]; moreover, being analytical functions they are infinitely differentiable. Thus, enabling us to define the so-called connection coefficients [5–7] for any order derivative. Connection coefficients are an expedient tool for the projection of differential operators, useful for computing the wavelet solution of integrodifferential equations [8–13].

Wavelets are localized functions, in time and/or frequency, which are the basis for energy-bounded functions and in particular for $L_2(\mathbb{R})$-functions. So that localized pulse problems [14, 15] can be easily approached and analyzed. Moreover, wavelet allows the multiscale decomposition of problems, thus emphasizing the contribution of each scale. By defining a suitable inner product on the orthogonal family of scaling/wavelet functions, any $L_2(\mathbb{R})$-function can be approximated at a fixed scale, by a truncated series having, as basis, the scaling functions and the wavelet functions. The wavelet coefficients of these series represent the contribution of each scale.

Shannon wavelets are related to the harmonic wavelets [3, 5, 8], being the real part thereof, and to the well-known sinc function, which is the basic function in signal analysis. It should be also noticed that, as compared with other wavelet families, the main advantage

of Shannon wavelets is that they are analytical functions, thus being infinitely differentiable. Moreover, they are sharply bounded in the frequency domain, so that, by taking into account the Parseval identity, any computation can be easily performed by their Fourier transforms.

The theory of connection coefficients was initially given [10, 13] for the compactly supported wavelet families, such as the Daubechies wavelets [4]. The computation of these connection coefficients was based on the recursive equations of the wavelet theory and the explicit forms of these coefficients were given only up to the second order derivatives. The connection coefficients are the wavelet coefficients of the derivatives of the wavelet basis. These coefficients are a fundamental tool for the approximation of differential operators, with respect to the wavelet basis.

In some recent papers, the connection coefficients for Shannon wavelets have been explicitly computed up to any order derivative with a finite analytical form. This is due to the analytical form of Shannon wavelets and the discovery by Cattani of a suitable series expansion for the connection coefficients [2, 6, 7].

In the following, we will define the wavelet representation of fractional derivative, so that the fractional derivative of an $L_2(\mathbb{R})$-function can be easily computed by knowing the connection coefficients. The fractional derivatives of the Shannon scaling/wavelet basis are defined and the error of the approximation will be explicitly computed. Moreover, a comparison with the classical definition of Grünwald formula [16, 17] is given, by showing the major performance of wavelets, in terms of rate of convergence.

In particular, Section 2 gives some preliminary remarks, definitions, and properties about Shannon wavelets. Their corresponding connection coefficients are discussed in Section 3. This Section deals with some properties of connection coefficients, functional equalities, and error of approximation. Fractional derivatives of the Shannon scaling function and wavelets are given in Section 4. In this section, it is also shown that the fractional derivative is a semigroup. The error of the approximation is explicitly computed and compared with classical definitions of the fractional derivative, and in particular with the Grünwald formula.

2. Preliminary Remarks

In this section, some remarks on Shannon wavelets and connection coefficients are given (see also [7]).

Shannon wavelet theory (see e.g. [1, 2, 6, 7, 9]) is based on the scaling function $\varphi(x)$, also known as sinc function, and the wavelet function $\psi(x)$, respectively, defined as

$$
\begin{aligned}
\varphi(x) = \operatorname{sinc} x &\overset{\text{def}}{=} \frac{\sin \pi x}{\pi x} = \frac{e^{\pi i x} - e^{-\pi i x}}{2\pi i x}, \\
\psi(x) &= \frac{\sin 2\pi(x - (1/2)) - \sin \pi(x - (1/2))}{\pi(x - (1/2))} \\
&= \frac{e^{-2 i \pi x}\left(-i + e^{i \pi x} + e^{3 i \pi x} + i\, e^{4 i \pi x}\right)}{2\pi(x - (1/2))}.
\end{aligned}
\tag{2.1}
$$

The corresponding families of translated and dilated instances wavelet [1, 2, 6, 7, 9], on which is based the multiscale analysis [4], are

$$
\begin{aligned}
\varphi_k^n(x) &= 2^{n/2}\varphi(2^n x - k) \;\; = 2^{n/2}\frac{\sin \pi(2^n x - k)}{\pi(2^n x - k)} \\
&= 2^{n/2}\frac{e^{\pi i(2^n x - k)} - e^{-\pi i(2^n x - k)}}{2\pi i(2^n x - k)}, \\
\psi_k^n(x) &= 2^{n/2}\frac{\sin 2\pi(2^n x - k - (1/2)) - \sin \pi(2^n x - k - (1/2))}{\pi(2^n x - k - (1/2))} \\
&= \frac{2^{n/2}}{2\pi(2^n x - k - (1/2))}\sum_{s=1}^{2} i^{1+s} e^{s\pi i(2^n x - k)} - i^{1-s} e^{-s\pi i(2^n x - k)},
\end{aligned}
\tag{2.2}
$$

being, in particular,

$$
\varphi_0^0(x) = \varphi(x), \quad \psi_0^0(x) = \psi(x), \quad \varphi_k^0(x) = \varphi_k(x) = \varphi(x - k),
$$
$$
\psi_k^0(x) = \psi_k(x) = \psi(x - k).
\tag{2.3}
$$

Let

$$
\widehat{f}(\omega) = \widehat{f(x)} \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \qquad f(x) = \int_{-\infty}^{\infty}\widehat{f}(\omega)e^{i\omega x}\,d\omega
\tag{2.4}
$$

be the Fourier transform of the function $f(x) \in L_2(\mathbb{R})$, and its inverse transform, respectively. The Fourier transform of (2.1) give us [2]

$$
\widehat{\varphi}(\omega) = \frac{1}{2\pi}\chi(\omega + 3\pi) =
\begin{cases}
\dfrac{1}{2\pi}, & -\pi \le \omega < \pi \\
0, & \text{elsewhere}
\end{cases}
\tag{2.5}
$$

$$
\widehat{\psi}(\omega) = \frac{1}{2\pi}e^{i\omega/2}\big[\chi(2\omega) + \chi(-2\omega)\big],
$$

with

$$
\chi(\omega) =
\begin{cases}
1, & 2\pi \le \omega < 4\pi \\
0, & \text{elsewhere.}
\end{cases}
\tag{2.6}
$$

Analogously for the dilated and translated instances of scaling/wavelet function, in the frequency domain, it is

$$
\widehat{\varphi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi}e^{i\omega k/2^n}\chi\left(\frac{\omega}{2^n} + 3\pi\right)
$$

$$
\widehat{\psi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi}e^{i\omega(k+1/2)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right) + \chi\left(-\frac{\omega}{2^{n-1}}\right)\right].
\tag{2.7}
$$

Both families of Shannon scaling and wavelet are $L_2(\mathbb{R})$-functions therefore, for each $f(x) \in L_2(\mathbb{R})$ and $g(x) \in L_2(\mathbb{R})$, the inner product is defined as

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = 2\pi \int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}d\omega = 2\pi \left\langle \widehat{f}, \widehat{g} \right\rangle, \tag{2.8}$$

where the bar stands for the complex conjugate.

Shannon wavelets fulfill the following orthogonality properties (for the proof see e.g., [2, 7]):

$$\left\langle \psi_k^n(x), \psi_h^m(x) \right\rangle = \delta^{nm}\delta_{hk}, \quad \left\langle \varphi_k^0(x), \varphi_h^0(x) \right\rangle = \delta_{kh}, \quad \left\langle \varphi_k^0(x), \psi_h^m(x) \right\rangle = 0, \quad m \geq 0, \tag{2.9}$$

δ^{nm}, δ_{hk} being the Kronecker symbols.

2.1. Properties of the Shannon Wavelet

According to (2.2), Shannon wavelets can be easily computed at some special points, being in particular

$$\varphi_k(h) = \varphi_h(k) = \varphi(h - k) = \varphi(k - h) = \delta_{kh}, \quad (h, k \in \mathbb{Z}), \tag{2.10}$$

so that

$$\varphi_k(x) = \begin{cases} 0, & x = h \neq k, \quad (h, k \in \mathbb{Z}) \\ 1, & x = h = k, \quad (h, k \in \mathbb{Z}). \end{cases} \tag{2.11}$$

It is also [7]

$$\psi_k^n(h) = (-1)^{2^n h - k} \frac{2^{1+n/2}}{(2^{n+1}h - 2k - 1)\pi}, \quad \left(2^{n+1}h - 2k - 1 \neq 0\right)$$

$$\psi_k^n(x) = 0, \quad x = 2^{-n}\left(k + \frac{1}{2} \pm \frac{1}{3}\right), \quad (n \in \mathbb{N}, k \in \mathbb{Z}) \tag{2.12}$$

$$\lim_{x \to 2^{-n}(h+(1/2))} \psi_k^n(x) = -2^{n/2}\delta_{hk}.$$

In the following, we will be interested on the maximum values of these functions which can be easily computed. The maximum value of the scaling function $\varphi_k(x)$ can be found at the integers $x = k$

$$\max[\varphi_k(x_M)] = 1, \quad x_M = k, \tag{2.13}$$

and the max values of $\psi_k^n(x)$ are

$$\max\left[\psi_k^n(x_M)\right] = 2^{n/2}\frac{3\sqrt{3}}{\pi}, \qquad x_M = \begin{cases} -2^{-n}\left(k + \dfrac{1}{6}\right) \\ \dfrac{2^{-n-1}}{3}(18k + 7). \end{cases} \tag{2.14}$$

Both families of scaling and wavelet functions belong to $L_2(\mathbb{R})$, thus having a bounded range and (slow) decay to zero

$$\lim_{x \to \pm\infty} \varphi_k^n(x) = 0, \qquad \lim_{x \to \pm\infty} \psi_k^n(x) = 0. \tag{2.15}$$

Let $\mathcal{B} \subset L_2(\mathbb{R})$ the set of functions $f(x)$ in $L_2(\mathbb{R})$ such that the integrals

$$\alpha_k \overset{\text{def}}{=} \langle f(x), \varphi_k(x) \rangle \overset{(2.8)}{=} \int_{-\infty}^{\infty} f(x)\overline{\varphi_k^0(x)}dx$$

$$\beta_k^n \overset{\text{def}}{=} \langle f(x), \psi_k^n(x) \rangle \overset{(2.8)}{=} \int_{-\infty}^{\infty} f(x)\overline{\psi_k^n(x)}dx \tag{2.16}$$

exist with finite values, then it can be shown [2–4, 7] that the series

$$f(x) = \sum_{h=-\infty}^{\infty} \alpha_h\, \varphi_h(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x) \tag{2.17}$$

converges to $f(x)$.

According to (2.8), the coefficients can be also computed in the Fourier domain [7] so that

$$\alpha_k = \int_{-\pi}^{\pi} \hat{f}(\omega)\, e^{i\omega k} d\omega,$$

$$\beta_k^n = 2^{-n/2}\left[\int_{2^n\pi}^{2^{n+1}\pi} \hat{f}(\omega)e^{i\omega(k+1/2)/2^n}d\omega + \int_{-2^{n+1}\pi}^{-2^n\pi} \hat{f}(\omega)e^{i\omega(k+1/2)/2^n}d\omega\right]. \tag{2.18}$$

In the frequency domain, (2.17) gives [7]

$$\hat{f}(\omega) = \frac{1}{2\pi}\chi(\omega + 3\pi)\sum_{h=-\infty}^{\infty} \alpha_h e^{i\,\omega h}$$

$$+ \frac{1}{2\pi}\chi\left(\frac{\omega}{2^{n-1}}\right)\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-n/2}\beta_k^n e^{i\omega(k+1/2)/2^n}$$

$$+ \frac{1}{2\pi}\chi\left(-\frac{\omega}{2^{n-1}}\right)\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-n/2}\beta_k^n e^{i\omega(k+1/2)/2^n}. \tag{2.19}$$

When the upper bound for the series of (2.17) is finite, then we have the approximation

$$f(x) \cong \sum_{h=-K}^{K} \alpha_h \varphi_h(x) + \sum_{n=0}^{N} \sum_{k=-S}^{S} \beta_k^n \psi_k^n(x). \tag{2.20}$$

The error of the approximation has been estimated in [7].

2.2. Reconstruction of the Derivatives

In order to represent the differential operators in wavelet bases, we have to compute the wavelet decomposition of the derivatives. It can be shown [2, 7] that the derivatives of the Shannon wavelets are orthogonal functions:

$$\frac{d^\ell}{dx^\ell}\varphi_h(x) = \sum_{k=-\infty}^{\infty} \lambda_{hk}^{(\ell)} \varphi_k(x),$$

$$\frac{d^\ell}{dx^\ell}\psi_h^m(x) = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \gamma^{(\ell)mn}_{hk} \psi_k^n(x), \tag{2.21}$$

being

$$\lambda_{kh}^{(\ell)} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell}\varphi_k^0(x), \varphi_h^0(x) \right\rangle, \qquad \gamma^{(\ell)mn}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell}\psi_k^n(x), \psi_h^m(x) \right\rangle, \tag{2.22}$$

the connection coefficients [2, 5, 6, 8–13].

The computation of connection coefficients can be easily performed in the Fourier domain, thanks to the equality (2.8)

$$\lambda_{kh}^{(\ell)} = 2\pi \left\langle \widehat{\frac{d^\ell}{dx^\ell}\varphi_k(x)}, \widehat{\varphi_h(x)} \right\rangle, \qquad \gamma^{(\ell)mn}_{kh} = 2\pi \left\langle \widehat{\frac{d^\ell}{dx^\ell}\psi_k^n(x)}, \widehat{\psi_h^m(x)} \right\rangle. \tag{2.23}$$

In fact, in the Fourier domain, the ℓ-order derivative of the (scaling) wavelet functions are simply

$$\widehat{\frac{d^\ell}{dx^\ell}\varphi_k^n}(x) = (i\omega)^\ell \widehat{\varphi}_k^n(\omega), \qquad \widehat{\frac{d^\ell}{dx^\ell}\psi_k^n}(x) = (i\omega)^\ell \widehat{\psi}_k^n(\omega), \tag{2.24}$$

and, according to (2.7),

$$\widehat{\frac{d^\ell}{dx^\ell}\varphi_k^n}(x) = (i\omega)^\ell \frac{2^{-n/2}}{2\pi} e^{i\omega k/2^n} \chi\left(\frac{\omega}{2^n} + 3\pi\right),$$

$$\widehat{\frac{d^\ell}{dx^\ell}\psi_k^n}(x) = (i\omega)^\ell \frac{2^{-n/2}}{2\pi} e^{i\omega(k+(1/2))/2^n} \left[\chi\left(\frac{\omega}{2^{n-1}}\right) + \chi\left(-\frac{\omega}{2^{n-1}}\right)\right]. \tag{2.25}$$

It has been shown [2, 6, 7] that the any order connection coefficients $(2.22)_1$ of the Shannon scaling functions $\varphi_k(x)$ are

$$\lambda_{kh}^{(\ell)} = \begin{cases} (-1)^{k-h} \dfrac{i^\ell}{2\pi} \displaystyle\sum_{s=1}^{\ell} \dfrac{\ell!\pi^s}{s![i(k-h)]^{\ell-s+1}} \left[(-1)^s - 1\right], & k \neq h \\[6pt] \dfrac{i^\ell \pi^{\ell+1}}{2\pi(\ell+1)} \left[1 + (-1)^\ell\right], & k = h, \end{cases} \tag{2.26}$$

or, by defining

$$\mu(m) = \text{sign}(m) = \begin{cases} 1, & m > 0 \\ -1, & m < 0 \\ 0, & m = 0, \end{cases} \tag{2.27}$$

shortly as,

$$\begin{aligned} \lambda_{kh}^{(\ell)} =\ & \frac{i^\ell \pi^\ell}{2(\ell+1)} \left[1 + (-1)^\ell\right] \left(1 - |\mu(k-h)|\right) \\[6pt] & + (-1)^{k-h} |\mu(k-h)| \frac{i^\ell}{2\pi} \sum_{s=1}^{\ell} \frac{\ell!\pi^s}{s![i(k-h)]^{\ell-s+1}} \left[(-1)^s - 1\right], \end{aligned} \tag{2.28}$$

when $\ell \geq 1$, and for $\ell = 0$,

$$\lambda_{kh}^{(0)} = \delta_{kh}. \tag{2.29}$$

For the proof see [2].

Analogously for the connection coefficients $(2.22)_2$ we have that the any order connection coefficients of the Shannon scaling wavelets $\psi_k^n(x)$ are

$$\gamma_{kh}^{(\ell)nm} = \mu(h-k)\delta^{nm} \left\{ \sum_{s=1}^{\ell+1} (-1)^{[1+\mu(h-k)](2\ell-s+1)/2} \frac{\ell! i^{\ell-s} \pi^{\ell-s}}{(\ell-s+1)!|h-k|^s} (-1)^{-s-2(h+k)} 2^{n\ell-s-1} \right.$$

$$\left. \times \left\{ 2^{\ell+1}\left[(-1)^{4h+s} + (-1)^{4k+\ell}\right] - 2^s\left[(-1)^{3k+h+\ell} + (-1)^{3h+k+s}\right]\right\} \right\}, \quad k \neq h$$

$$\gamma_{kh}^{(\ell)nm} = \delta^{nm} \left[i^\ell \frac{\pi^\ell 2^{n\ell-1}}{\ell+1} \left(2^{\ell+1} - 1\right) 1 + \left((-1)^\ell\right) \right], \quad k = h,$$

$$\tag{2.30}$$

or, shortly

$$
\gamma^{(\ell)^{nm}_{kh}} = \delta^{nm} \Bigg\{ i^{\ell} \left(1 - |\mu(h-k)|\right) \frac{\pi^{\ell} 2^{n\ell-1}}{\ell+1} \left(2^{\ell+1} - 1\right)\left(1 + (-1)^{\ell}\right)
$$

$$
+ \mu(h-k) \sum_{s=1}^{\ell+1} (-1)^{[1+\mu(h-k)](2\ell-s+1)/2} \frac{\ell! i^{\ell-s} \pi^{\ell-s}}{(\ell-s+1)! \, |h-k|^{s}} (-1)^{-s-2(h+k)} 2^{n\ell-s-1}
$$

$$
\times \left\{ 2^{\ell+1} \left[(-1)^{4h+s} + (-1)^{4k+\ell} \right] - 2^{s} \left[(-1)^{3k+h+\ell} + (-1)^{3h+k+s} \right] \right\} \Bigg\},
$$

$$
\tag{2.31}
$$

for $\ell \geq 1$, and

$$
\gamma^{(0)^{nm}_{kh}} = \delta_{kh} \delta^{nm}, \tag{2.32}
$$

$\ell = 0$, respectively.

For the proof see [2].

3. Remarks on Connection Coefficients

3.1. Recursiveness

The connection coefficients fulfill some recursive formula as follows.

Theorem 3.1. *The connection coefficients* (2.26) *are recursively given by*

$$
\lambda_{kh}^{(\ell+1)} = \begin{cases} \dfrac{\ell+1}{k-h} \lambda_{kh}^{(\ell)} - (-1)^{k-h} \dfrac{i^{\ell} \pi^{\ell+1}}{k-h} \left[(-1)^{\ell} + 1 \right], & k \neq h \\[3mm] i\pi \dfrac{\ell+1}{\ell+2} \lambda_{kh}^{(\ell)} + \dfrac{(-i)^{\ell+1} \pi^{\ell+1}}{\ell+2}, & k = h, \end{cases} \tag{3.1}
$$

Proof. Let us show first when $k = h$. From the definition (2.26), it is

$$
\lambda_{kk}^{(\ell+1)} = \frac{i^{\ell+1} \pi^{\ell+2}}{2\pi(\ell+2)} \left[1 + (-1)^{\ell+1} \right]
$$

$$
= i\pi \frac{(\ell+1)}{(\ell+2)} \frac{i^{\ell} \pi^{\ell+1}}{2\pi(\ell+1)} \left[1 + (-1)^{\ell+1} + (-1)^{\ell} - (-1)^{\ell} \right] \tag{3.2}
$$

$$
= i\pi \frac{(\ell+1)}{(\ell+2)} \frac{i^{\ell} \pi^{\ell+1}}{2\pi(\ell+1)} \left[1 + (-1)^{\ell} + 2(-1)^{\ell+1} \right],
$$

from where (3.1)$_2$ follows. Analogously with simple computation we obtain (3.1)$_1$. $\square$

Shorty and with some caution, (3.1) can be written as

$$
\begin{aligned}
\lambda_{kh}^{(\ell+1)} = {} & (1 - \delta_{kh}) \left[\frac{\ell+1}{k-h} \lambda_{kh}^{(\ell)} - (-1)^{k-h} \frac{i^{\ell} \pi^{\ell+1}}{k-h} \left[(-1)^{\ell} + 1 \right] \right] \\
& + \delta_{kh} \left[i\pi \frac{\ell+1}{\ell+2} \lambda_{kh}^{(\ell)} + \frac{(-i)^{\ell+1} \pi^{\ell+1}}{\ell+2} \right],
\end{aligned}
\tag{3.3}
$$

that is,

$$
\begin{aligned}
\lambda_{kh}^{(\ell+1)} = {} & \left[(1 - \delta_{kh}) \frac{\ell+1}{k-h} + \delta_{kh} i\pi \frac{\ell+1}{\ell+2} \right] \lambda_{kh}^{(\ell)} \\
& - (1 - \delta_{kh})(-1)^{k-h} \frac{i^{\ell} \pi^{\ell+1}}{k-h} \left[(-1)^{\ell} + 1 \right] + \delta_{kh} \frac{(-i)^{\ell+1} \pi^{\ell+1}}{\ell+2}.
\end{aligned}
\tag{3.4}
$$

It is not so easy to find out a similar property also for the γ-coefficients as a function of ℓ however, there is a simple rule for the recursiveness of the scale (upper) indexes, as follows.

Theorem 3.2. *The connection coefficients* (2.30) *are recursively given by the matrix at the lowest scale level:*

$$
\gamma^{(\ell)\,nn}_{\quad kh} = 2^{\ell(n-1)} \gamma^{(\ell)\,11}_{\quad kh} .
\tag{3.5}
$$

Proof. As can be seen from (2.30) parameter n appears only in the factor

$$
2^{n\ell-1},
\tag{3.6}
$$

so that (3.5) follows from the identity

$$
2^{n\ell-1} = 2^{\ell(n-1)} 2^{\ell-1}.
\tag{3.7}
$$

□

Moreover, it can be shown also that

$$
\gamma^{(2\ell+1)\,nn}_{\quad\quad kh} = -\gamma^{(2\ell+1)\,nn}_{\quad\quad hk}, \qquad \gamma^{(2\ell)\,nn}_{\quad kh} = \gamma^{(2\ell)\,nn}_{\quad hk}.
\tag{3.8}
$$

3.2. Taylor Series

By using the connection coefficients, it is easy to show the following theorem.

Theorem 3.3. *If $f(x) \in B_\psi \subset L_2(\mathbb{R})$ and $f(x) \in \mathcal{C}^S$ the Taylor series of $f(x)$ in x_0 is*

$$
f(x) = f(x_0)
$$

$$
+ \sum_{r=1}^{S} \left[\sum_{h,k=-\infty}^{\infty} \alpha_h \, \lambda_{hk}^{(r)} \varphi_k(x_0) + \sum_{n=0}^{\infty} \sum_{k,s=-\infty}^{\infty} 2^{r(n-1)} \beta_k^n \gamma_{sk}^{(r)\,11} \psi_s^n(x_0) \right] \frac{(x-x_0)^r}{r!} + R^S(x, x_0),
$$

(3.9)

being α_h and β_k^n given by (2.16), (2.18) and $R^S(x, x_0)$ the error.

Proof. From (2.17), the ℓ-order derivative ($\ell \leq S$) is

$$
f^{(\ell)}(x) = \sum_{h=-\infty}^{\infty} \alpha_h \, \frac{d^\ell}{dx^\ell} \varphi_h(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \frac{d^\ell}{dx^\ell} \psi_k^n(x),
$$

$$
\overset{(2.21)}{=} \sum_{h=-\infty}^{\infty} \alpha_h \sum_{k=-\infty}^{\infty} \lambda_{hk}^{(\ell)} \varphi_k(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \gamma_{sk}^{(\ell)\,mn} \psi_s^m(x), \qquad (3.10)
$$

$$
= \sum_{h,k=-\infty}^{\infty} \alpha_h \, \lambda_{hk}^{(\ell)} \varphi_k(x) + \sum_{n,m=0}^{\infty} \sum_{k,s=-\infty}^{\infty} \beta_k^n \gamma_{sk}^{(\ell)\,mn} \psi_s^m(x),
$$

so that by taking into account (3.5) the proof follows. $\square$

In particular, by a suitable choice of the initial point x_0, (3.9) can be simplified. For instance, at the integers, $x_0 = h$, $(h \in \mathbb{Z})$, according to (2.10), (2.12) and (3.5), it is

$$
f(x) \cong f(h) + \sum_{r=1}^{S} \left[\sum_{h=-\infty}^{\infty} \alpha_h \, \lambda_{hh}^{(r)} + \sum_{n=0}^{\infty} \sum_{k,s=-\infty}^{\infty} (-1)^{2^n h - s} \frac{2^{r(n-1)+1+n/2}}{(2^{n+1}h - 2s - 1)\pi} \beta_k^n \gamma_{sk}^{(r)\,11} \right] \frac{(x-h)^r}{r!}.
$$

(3.11)

3.3. Functional Equations

The connection coefficients fulfill some identities as follows.

Theorem 3.4. *For any $k \in \mathbb{Z}$ and $\ell \in \mathbb{N}$, it is*

$$
(i\omega)^\ell e^{-i\omega k} = \sum_{h=-\infty}^{\infty} \lambda_{kh}^{(\ell)} e^{-i\omega h}, \quad -\pi \leq \omega \leq \pi,
$$

(3.12)

or

$$
(i\omega)^\ell = \sum_{h=-\infty}^{\infty} \lambda_{kh}^{(\ell)} e^{-i\omega(h-k)}, \quad -\pi \leq \omega \leq \pi, \; \forall k \in \mathbb{Z}.
$$

(3.13)

Proof. From (2.21), by a Fourier transform of both sides and taking into account (2.24), we get

$$(i\omega)^\ell \widehat{\varphi}_k(\omega) = \sum_{h=-\infty}^{\infty} \lambda_{kh}^{(\ell)} \widehat{\varphi}_h(\omega)$$

$$(i\omega)^\ell e^{-i\omega k} \chi(\omega + 3\pi) \overset{(2.7)}{=} \sum_{h=-\infty}^{\infty} \lambda_{kh}^{(\ell)} e^{-i\omega h} \chi(\omega + 3\pi),$$

(3.14)

from where the identity (3.12) follows. □

In particular, by assuming, without restrictions, $k = 0$, we have the following (see Figure 1).

Corollary 3.5. *For any $\ell \in \mathbb{N}$ it is*

$$(i\omega)^\ell = \sum_{h=-\infty}^{\infty} \lambda_{0h}^{(\ell)} e^{-i\omega h}, \quad -\pi \leq \omega \leq \pi,$$

(3.15)

so that $\lambda_{0h}^{(\ell)}$ are the Fourier coefficients of the power $(i\omega)^\ell$.

Analogously, from (2.21)$_2$, we have the following.

Theorem 3.6. *For any $k \in \mathbb{Z}$ and $\ell, n \in \mathbb{N}$ it is*

$$(i\omega)^\ell e^{-i\omega(k+1/2)/2^n} = \sum_{h=-\infty}^{\infty} \gamma_{kh}^{(\ell)nn} e^{-i\omega(h+1/2)/2^n}, \quad \omega \in \left[-2^{n+1}\pi, -2^n\pi\right] \cup \left[2^n\pi, 2^{n+1}\pi\right],$$

(3.16)

or

$$(i\omega)^\ell = \sum_{h=-\infty}^{\infty} \gamma_{kh}^{(\ell)nn} e^{-i\omega(h-k)/2^n}, \quad \omega \in \left[-2^{n+1}\pi, -2^n\pi\right] \cup \left[2^n\pi, 2^{n+1}\pi\right].$$

(3.17)

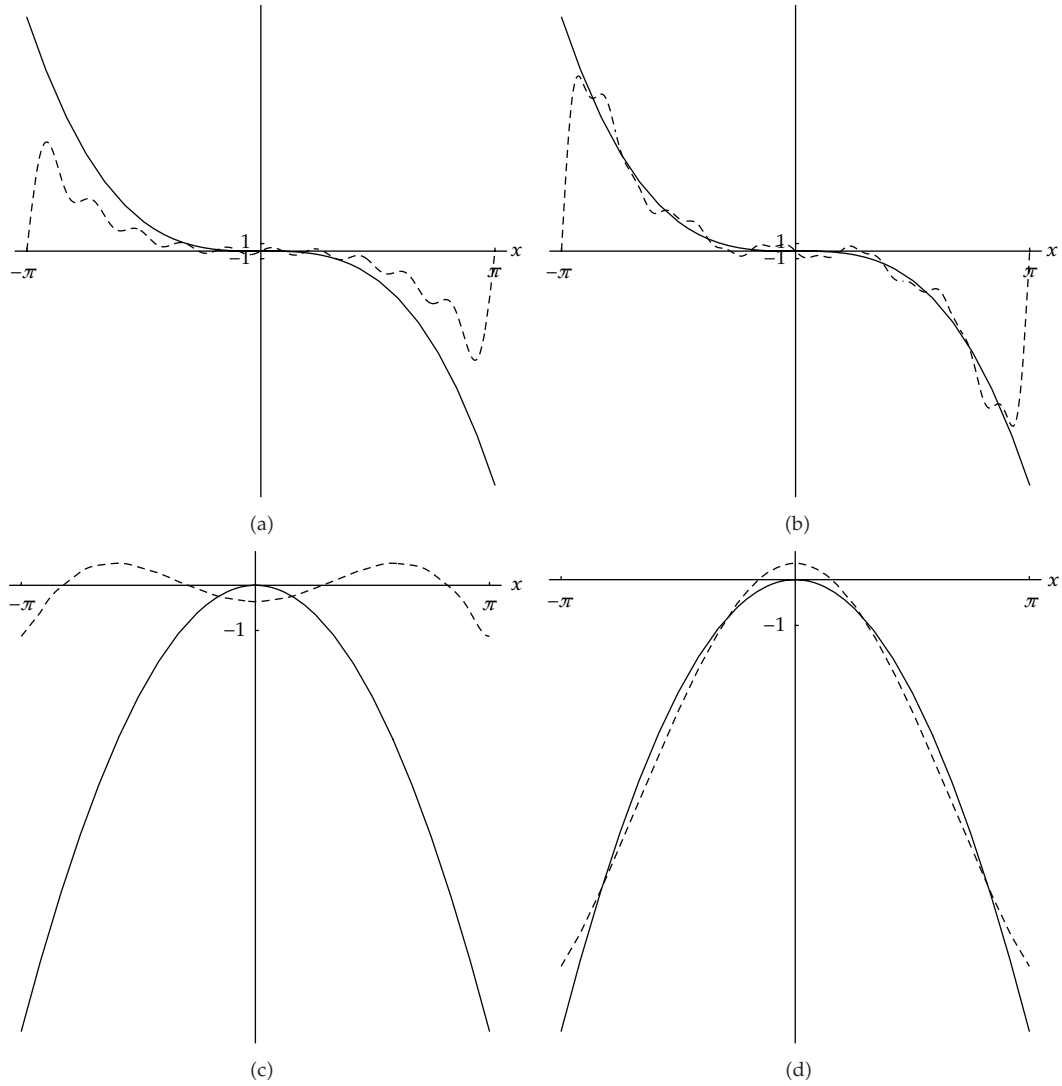In particular, with $k = 0$, and taking into account (3.5), we have the following.

Corollary 3.7. *For any $\ell, n \in \mathbb{N}$ it is*

$$(i\omega)^\ell = 2^{\ell(n-1)} \sum_{h=-\infty}^{\infty} \gamma_{0h}^{(\ell)11} e^{-i\omega h/2^n}, \quad \omega \in \left[-2^{n+1}\pi, -2^n\pi\right] \cup \left[2^n\pi, 2^{n+1}\pi\right].$$

(3.18)

As a consequence of the previous theorems we have the following.

Theorem 3.8. *For any $\ell, n \in \mathbb{N}$ it is*

$$(i\omega)^\ell = \begin{cases} \displaystyle\sum_{h=-\infty}^{\infty} \lambda_{0h}^{(\ell)} e^{-i\omega h}, & -\pi \leq \omega \leq \pi \\ \displaystyle 2^{\ell(n-1)} \sum_{h=-\infty}^{\infty} \gamma_{0h}^{(\ell)11} e^{-i\omega h/2^n}, & \omega \in \left[-2^{n+1}\pi, -2^n\pi\right] \cup \left[2^n\pi, 2^{n+1}\pi\right]. \end{cases}$$

(3.19)

(a)   (b)

(c)   (d)

Figure 1: Approximation of $(i\omega)^{\ell}$ (plain) by the r.h.s of (3.15) at different scale: (a) $\Im[(i\omega)^3]$, $k = 5$, $|h_{\max}| = 5$; (b) $\Im[(i\omega)^3]$, $k = 5$, $|h_{\max}| = 10$; (c) $\Re[(i\omega)^2]$, $k = 7$, $|h_{\max}| = 5$; (d) $\Re[(i\omega)^2]$, $k = 7$, $|h_{\max}| = 8$.

There we have the following.

Corollary 3.9. *The Fourier transform of the derivatives of a function is*

$$\widehat{\frac{d^{\ell}}{dx^{\ell}}f(x)} = \widehat{f}(\omega) \times \begin{cases} \displaystyle\sum_{h=-\infty}^{\infty} \lambda_{0h}^{(\ell)} e^{-i\omega h}, & -\pi \leq \omega \leq \pi \\ \displaystyle 2^{\ell(n-1)} \sum_{h=-\infty}^{\infty} \gamma_{kh}^{(\ell)\,11} e^{-i\omega h/2^n}, & \omega \in \left[-2^{n+1}\pi, -2^n\pi\right] \cup \left[2^n\pi, 2^{n+1}\pi\right]. \end{cases} \tag{3.20}$$

If we express $e^{i\omega}$ as a Taylor series we have

$$e^{i\omega} = \sum_{\ell=0}^{\infty} \frac{(i\omega)^\ell}{\ell!}, \tag{3.21}$$

so that $e^{i\omega}$ with $-\pi \le \omega \le \pi$ is the solution of the functional equation

$$X = \sum_{\ell=0}^{\infty} \sum_{h=-\infty}^{\infty} \frac{1}{\ell!} \lambda_{0h}^{(\ell)} X^{-h}. \tag{3.22}$$

Moreover, the theorem of moments

$$\int_{\mathbb{R}} x^\ell f(x) dx = i^\ell \frac{d\widehat{f}(\omega)}{d\omega^\ell} \tag{3.23}$$

can be written as

$$\int_{\mathbb{R}} x^\ell f(x) dx = i^\ell \widehat{f}(\omega) \times \begin{cases} \displaystyle\sum_{h=-\infty}^{\infty} \lambda_{kh}^{(\ell)} e^{-i\omega h}, & -\pi \le \omega \le \pi \\ \displaystyle\sum_{h=-\infty}^{\infty} \gamma_{kh}^{(\ell)nn} e^{-i\omega(h-k)/2^n}, & \omega \in \left[-2^{n+1}\pi, -2^n\pi\right] \cup \left[2^n\pi, 2^{n+1}\pi\right]. \end{cases} \tag{3.24}$$

3.4. Error of the Approximation by Connection Coefficients

For a fixed scale of approximation in (2.21), it is possible to estimate the error as follows. It should be noticed that the approximation depends on a the upper bound of the limits in the sums.

Theorem 3.10 (error of the approximation of scaling functions derivatives). *The error of the approximation in* $(2.21)_1$ *is given by*

$$\left| \frac{d^\ell}{dx^\ell} \varphi_h(x) - \sum_{k=-N}^{N} \lambda_{hk}^{(\ell)} \varphi_k(x) \right| \le \left| \lambda_{h(-N-1)}^{(\ell)} + \lambda_{h(N+1)}^{(\ell)} \right|. \tag{3.25}$$

Proof. The error of the approximation $(2.21)_1$ is defined as

$$\frac{d^\ell}{dx^\ell} \varphi_h(x) - \sum_{k=-N}^{N} \lambda_{hk}^{(\ell)} \varphi_k(x) = \sum_{k=-\infty}^{-N-1} \lambda_{hk}^{(\ell)} \varphi_k(x) + \sum_{k=N+1}^{\infty} \lambda_{hk}^{(\ell)} \varphi_k(x). \tag{3.26}$$

Concerning the r.h.s, and according to (2.13), it is

$$\sum_{k=-\infty}^{-N-1} \lambda_{hk}^{(\ell)}\, \varphi_k(x) + \sum_{k=N+1}^{\infty} \lambda_{hk}^{(\ell)}\, \varphi_k(x)$$

$$\leq \max_{x\in\mathbb{R}} \left[\sum_{k=-\infty}^{-N-1} \lambda_{hk}^{(\ell)}\, \varphi_k(x) + \sum_{k=N+1}^{\infty} \lambda_{hk}^{(\ell)}\, \varphi_k(x) \right] \tag{3.27}$$

$$= \lambda_{h(-N-1)}^{(\ell)}\, \varphi_{-N-1}(x) + \lambda_{h(N+1)}^{(\ell)}\, \varphi_{N+1}(x) \leq \lambda_{h(-N-1)}^{(\ell)} + \lambda_{h(N+1)}^{(\ell)}.$$

$\square$

Theorem 3.11 (error of the approximation of wavelet functions derivatives). *The error of the approximation in* $(2.21)_2$ *is given by*

$$\left| \frac{d^\ell}{dx^\ell} \psi_h^m(x) - \sum_{n=0}^{N} \sum_{k=-S}^{S} \gamma^{(\ell)mn}_{hk}\, \psi_k^n(x) \right| \leq \left| 2^{\ell(m-1)+m/2} \frac{3\sqrt{3}}{\pi} \left[\gamma^{(\ell)11}_{h(-S-1)} + \gamma^{(\ell)11}_{h(S+1)} \right] \right|. \tag{3.28}$$

Proof. The error of the approximation is

$$\frac{d^\ell}{dx^\ell} \psi_h^m(x) - \sum_{n=0}^{N} \sum_{k=-S}^{S} \gamma^{(\ell)mn}_{hk}\, \psi_k^n(x) = \sum_{n=N+1}^{\infty} \left[\sum_{k=-\infty}^{-S-1} \gamma^{(\ell)mn}_{hk}\, \psi_k^n(x) + \sum_{k=S+1}^{\infty} \gamma^{(\ell)mn}_{hk}\, \psi_k^n(x) \right]. \tag{3.29}$$

If $m < N$, the r.h.s. according to (2.30) is zero; therefore, we assume that $m > N$ so that the last equation becomes

$$\frac{d^\ell}{dx^\ell} \psi_h^m(x) - \sum_{n=0}^{N} \sum_{k=-S}^{S} \gamma^{(\ell)mn}_{hk}\, \psi_k^n(x) = \left[\sum_{k=-\infty}^{-S-1} \gamma^{(\ell)mm}_{hk}\, \psi_k^n(x) + \sum_{k=S+1}^{\infty} \gamma^{(\ell)mm}_{hk}\, \psi_k^n(x) \right]$$

$$\stackrel{(3.5)}{=} 2^{\ell(m-1)} \left[\sum_{k=-\infty}^{-S-1} \gamma^{(\ell)11}_{hk} + \sum_{k=S+1}^{\infty} \gamma^{(\ell)11}_{hk} \right] \psi_k^m(x)$$

$$\leq 2^{\ell(m-1)} \max\left\{ \left[\sum_{k=-\infty}^{-S-1} \gamma^{(\ell)11}_{hk} + \sum_{k=S+1}^{\infty} \gamma^{(\ell)11}_{hk} \right] \psi_k^m(x) \right\}$$

$$= 2^{\ell(m-1)} \left[\gamma^{(\ell)11}_{h(-S-1)} \psi_{(-S-1)}^m(x) + \gamma^{(\ell)11}_{h(S+1)} \psi_{(S+1)}^m(x) \right]$$

$$\leq 2^{\ell(m-1)} \left[\gamma^{(\ell)11}_{h(-S-1)} \max \psi_{(-S-1)}^m(x) + \gamma^{(\ell)11}_{h(S+1)} \max \psi_{(S+1)}^m(x) \right]$$

$$\stackrel{(2.14)}{=} 2^{\ell(m-1)} 2^{m/2} \frac{3\sqrt{3}}{\pi} \left[\gamma^{(\ell)11}_{h(-S-1)} + \gamma^{(\ell)11}_{h(S+1)} \right]. \tag{3.30}$$

$\square$

4. Fractional Derivatives of the Wavelet Basis

The simplest way to define the fractional derivative is based on the assumption that the noninteger derivative of the exponential function formally coincides with the derivative with integer order so that

$$\frac{d^v}{dx^v}e^{ax} = a^v e^{ax} \quad v \in \mathbb{Q}. \tag{4.1}$$

For negative values of v, this formula still holds true and it represents the integration.

It is known that the fractional derivative cannot be analytically computed except for some special functions, such as (see e.g., [16–18]) the following:

$$\frac{d^v}{dx^v}e^{ax} = a^v e^{ax}, \qquad \frac{d^v}{dx^v}\cos ax = a^v \cos\left(ax + \frac{\pi}{2}v\right),$$

$$\frac{d^v}{dx^v}\sin ax = a^v \sin\left(ax + \frac{\pi}{2}v\right). \tag{4.2}$$

From these, classical examples, we can see that the fractional derivative can be also interpreted as an interpolating function between derivatives with integer order, so that

$$\frac{d^v}{dx^v}f(x) = (1-v)f(x) + v\frac{d}{dx}f(x), \quad 0 \le v \le 1. \tag{4.3}$$

More in general, let $f(x)$ be a single-valued real function, then the Riemann-Liouville fractional order derivative is defined as [16]

$$\frac{d^v}{dx^v}f(x) \stackrel{\text{def}}{=} \frac{1}{\Gamma(1-v)}\frac{d}{dx}\int_0^x \frac{f(\xi)}{(x-\xi)^v}d\xi, \quad (0 < v < 1, \; x > 0), \tag{4.4}$$

$\Gamma(v)$ being the gamma function.

Other equivalent representations were given by Caputo (for a differentiable function)

$$\frac{d^v}{dx^v}f(x) \stackrel{\text{def}}{=} \frac{1}{\Gamma(1-v)}\int_0^x \frac{f'(\xi)}{(x-\xi)^v}d\xi, \quad 0 < v < 1, \tag{4.5}$$

and by Grünwald (see e.g., [17, 18])

$$\frac{d^v}{dx^v}f(x) = \lim_{N\to\infty}\frac{1}{\Gamma(-v)}\left(\frac{x}{N}\right)^{-v}\sum_{k=0}^{N-1}\frac{\Gamma(k-v)}{\Gamma(k+1)}f\left[\left(1-\frac{k}{N}\right)x\right], \quad (0 < v < 1, \; x > 0). \tag{4.6}$$

However, a drawback in the Grünwald definition, as well as in the Riemann-Liouville, is that it cannot be computed for negative values of the variable ($x < 0$).

4.1. Fractional Derivative of the Shannon Scaling Function

Let us assume that the fractional order derivative is defined by a linear interpolation of the integer order derivatives, so that the fractional derivative of the scaling-wavelet basis

$$\frac{d^{\ell+v}}{dx^{\ell+v}}\varphi_h(x), \qquad \frac{d^{\ell+v}}{dx^{\ell+v}}\psi_h^m(x). \tag{4.7}$$

with

$$0 \le v \le 1, \tag{4.8}$$

can be defined as

$$\frac{d^{\ell+v}}{dx^{\ell+v}}\varphi_h(x) \stackrel{\text{def}}{=} (1-v)\frac{d^{\ell}}{dx^{\ell}}\varphi_h(x) + v\frac{d^{\ell+1}}{dx^{\ell+1}}\varphi_h(x),$$

$$\frac{d^{\ell+v}}{dx^{\ell+v}}\psi_h^m(x) \stackrel{\text{def}}{=} (1-v)\frac{d^{\ell}}{dx^{\ell}}\psi_h^m(x) + v\frac{d^{\ell+1}}{dx^{\ell+1}}\psi_h^m(x). \tag{4.9}$$

Let us show the following.

Theorem 4.1. *The fractional derivative of the Shannon scaling functions is*

$$\frac{d^{\ell+v}}{dx^{\ell+v}}\varphi_h(x) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty}\lambda_{hk}^{(\ell+v)}\varphi_k(x) = \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty}\left[(1-v)\lambda_{hk}^{(\ell)} + v\lambda_{hk}^{(\ell+1)}\right]\varphi_k(x), & \ell > 0 \\ \displaystyle\sum_{k=-\infty}^{\infty}\left[(1-v)\delta_{hk} + v\lambda_{hk}^{(1)}\right]\varphi_k(x), & \ell = 0. \end{cases} \tag{4.10}$$

Proof. From (4.9), by taking into account (2.21), it is

$$\frac{d^{\ell+v}}{dx^{\ell+v}}\varphi_h(x) \stackrel{\text{def}}{=} (1-v)\sum_{k=-\infty}^{\infty}\lambda_{hk}^{(\ell)}\varphi_k(x) + v\sum_{k=-\infty}^{\infty}\lambda_{hk}^{(\ell+1)}\varphi_k(x)$$

$$\stackrel{(3.1)}{=} \sum_{k=-\infty}^{\infty}\left[(1-v)\lambda_{hk}^{(\ell)} + v\lambda_{hk}^{(\ell+1)}\right]\varphi_k(x), \tag{4.11}$$

and, when $\ell = 0$,

$$\frac{d^v}{dx^v}\varphi_h(x) = \sum_{k=-\infty}^{\infty}\left[(1-v)\delta_{hk} + v\lambda_{hk}^{(1)}\right]\varphi_k(x). \tag{4.12}$$

$\square$

With this definition, the fractional order derivative of the scaling functions is a commutative operator according to the following.

Theorem 4.2. *The operator* (4.10) *is a semigroup, so that*

$$\frac{d^\mu}{dx^\mu}\frac{d^\nu}{dx^\nu}\varphi_h(x) = \frac{d^\nu}{dx^\nu}\frac{d^\mu}{dx^\mu}\varphi_h(x) = \frac{d^{\mu+\nu}}{dx^{\mu+\nu}}\varphi_h(x). \tag{4.13}$$

Proof. Without loss of generality, let us show that

$$\frac{d^\mu}{dx^\mu}\frac{d^\nu}{dx^\nu}\varphi(x) = \frac{d^\nu}{dx^\nu}\frac{d^\mu}{dx^\mu}\varphi(x). \tag{4.14}$$

According to $(4.10)_2$, it is

$$\frac{d^\nu}{dx^\nu}\varphi_0(x) = \sum_{k=-\infty}^{\infty}\left[(1-\nu)\delta_{0k} + \nu\lambda_{0k}^{(1)}\right]\varphi_k(x), \tag{4.15}$$

that is

$$\frac{d^\nu}{dx^\nu}\varphi(x) = (1-\nu)\varphi(x) + \nu\left[\lambda_{00}^{(1)}\varphi(x) + \sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\lambda_{0k}^{(1)}\varphi_k(x)\right], \tag{4.16}$$

and, taking into account (2.26), by explicit computation we have

$$\frac{d^\nu}{dx^\nu}\varphi(x) = (1-\nu)\varphi(x) + \nu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\varphi_k(x). \tag{4.17}$$

By deriving, with respect to μ, we have

$$\frac{d^\mu}{dx^\mu}\frac{d^\nu}{dx^\nu}\varphi(x) = (1-\nu)\frac{d^\mu}{dx^\mu}\varphi(x) + \nu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\frac{d^\mu}{dx^\mu}\varphi_k(x)$$

$$\overset{(4.17)}{=} (1-\nu)\left[(1-\mu)\varphi(x) + \mu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\varphi_k(x)\right] \tag{4.18}$$

$$+ \nu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\frac{d^\mu}{dx^\mu}\varphi_k(x),$$

that is, according to (2.26),

$$
\begin{aligned}
\frac{d^\mu}{dx^\mu}\frac{d^\nu}{dx^\nu}\varphi(x) &= (1-\nu)\left[(1-\mu)\varphi(x) + \mu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\varphi_k(x)\right] \\
&\quad + \nu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\sum_{s=-\infty}^{\infty}\left[(1-\mu)\delta_{sk} + \mu\lambda_{sk}^{(1)}\right]\varphi_s(x) \\
&= (1-\nu)\left[(1-\mu)\varphi(x) + \mu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\varphi_k(x)\right] \\
&\quad + \nu(1-\mu)\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\varphi_k(x) + \nu\mu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\sum_{s=-\infty}^{\infty}\lambda_{sk}^{(1)}\varphi_s(x).
\end{aligned}
\tag{4.19}
$$

From where,

$$
\begin{aligned}
\frac{d^\mu}{dx^\mu}\frac{d^\nu}{dx^\nu}\varphi(x) &= (1-\nu)(1-\mu)\varphi(x) + \left[(1-\nu)\mu + \nu(1-\mu)\right]\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\varphi_k(x) \\
&\quad + \nu\mu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k}\sum_{s=-\infty}^{\infty}\lambda_{sk}^{(1)}\varphi_s(x),
\end{aligned}
\tag{4.20}
$$

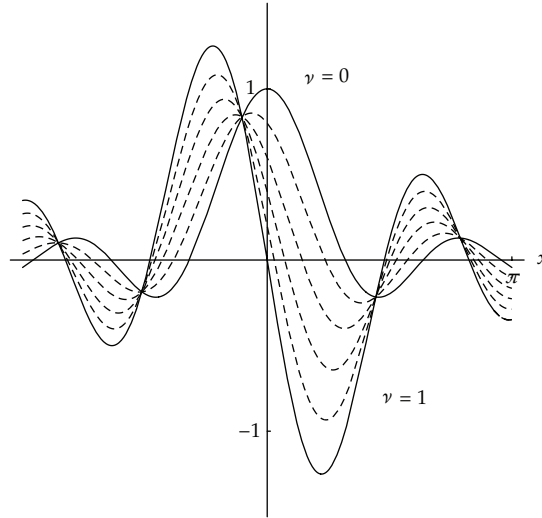the proof follows due to the symmetry of the change $\mu \to \nu$. $\square$

It can be easily seen that together with (4.17) also the following equations hold:

$$
\begin{aligned}
\frac{d^\nu}{dx^\nu}\varphi_1(x) &= (1-\nu)\varphi_1(x) + \nu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k-1}\varphi_k(x) \\
\frac{d^\nu}{dx^\nu}\varphi_{-1}(x) &= (1-\nu)\varphi_{-1}(x) + \nu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{1+k}\varphi_k(x),
\end{aligned}
\tag{4.21}
$$

and, in general,

$$
\frac{d^\nu}{dx^\nu}\varphi_h(x) = (1-\nu)\varphi_h(x) + \nu\sum_{\substack{k\neq 0 \\ k=-\infty}}^{\infty}\frac{(-1)^k}{k-h}\varphi_k(x).
\tag{4.22}
$$

Moreover, when $\mu + \nu = 1$, then we can see that the definition (2.26) reduces to the ordinary derivative, according to the following.

Figure 2: Fractional derivative of the scaling functions $(d^\nu/dx^\nu)\varphi(x)$ with upper limit $N = 4$ at different values of $\nu = 0, 1/5, 2/5, 3/5, 4/5, 1$.

Theorem 4.3. *When $\mu + \nu = 1$, then*

$$\frac{d^\mu}{dx^\mu}\frac{d^\nu}{dx^\nu}\varphi_h(x) = \frac{d^{\mu+\nu}}{dx^{\mu+\nu}}\varphi_h(x) = \frac{d}{dx}\varphi_h(x). \tag{4.23}$$

Proof. If we restrict to $\varphi(x)$, according to the definition (2.26), it is

$$\frac{d^\mu}{dx^\mu}\frac{d^\nu}{dx^\nu}\varphi(x) = \sum_{k=-\infty}^{\infty}\left[(1-(\mu+\nu))\delta_{0k}+(\mu+\nu)\lambda_{0k}^{(1)}\right]\varphi_k(x), \tag{4.24}$$

and since $(\mu + \nu) = 1$ we have

$$\frac{d^\mu}{dx^\mu}\frac{d^\nu}{dx^\nu}\varphi(x) = \frac{d}{dx}\varphi(x) = \sum_{k=-\infty}^{\infty}\lambda_{0k}^{(1)}\varphi_k(x), \tag{4.25}$$

$\square$

According to the definition (4.10), the fractional derivative is an interpolation between integer order derivative (see Figure 2).

4.2. Error of the Approximation of (4.10)

In the definition (4.10), the fractional derivative depends on a fixed bound N of the infinite series. In this section, it will be shown that the rate of convergence of the series, on the r.h.s of (4.10), is quite fast; already with low values of N, the approximation is quite good (Figure 3).

(a)   (b)
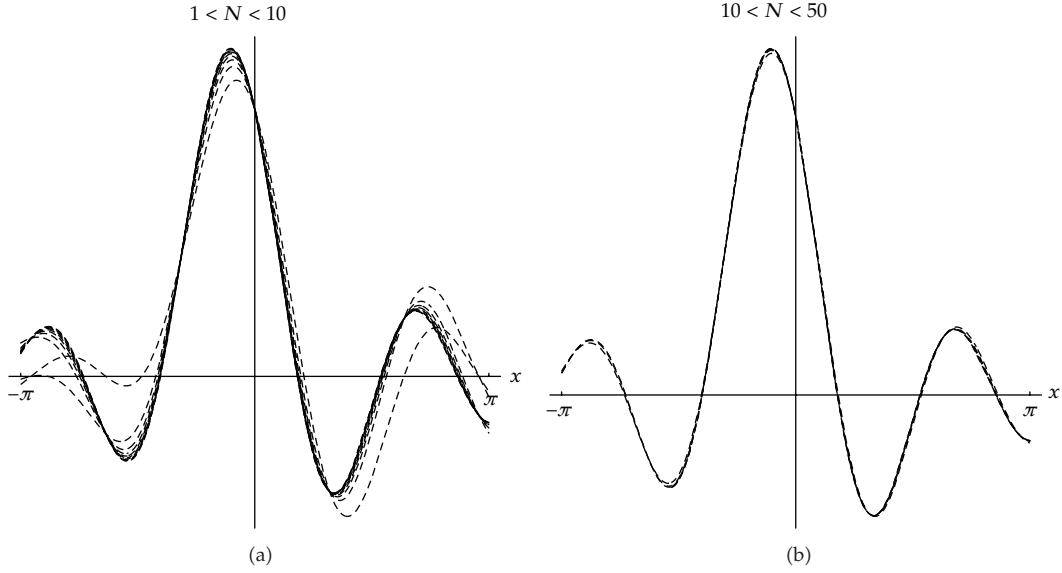
Figure 3: Fractional derivative of the scaling functions $(d^{3/10}/dx^{3/10})\varphi(x)$ with upper limit $N = 1, \ldots, 10$ (a) and $N = 10, \ldots, 50$ (b).

4.2.1. Rate of Convergence

If we compare the fractional derivative $(d^\nu/dx^\nu)\varphi_h(x)$ given by (4.10) with the Grünwald definition (4.6), we can see that the approximation by connection coefficients is good (see Figure 4), with a lower number of terms. Moreover, the definition based on connection coefficients can be extended also to negative values of the variable.

Since we have defined the fractional derivative on an infinite series $N \to \infty$, as well as the Grünwald formula, we can explicitly compute the error of the approximation as the difference between the approximated value at $N + 1$ and the corresponding value of the infinite series at N. For instance, with respect to (4.10), it is

$$\varepsilon_N^\nu = \max_{x \in \mathbb{R}} \left| \sum_{k=-(N+1)}^{N+1} \lambda_{hk}^{(\ell+\nu)} \varphi_k(x) - \sum_{k=-N}^{N} \lambda_{hk}^{(\ell+\nu)} \varphi_k(x) \right|, \tag{4.26}$$

while for the Grünwald formula (4.6) we have

$$\epsilon_N^\nu = \max_{x>0} \left| \frac{1}{\Gamma(-\nu)} \left(\frac{x}{N+1} \right)^{-\nu} \sum_{k=0}^{N} \frac{\Gamma(k-\nu)}{\Gamma(k+1)} f\left[\left(1 - \frac{k}{N+1} \right) x \right] \right.$$
$$\left. - \frac{1}{\Gamma(-\nu)} \left(\frac{x}{N} \right)^{-\nu} \sum_{k=0}^{N-1} \frac{\Gamma(k-\nu)}{\Gamma(k+1)} f\left[\left(1 - \frac{k}{N} \right) x \right] \right|, \tag{4.27}$$

Let us show the following.
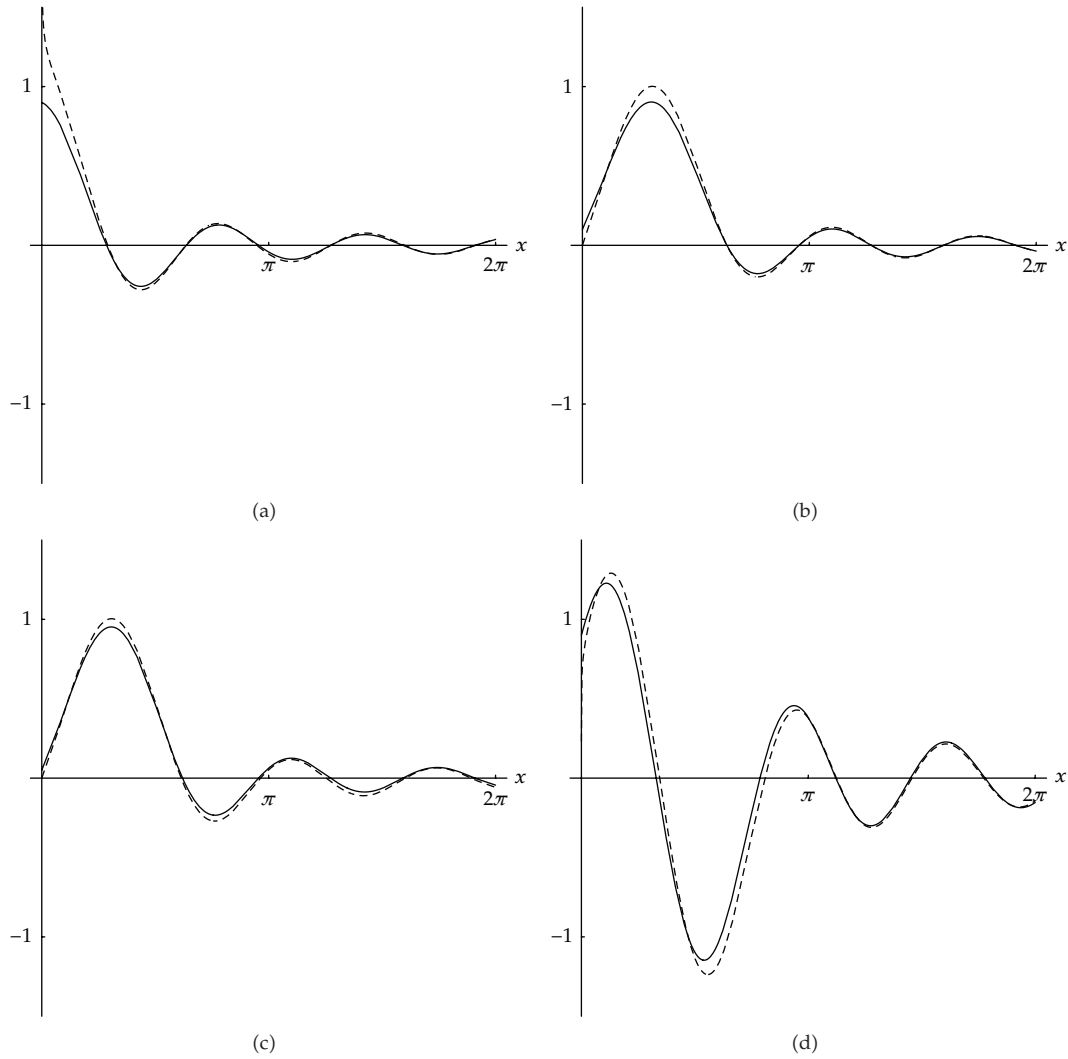
Figure 4: Fractional derivative of the scaling functions $(d^\nu/dx^\nu)\varphi_h(x)$ by Grünwald approximation (4.6) (shaded) and connection coefficients interpolation $(4.10)_2$ (plain): (a) $\nu = 1/10$, $h = 0$ with upper limit $N = 1$ (connection coefficients) and $N = 4$ (Grünwald); (b) $\nu = 1/10$, $h = 1$ with upper limit $N = 1$ (connection coefficients) and $N = 1$ (Grünwald); (c) $\nu = 1/20$, $h = 1$ with upper limit $N = 2$ (connection coefficients) and $N = 8$ (Grünwald); (d) $\nu = 9/10$, $h = 1$ with upper limit $N = 10$ (connection coefficients) and $N = 50$ (Grünwald).

Theorem 4.4. *For $\ell = 0$, the approximation error of $(4.10)_2$ is given by*

$$\varepsilon_N^\nu = 2\nu \left| \frac{(-1)^{N+1}h}{(N+1)^2 - h^2} \right|. \tag{4.28}$$

Proof. By taking into account (4.22), it is

$$
\sum_{k=-(N+1)}^{N+1} \lambda_{hk}^{(\ell+v)} \varphi_k(x) - \sum_{k=-N}^{N} \lambda_{hk}^{(\ell+v)} \varphi_k(x) = v \left[\frac{(-1)^{N+1}}{-(N+1)-h} \varphi_{-(N+1)}(x) \right.
$$

$$
\left. + \frac{(-1)^{N+1}}{(N+1)-h} \varphi_{(N+1)}(x) \right]
$$

$$
\overset{(2.13)}{<} v \left[\frac{(-1)^{N+1}}{-(N+1)-h} + \frac{(-1)^{N+1}}{(N+1)-h} \right]
$$

$$
= \frac{2v(-1)^{N+1}h}{(N+1)^2 - h^2}. \qquad \square
$$

(4.29)

Analogously, the following can be shown.

Theorem 4.5. *For $x > 0$, the approximation error of $(4.6)_2$ is given by*

$$
\epsilon_N^v = \frac{N^v}{\Gamma(-v)} \frac{\Gamma(N-v)}{\Gamma(N+1)}.
$$

(4.30)

Proof. At the integer $x = 1$, it is

$$
\frac{1}{\Gamma(-v)} \left(\frac{1}{N+1} \right)^{-v} \sum_{k=0}^{N} \frac{\Gamma(k-v)}{\Gamma(k+1)} f\left[\left(1 - \frac{k}{N+1} \right) \right] - \frac{1}{\Gamma(-v)} \left(\frac{1}{N} \right)^{-v} \sum_{k=0}^{N-1} \frac{\Gamma(k-v)}{\Gamma(k+1)} f\left[\left(1 - \frac{k}{N} \right) \right]
$$

$$
< \frac{1}{\Gamma(-v)} N^v \sum_{k=0}^{N} \frac{\Gamma(k-v)}{\Gamma(k+1)} f\left[\left(1 - \frac{k}{N+1} \right) \right] - \frac{1}{\Gamma(-v)} N^v \sum_{k=0}^{N-1} \frac{\Gamma(k-v)}{\Gamma(k+1)} f\left[\left(1 - \frac{k}{N} \right) \right]
$$

$$
\overset{(2.13)}{<} \frac{N^v}{\Gamma(-v)} \frac{\Gamma(N-v)}{\Gamma(N+1)}.
$$

(4.31)

$\square$

4.3. Fractional Derivative of the Shannon Wavelet

Analogously to (4.10), the following can be proved.

Theorem 4.6. *The fractional derivative of the Shannon wavelet functions is*

$$
\frac{d^{\ell+v}}{dx^{\ell+v}} \psi_h^m(x) \overset{\text{def}}{=} \sum_{k=-\infty}^{\infty} \gamma^{(\ell+v)_{hk}^{mm}} \psi_k^m(x)
$$

$$
= \begin{cases} 2^{\ell(m-1)} \left[\sum_{k=-\infty}^{\infty} (1-v) \gamma^{(\ell)_{hk}^{11}} + v \, 2^{m-1} \gamma^{(\ell+1)_{hk}^{11}} \right] \psi_k^m(x), & \ell > 0 \\ \left[\sum_{k=-\infty}^{\infty} (1-v) \delta_{hk} + v 2^{m-1} \gamma^{(1)_{hk}^{11}} \right] \psi_k^m(x), & \ell = 0. \end{cases}
$$

(4.32)

Proof. From (4.9), by taking into account (2.21)

$$
\begin{aligned}
\frac{d^{\ell+v}}{dx^{\ell+v}}\psi_h^m(x) &= (1-v)\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\gamma^{(\ell)_{hk}^{mn}}\psi_k^n(x) + v\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\gamma^{(\ell+1)_{hk}^{mn}}\psi_k^n(x) \\
&\overset{(3.5)}{=} \left[(1-v)\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\delta^{mn}2^{\ell(n-1)}\gamma^{(\ell)_{hk}^{11}} + v\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\delta^{mn}2^{(\ell+1)(n-1)}\gamma^{(\ell+1)_{hk}^{11}}\right]\psi_k^n(x) \\
&= \left[(1-v)\sum_{k=-\infty}^{\infty}2^{\ell(m-1)}\gamma^{(\ell)_{hk}^{11}} + v\sum_{k=-\infty}^{\infty}2^{(\ell+1)(m-1)}\gamma^{(\ell+1)_{hk}^{11}}\right]\psi_k^m(x) \\
&= 2^{\ell(m-1)}\left[\sum_{k=-\infty}^{\infty}(1-v)\gamma^{(\ell)_{hk}^{11}} + v\,2^{m-1}\gamma^{(\ell+1)_{hk}^{11}}\right]\psi_k^m(x).
\end{aligned}
\tag{4.33}
$$

$\square$

Analogously to the fractional derivative of the scaling function, also for the wavelet function, the fractional order derivatives are enveloped by the integer order derivatives (Figure 5).

4.4. Fractional Derivative of an $L_2(\mathbb{R})$ Function

Let $f(x) \in \mathcal{B} \subset L_2(\mathbb{R})$ be a function such that (2.17) holds, then its fractional derivative can be computed as

$$
\frac{d^v}{dx^v}f(x) = \sum_{h=-\infty}^{\infty}\alpha_h\frac{d^v}{dx^v}\varphi_h(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\beta_k^n\frac{d^v}{dx^v}\psi_k^n(x),
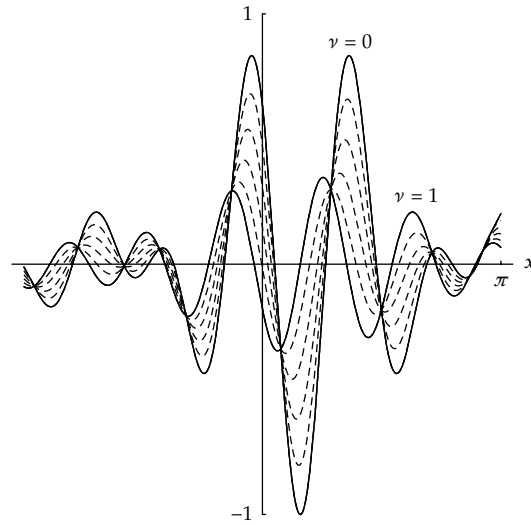\tag{4.34}
$$

where the fractional derivatives of the scaling functions $\varphi_h(x)$ and wavelets $\psi_k^n(x)$ are given by (4.10) and (4.32), respectively.

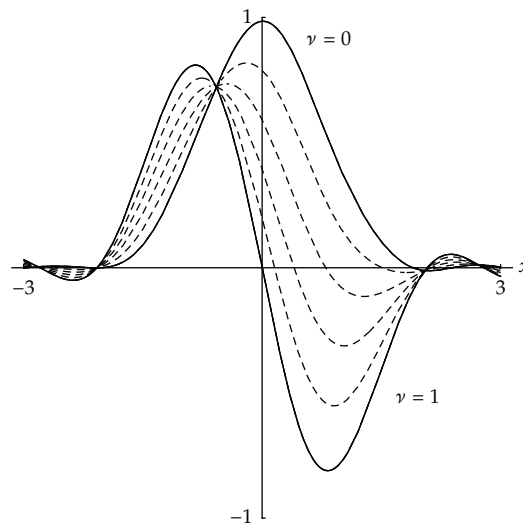For instance, a good approximation of $y = e^{-x^2}$ is (Figure 6)

$$
e^{-x^2} \cong 0.97\varphi(x) + 0.39\left[\varphi_{-1}(x) + \varphi_1(x)\right].
\tag{4.35}
$$

The fractional derivative is

$$
\frac{d^v}{dx^v}e^{-x^2} \cong 0.97\frac{d^v}{dx^v}\varphi(x) + 0.39\frac{d^v}{dx^v}\left[\varphi_{-1}(x) + \varphi_1(x)\right],
\tag{4.36}
$$

Figure 5: Fractional derivative of the wavelet functions $(d^v/dx^v)\psi_0^0(x)$ with upper limit $N = 4$ at different values of $v = 0, 1/5, 2/5, 3/5, 4/5, 1$.

Figure 6: Fractional derivative of the function $y = e^{-x^2}$ with upper limit $N = 4$ at different values of $v = 0, 1/5, 2/5, 3/5, 4/5, 1$.

so that by using (4.17) and (4.21) we have

$$
\frac{d^v}{dx^v} e^{-x^2} \cong 0.97 \left[(1-v)\varphi(x) + v \sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} \frac{(-1)^k}{k} \varphi_k(x) \right]
$$

$$
+ 0.39(1-v)\left[\varphi_{-1}(x) + \varphi_1(x)\right] \tag{4.37}
$$

$$
+ 0.39v \left[\sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} \frac{(-1)^{k+1}}{k} \varphi_k(x) + \sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} \frac{(-1)^k}{k-1} \varphi_k(x) \right],
$$

5. Conclusion

In this paper, fractional calculus has been revised by using Shannon wavelets. Fractional derivatives of the Shannon scaling/wavelet functions, based on connection coefficients, are explicitly computed and the approximation error is estimated. In the comparison with the classical Grünwald formula of fractional derivative, Shannon wavelets and connection coefficients make a better approximation and rate of convergence.

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