Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 504580, 13 pages
doi:10.1155/2012/504580

Research Article

Sampling in the Linear Canonical Transform Domain

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Received 28 March 2012; Accepted 20 May 2012

Academic Editor: Carlo Cattani

This paper investigates the interpolation formulae and the sampling theorem for bandpass signals in the linear canonical transform (LCT) domain. Firstly, one of the important relationships between the bandpass signals in the Fourier domain and the bandpass signals in the LCT domain is derived. Secondly, two interpolation formulae from uniformly sampled points at half of the sampling rate associated with the bandpass signals and their generalized Hilbert transform or the derivatives in the LCT domain are obtained. Thirdly, the interpolation formulae from nonuniform samples are investigated. The simulation results are also proposed to verify the correctness of the derived results.

1. Introduction

Sampling is one of the critical steps in the digital signal processing community because it serves as a bridge between the continuous physical signals and the discrete signals. The well-known sampling theory of Shannon and the sampling condition of Nyquist play an important role in modern sampling theory. Practical signal recovery from uniform or nonuniform samples at rates above or below the Nyquist in the traditional Fourier transform domain is well known [1, 2]. Nowadays, with the rapid developments of the classical sampling theories, more and more results of the sampling theories and their applications are proposed [3–6]; it is shown both in practice and in theory that all of the above-mentioned sampling formulae work well for the stationary signals in the Fourier transform domain.

However, most of the signals we meet in real applications (e.g., the Radar signals) are nonstationary and non-band-limited in the Fourier transform domain. We will obtain the suboptimal or wrong results if we still use the classical theories of Fourier domain.

It is therefore worthwhile looking for novel and useful tools to analyze and process these signals. Focusing on the problems of nonstationary signal processing, more and more signal processing tools are proposed, such as the wavelet transform [7], the fractional Fourier transform (FrFT) [8–10], the fractional calculus and fractional differential equations [11–13], and the linear canonical transform (LCT) [14].

The LCT [14], which was introduced in the 1970s with four parameters, has been proven to be one of the most powerful tools for nonstationary signal processing. The well-known signal processing operations, such as the Fourier transform (FT), the FrFT, the Fresnel transform, and the scaling operations, are all special cases of the LCT [14]. Therefore, understanding the LCT may help to gain more insight into its special cases and to carry the knowledge gained from one subject to others. The well-known concepts associated with the traditional Fourier transform, such as the uncertainty principles [15–17], the eigenfunctions [18], the convolution and product theorem [19], the Hilbert transform [20, 21], and the optical filtering [22, 23], are well studied and extended in the LCT domain.

Recently, with the rapid developments of the LCT in spectral analysis [24] and digital realization [25, 26], the expansion of the classical uniform or nonuniform sampling theorem for the band-limited or time-limited signal in the LCT domain has been obtained [27–31]. These sampling theorems establish the fact that a band-limited or time-limited continuous signal in the LCT domain can be completely reconstructed by a set of signal samples. However, there still exist many problems associated with the sampling of the signals in the LCT domain. One of them relates to the signal recovery methods and interpolation formulae for the bandpass signals from uniform or nonuniform samples in the LCT domain.

Focusing on the sampling of the bandpass signals in the LCT domain, this paper first derives a useful relationship between the bandpass signals in the Fourier domain and the bandpass signals in the LCT domain. Then, we discuss interpolation problems of bandpass signals below the Nyquist rate in the LCT domain. The experimental results are also proposed to show the correctness of the derived results.

2. Preliminaries

2.1. *The Linear Canonical Transform*

The linear canonical transform (LCT) with parameters $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of a signal $f(t)$ is defined as

$$Ł_f^A(u) = \begin{cases} \sqrt{\dfrac{1}{jb2\pi}} \displaystyle\int_{-\infty}^{+\infty} f(t) \exp\left\{ j\frac{1}{2}\left[\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right]\right\} dt, & b \neq 0, \\ \sqrt{d} \exp\left\{ j\dfrac{1}{2}cdu^2\right\} f(du), & b = 0, \end{cases} \tag{2.1}$$

where $\det(A) = ad - bc = 1$.

In this paper, we restrict ourselves to the class of the LCT with real parameters $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, in such cases the LCT operator is unitary in $L^2(R)$ and the inverse LCT is given by the

LCT having parameters $B = A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. In other words, we can reconstruct the original signal $f(t)$ from $L_f^A(t)$ via the following:

$$f(t) = \begin{cases} \sqrt{\dfrac{1}{-jb2\pi}} \displaystyle\int_{-\infty}^{+\infty} L_f^A(u) \exp\left\{ j\frac{1}{2}\left[-\frac{d}{b}u^2 + \frac{2}{b}tu - \frac{a}{b}t^2 \right] \right\} du, & b \neq 0, \\ \sqrt{a} \exp\left\{ -j\dfrac{1}{2}cdt^2 \right\} L_f^A(at), & b = 0. \end{cases} \tag{2.2}$$

It should be noted that, when $b = 0$, the LCT of a signal is essentially a chirp multiplication and it is of no particular interest to our objective in this work, so it will not be discussed in this paper, and, without loss of generality, we assume $b > 0$ in the following sections.

It can be verified [10] that the LCT with parameters$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ reduces to FrFT and to FT when $\theta = \pi/2$. The LCT also reduces to the Fresnel transform if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Multiplication by a Gaussian or chirp function is obtained with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. The scaling operator can be viewed as a special case of the LCT with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}$. For further details about the definition and properties of the LCT, readers can refer to [10].

2.2. The Hilbert Transform Associated with the LCT

The Hilbert transform (HT) is the quintessential singular integral operator as follows:

$$Hf(t) = \frac{1}{\pi} \int \frac{f(t)}{t - x} dx, \tag{2.3}$$

where $f(t)$ is a real signal and the integral is understood in the Cauchy principal value sense. The HT is turned to play an important role in signal processing and optical analysis. The analytic signal associated with signal $f(t)$ can be defined by the HT as follows:

$$z(t) = f(t) + jHf(t). \tag{2.4}$$

The analytic signal contains no negative frequency components in the Fourier domain, and it can be used in signal analysis applications and modulations. There exist many works on the analytic signals associated with the Fourier transform and fractional Fourier transform.

The generalized Hilbert transform associated with the LCT of a signal $f(t)$ is defined as [20, 21]

$$L_A Hf(t) = \frac{e^{-j(d/2b)t^2}}{\pi} \int_{-\infty}^{+\infty} \frac{f(x)e^{j(a/2b)x^2}}{t - x} dx, \tag{2.5}$$

and the generalized analytic part of signal $f(t)$ associated with the above definition is proposed as

$$L_A Zf(t) = f(t) + jL_A Hf(t). \tag{2.6}$$

It was shown in [20, 21] that the generalized Hilbert transform remains the similar properties as the classical Hilbert transform. That is to say, the generalized analytic signal contains no negative LCT frequency components in the LCT domain.

3. Interpolation of Bandpass Signals below the Nyquist Rate

The sampling formulae and the interpolation methods for the band-limited signals in the LCT domain from the uniform sampling sets are well studied and derived in [26–31]. The spectral analysis of uniformly or nonuniformly sampled signals was proposed recently in [24]. In this section, we will investigate the interpolation problems for bandpass signals in the LCT domain in detail.

Recall that a bandpass signal is a signal that contains no high and no low frequencies in the Fourier domain; similarly we can also define the bandpass signal of the LCT domain as a signal that contains no high and no low frequencies in the LCT domain. Mathematically, the bandpass signal is a signal whose LCT is supported in $[w_c-\sigma, w_c+\sigma]$, where w_c is the carrier frequency assumed to be at the center of the bandpass signal. The relationship between the bandpass signals in the LCT domain and the Fourier transform domain can be reflected by the following lemma.

Lemma 3.1. *Assume a signal $f(t)$ is bandpass to $[w_c-\sigma, w_c+\sigma]$ in the LCT domain with parameter $A=\{a,b,c,d\}$ and $b>0$. Let*

$$g(t)=\int_{-\infty}^{+\infty} L_f^A(u)e^{-j(d/2b)u^2+j(1/b)ut}du. \tag{3.1}$$

Then, $g(t)$ is a signal bandpass to $[(w_c-\sigma)(1/b),(w_c+\sigma)(1/b)]$ in the traditional Fourier transform domain.

Proof. Applying the Fourier transform to both sides of (3.1), it is easy to obtain

$$\begin{aligned}
F(g(t))(w)&=\int_{-\infty}^{+\infty} e^{-jwt}\int_{-\infty}^{+\infty} L_f^A(u)e^{-j(d/2b)u^2+j(1/b)ut}du\,dt\\
&=\int_{-\infty}^{+\infty} e^{-j(d/2b)u^2}L_f^A(u)du\int_{-\infty}^{+\infty} e^{j(1/b)ut-jwt}dt\\
&=\int_{-\infty}^{+\infty} e^{-j(d/2b)u^2}L_f^A(u)\frac{1}{2\pi}\delta\left(\frac{u}{b}-w\right)du\\
&=\frac{1}{2\pi}L_f^A(bw)e^{-j(db/2)w^2}.
\end{aligned}\tag{3.2}$$

Since the signal $f(t)$ is bandpass to $[w_c-\sigma, w_c+\sigma]$ in the LCT domain, that is, the LCT of the signal $f(t)$, $L_f^A(u)$, is supported in $[w_c-\sigma, w_c+\sigma]$. It is easy to show that $L_f^A(bw)$ will be supported in $[(w_c-\sigma)(1/b), (w_c+\sigma)(1/b)]$, from which we can obtain the final results. □

3.1. Interpolation of Bandpass Signals below the Nyquist Rate from Uniform Samples

Based on the facts shown in Lemma 3.1 and the definition of the bandpass signal in the Fourier domain, the following interesting theorem about the reconstruction of the original signal below the Nyquist rate from uniform sampling points can be obtained.

Theorem 3.2. *Let a signal $f(t)$ be bandpass to $[w_c-\sigma, w_c+\sigma]$ in the LCT domain; then $f(t)$ can be represented by the uniform samples of $f(t)$ and its generalized Hilbert transform $L_AHf(t)$ at half of the Nyquist rate as follows:*

$$\begin{aligned} f(t) = e^{-j(a/2b)t^2-j(w_c/b)t}\sum_{k=-\infty}^{+\infty}\Bigg\{ & f\left(\frac{2kb\pi}{\sigma}\right)e^{j((a/2b)(2kb\pi/\sigma)^2+2k\pi w_c/\sigma)}\cos\left(\frac{1}{2}\left(\frac{\sigma}{b}t-2k\pi\right)\right) \\ & - e^{j(d/2b)(2kb\pi/\sigma)^2}L_AH\left(f(v)e^{j(w_c/b)v}\right) \\ & \times\left(\frac{2kb\pi}{\sigma}\right)\sin\left(\frac{1}{2}\left(\frac{\sigma}{b}t-2k\pi\right)\right)\Bigg\}\operatorname{sinc}\left(\frac{1}{2\pi}\left(\frac{\sigma}{b}t-2k\pi\right)\right). \end{aligned} \tag{3.3}$$

Proof. Let

$$g(t) = \int_{-\infty}^{+\infty} L_f^A(u)e^{-j(d/2b)u^2+j(1/b)ut}du. \tag{3.4}$$

By Lemma 3.1, the signal $g(t)$ is bandpass to $[(w_c-\sigma)(1/b), (w_c+\sigma)(1/b)]$ in the traditional Fourier domain. Let $s(t) = g(t)e^{j(w_ct/b)}$. It is easy to show that signal $s(t)$ is a band-limited signal in the Fourier domain and the bandwidth is σ/b. Applying the well-known formulae associated with the traditional Hilbert transform in the Fourier domain [1, 2], we obtain

$$\begin{aligned} s(t) = \sum_{k=-\infty}^{+\infty}\Bigg\{ & s\left(\frac{2kb\pi}{\sigma}\right)\cos\left(\frac{1}{2}\left(\frac{\sigma}{b}t-2k\pi\right)\right) \\ & -Hs\left(\frac{2kb\pi}{\sigma}\right)\sin\left(\frac{1}{2}\left(\frac{\sigma}{b}t-2k\pi\right)\right)\Bigg\} \\ & \times\operatorname{sinc}\left(\frac{1}{2\pi}\left(\frac{\sigma}{b}t-2k\pi\right)\right), \end{aligned} \tag{3.5}$$

where $Hs(\cdot)$ is the Hilbert transform of the signal $s(t)$, that is,

$$Hs(t) = \left(\frac{1}{\pi}\right)\int_{-\infty}^{+\infty}\left(\frac{s(v)}{t} - v\right)dv. \tag{3.6}$$

From the definition of $f(t)$, $g(t)$, and $s(t)$, we obtain

$$s(t) = \sqrt{-j2b\pi}e^{j(a/2b)t^2}e^{j(w_c/b)t}f(t). \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain

$$\begin{aligned} Hs(t) &= \left(\frac{1}{\pi}\right)\int_{-\infty}^{+\infty}\left(\frac{s(v)}{t} - v\right)dv \\ &= \left(\frac{1}{\pi}\right)\int_{-\infty}^{+\infty}\left(\frac{\sqrt{-j2b\pi}e^{j(a/2b)v^2}e^{j(w_c/b)v}f(v)}{t} - v\right)dv \\ &= \sqrt{-jb2\pi}e^{j(d/2b)t^2}L_A H\left(f(v)e^{j(w_c/b)v}\right)(t), \end{aligned} \tag{3.8}$$

and substituting (3.8) into (3.5), we obtain

$$\begin{aligned} s(t) = \sqrt{-jb2\pi}\sum_{k=-\infty}^{+\infty}&\left\{f\left(\frac{2kb\pi}{\sigma}\right)e^{j((a/2b)(2kb\pi/\sigma)^2+2k\pi w_c/\sigma)}\cos\left(\frac{1}{2}\left(\frac{\sigma}{b}t - 2k\pi\right)\right)\right. \\ &\left. - e^{j(d/2b)(2kb\pi/\sigma)^2}L_A H\left(f(v)e^{j(w_c/b)v}\right)\left(\frac{2kb\pi}{\sigma}\right)\sin\left(\frac{1}{2}\left(\frac{\sigma}{b}t - 2k\pi\right)\right)\right\} \\ \times \operatorname{sinc}&\left(\frac{1}{2\pi}\left(\frac{\sigma}{b}t - 2k\pi\right)\right). \end{aligned} \tag{3.9}$$

The final result can be obtained by the relationship between $s(t)$ and $f(t)$ reflected in (3.7). □

Theorem 3.2 can be looked at as the generalization of the classical bandpass signal sampling theorem to the LCT domain. It showed us that if we use the information of the generalized Hilbert transform associated with the LCT, the sampling rate of the derived results in Theorem 3.2 is the half of the traditional sampling rate [26–31]. From the relationship between the LCT and the FrFT, we give the corresponding results in the FrFT domain without proof.

Corollary 3.3. *Let a signal $f(t)$ be bandpass to $[w_\alpha - \sigma, w_\alpha + \sigma]$ in the FrFT domain with parameter α; then $f(t)$ can be represented by the uniform samples of $f(t)$ and its generalized Hilbert transform $L_\alpha Hf(t)$ at half of the Nyquist rate as follows:*

$$\begin{aligned} f(t) = e^{-j(\cot\alpha/2)t^2 - j(w_\alpha/\sin\alpha)t} \sum_{k=-\infty}^{+\infty} \Bigg\{ & f\left(\frac{\sin\alpha 2k\pi}{\sigma}\right) e^{j((\cot\alpha/2)(\sin\alpha 2k\pi/\sigma)^2 + 2k\pi w_\alpha/\sigma)} \\ & \times \cos\left(\frac{1}{2}\left(\frac{\sigma}{\sin\alpha}t - 2k\pi\right)\right) \\ & - e^{j(\cot\alpha/2)(\sin\alpha 2k\pi/\sigma)^2} L_\alpha H\left(f(v) e^{j(w_\alpha/\sin\alpha)v}\right) \\ & \times \left(\frac{2kb\pi}{\sigma}\right) \sin\left(\frac{1}{2}\left(\frac{\sigma}{\sin\alpha}t - 2k\pi\right)\right)\Bigg\} \\ \times \operatorname{sinc}\left(\frac{1}{2\pi}\left(\frac{\sigma}{\sin\alpha}t - 2k\pi\right)\right). & \end{aligned} \tag{3.10}$$

Along with this idea, we can also use the information of the derivative of the signal to reduce the sampling rate in real applications. We obtain and derive Theorem 3.4.

Theorem 3.4. *Assume a signal $f(t)$ and its first derivative are continuous and $f(t)$ is bandpass to $[w_c - \sigma, w_c + \sigma]$ in the LCT domain; then the following sampling formula for $f(t)$ holds:*

$$\begin{aligned} f(t) = {} & e^{-j(a/2b)t^2 - j(w_c/b)t} \sum_{k=-\infty}^{+\infty} e^{j(a/2b)(2kb\pi/\sigma)^2 + j(w_c/b)(2kb\pi/\sigma)} \\ & \times \left\{ \left(1 + j\left(t - \frac{2kb\pi}{\sigma}\right)\left(\frac{at + w_c}{b}\right)\right) f\left(\frac{2kb\pi}{\sigma}\right) + \left(t - \frac{2kb\pi}{\sigma}\right) f'\left(\frac{2kb\pi}{\sigma}\right)\right\} \\ & \times \left\{\operatorname{sinc}\frac{1}{2}\left(\frac{\sigma t}{b\pi} - 2k\right)\right\}^2. \end{aligned} \tag{3.11}$$

Proof. Similar to the proof of Theorem 3.2, the signal $s(t)$ can be shown to be a band-limited signal in the Fourier domain and the bandwidth is $\sigma/\sin\alpha$. Applying the well-known formulae associated with the traditional sampling results in the Fourier domain [1, 2], we obtain

$$s(t) = \sum_{k=-\infty}^{+\infty} \left\{ s\left(\frac{2kb\pi}{\sigma}\right) + \left(t - \frac{2kb\pi}{\sigma}\right) s'\left(\frac{2kb\pi}{\sigma}\right)\right\} \times \left\{\operatorname{sinc}\frac{1}{2}\left(\frac{\sigma t}{b\pi} - 2k\right)\right\}^2. \tag{3.12}$$

From (3.7), the first derivative of signal $s(t)$ can be calculated as follows:

$$s'(t) = \sqrt{-j2b\pi}\left[e^{j(a/2b)t^2} e^{j(w_c/b)t}\left(j\left(\frac{at}{b} + \frac{w_c}{b}\right) f(t) + f'(t)\right)\right]. \tag{3.13}$$

Substituting (3.7) and (3.13) into (3.12), we obtain

$$s(t) = \sum_{k=-\infty}^{+\infty} e^{j(a/2b)(2kb\pi/\sigma)^2 + j(w_c/b)(2kb\pi/\sigma)} \times \left\{ \left(1 + j\left(t - \frac{2kb\pi}{\sigma}\right)\left(\frac{at}{b} + \frac{w_c}{b}\right)\right) f\left(\frac{2kb\pi}{\sigma}\right) + \left(t - \frac{2kb\pi}{\sigma}\right) f'\left(\frac{2kb\pi}{\sigma}\right) \right\} \times \left\{ \operatorname{sinc}\frac{1}{2}\left(\frac{\sigma t}{b\pi} - 2k\right)\right\}^2. \tag{3.14}$$

The final result can be obtained by the relationship between signals $f(t)$ and $s(t)$ in (3.7). □

The result of Theorem 3.4 reduces to the following result in the FrFT domain when the parameter $A = (\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$.

Corollary 3.5. *Assume a signal $f(t)$ and its first derivative are continuous and $f(t)$ is bandpass to $[w_c - \sigma, w_c + \sigma]$ in the FrFT domain with parameter α; then the following sampling formula for $f(t)$ holds:*

$$f(t) = e^{-j(\cot\alpha/2)t^2 - j(w_c/\sin\alpha)t} \sum_{k=-\infty}^{+\infty} e^{j(\cot\alpha/2)(\sin\alpha 2k\pi/\sigma)^2 + j(w_c/\sin\alpha)(2\sin\alpha k\pi/\sigma)} \times \left\{ \left(1 + j\left(t - \frac{2k\sin\alpha\pi}{\sigma}\right)\left(\frac{\cos\alpha t + w_c}{\sin\alpha}\right)\right) f\left(\frac{2\sin\alpha k\pi}{\sigma}\right) + \left(t - \frac{2\sin\alpha k\pi}{\sigma}\right) f'\left(\frac{2\sin\alpha k\pi}{\sigma}\right) \right\} \times \left\{ \operatorname{sinc}\frac{1}{2}\left(\frac{\sigma t}{\sin\alpha\pi} - 2k\right)\right\}^2. \tag{3.15}$$

3.2. Interpolation of Bandpass Signals from Nonuniform Samples

This subsection focuses on the interpolation of bandpass signals from nonuniform sampling points. It is well known that the bandpass signal can be interpolated from the nonuniform sampling points if the the nonuniform sampling set $\{t_k\}$ satisfies the following conditions [32]:

$$|t_k - t_i| \geq \alpha \quad \text{for } k \neq i,\ \alpha > 0, \qquad \left| t_k - \delta\frac{k}{B} \right| \leq L < \infty, \tag{3.16}$$

where $k \in Z$, B is the bandwidth of the bandpass signal in the traditional Fourier domain, and $0 < \delta \leq 1$. Similar to the derivation of the uniform sampling formulae, we derive the interpolation formula of bandpass signal in the LCT domain in following theorem.

Theorem 3.6. *Let a signal $f(t)$ be bandpass to $[w_c - \sigma, w_c + \sigma]$ in the LCT domain; then $f(t)$ can be interpolated by the nonuniform samples sets $\{t_k\}$ of $f(t)$ and its generalized Hilbert transform $L_A H f(t)$ as follows:*

$$f(t) = e^{-j(a/2b)t^2 - j(w_c/b)t} \sum_{k=-\infty}^{+\infty} \left\{ f(t_k) e^{j((a/2b)(t_k)^2 + 2k\pi w_c/\sigma)} \cos\left(\frac{1}{2}\left(\frac{\sigma}{b}t - 2k\pi\right)\right) - e^{j(d/2b)(t_k)^2} L_A H\left(f(v)e^{jw_c v}\right)(t_k) \times \sin\left(\frac{1}{2}\left(\frac{\sigma}{b}t - 2k\pi\right)\right) \right\} \times \operatorname{sinc}\left(\frac{1}{2\pi}\left(\frac{\sigma}{b}t - 2k\pi\right)\right), \tag{3.17}$$

where the nonuniform sampling set $\{t_k\}$ must satisfy conditions reflected in (3.15).

Proof. Let

$$g(t) = \int_{-\infty}^{+\infty} L_f^A(u) e^{-j(d/2b)u^2 + j(1/b)ut} du. \tag{3.18}$$

Then, the signal $g(t)$ is bandpass to $[(w_c - \sigma)(1/b), (w_c + \sigma)(1/b)]$ in the traditional Fourier domain. Let $s(t) = g(t)e^{j(w_c t/b)}$. It is easy to show that signal $s(t)$ is a band-limited signal in the Fourier domain and the bandwidth is σ/b. If the nonuniform sampling points t_k satisfy the requirements of (3.15), then the well-known nonuniform sampling formulae associated with the traditional Hilbert transform in the Fourier domain [1, 2] can be applied to $s(t)$. We obtain

$$s(t) = \sum_{k=-\infty}^{+\infty} \left\{ s(t_k) \cos\left(\frac{1}{2}\left(\frac{\sigma}{b}t - 2k\pi\right)\right) - Hs(t_k) \sin\left(\frac{1}{2}\left(\frac{\sigma}{b}t - 2k\pi\right)\right) \right\} \times \operatorname{sinc}\left(\frac{1}{2\pi}\left(\frac{\sigma}{b}t - 2k\pi\right)\right). \tag{3.19}$$

From the relationship between $f(t)$, $g(t)$, $s(t)$ and (3.7), we obtain

$$f(t) = \frac{1}{\sqrt{-j2b\pi}} e^{-j(a/2b)t^2} e^{-j(w_c/b)t} s(t). \tag{3.20}$$

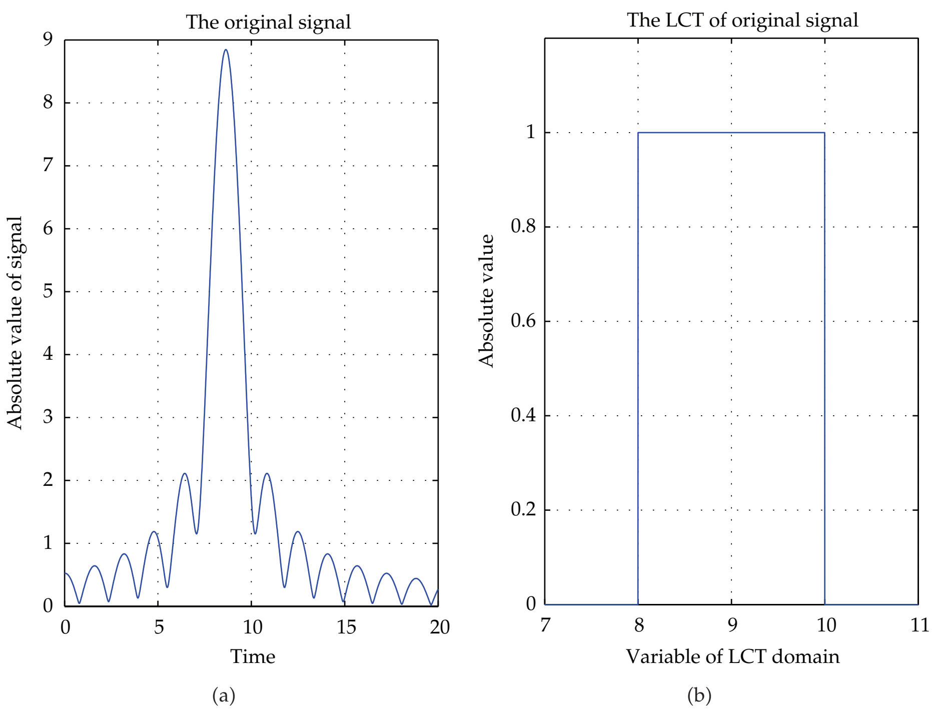

Figure 1: The LCT pair of a bandpass signal.

Substituting it into the right side of (3.17), we obtain

$$s(t) = \sqrt{-jb2\pi} \sum_{k=-\infty}^{+\infty} \left\{ f\left(\frac{2kb\pi}{\sigma}\right) e^{j((a/2b)(t_k)^2 + 2k\pi w_c/\sigma)} \cos\left(\frac{1}{2}\left(\frac{\sigma}{b}t - 2k\pi\right)\right) \right.$$
$$\left. - e^{j(d/2b)(t_k)^2} L_A H\left(f(v) e^{j(w_c/b)v}\right)(t_k) \times \sin\left(\frac{1}{2}\left(\frac{\sigma}{b}t - 2k\pi\right)\right) \right\} \tag{3.21}$$
$$\times \operatorname{sinc}\left(\frac{1}{2\pi}\left(\frac{\sigma}{b}t - 2k\pi\right)\right).$$

The final result can be obtained by substituting (3.20) into the right side of (3.19). □

4. Experimental Results

In order to show the correctness and effectiveness of the derived results in the LCT domain, we choose the sampling theory of passband signal associated with the generalized Hilbert transform as an example (Theorem 3.2) to perform the simulation in the LCT domain.

Figure 1 shows an LCT pair with $A = \{1, 2/\pi, 0, 1\}$ of a bandpass signal $f(t)$. The LCT of signal $f(t)$ is a real rectangular function confined to the interval $[8, 10]$, that is, the signal $f(t)$ is bandpass to $[9-1, 9+1]$ in the LCT domain.

Based on the derived results of Theorem 3.2, $w_c = 9$, $\sigma = 1$ and the sampling interval will be $2\pi b/\sigma = 4$. The real part of $f(t)$, the samples of $f(t)$, and the reconstructed signal

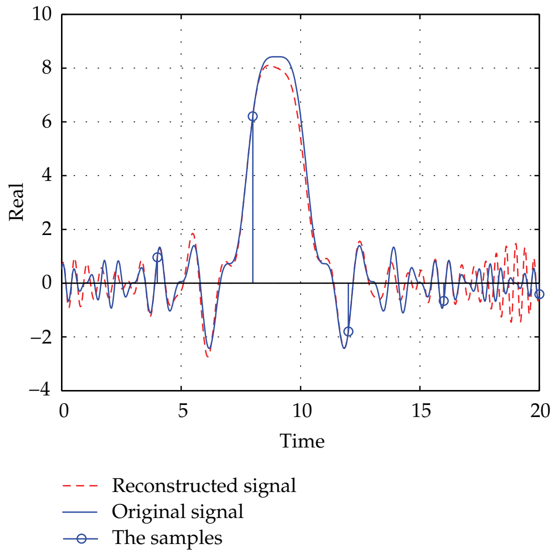

Figure 2: The real part of the original signal, the reconstructed signal, and the samples.

by samples of the original signal and its generalized Hilbert transform at half of the Nyquist rate of LCT domain is plotted in Figure 2, and the imaginary part of $f(t)$, the samples of $f(t)$, and the reconstructed signal by samples of the original signal and its generalized Hilbert transform at half of the Nyquist rate of LCT domain is plotted in Figure 3. It is noted from Figures 2 and 3 that the reconstructed signal denoted by dotted line overlaps almost exactly with the original signal denoted by continuous line.

It should be noted that the difference between the original and the reconstructed signal is due to the numerical error attributed to the fact that only a finite interval of the signal $f(t)$ and its generalized Hilbert transform is processed during numerical calculation.

5. Conclusion

Focusing on the interpolation problems for bandpass signals associated with the linear canonical transform, this paper investigates the interpolation formulae and the sampling theorem for bandpass signals in the LCT domain. Firstly, one of the important relationships between the bandpass signals in the Fourier domain and the bandpass signals in the LCT domain is derived. Secondly, two interpolation formulae from uniformly sampled points at half of the sampling rate associated with the bandpass signals and its generalized Hilbert transform or the derivatives in the LCT domain are obtained. Thirdly, the interpolation formulae from nonuniform samples are derived. The simulation results are proposed to verify the correctness of the derived results.

The proposed results can be seen as the expansion of the classical results in the Fourier domain to the LCT domain and can be used in the situations where the processed signals are not bandpass in the traditional Fourier domain but bandpass in the LCT domain. For example, in radar signal processing community, most of the echo signals that we encounter are non-band-limited signals in the Fourier domain; more suitable results may be obtained if

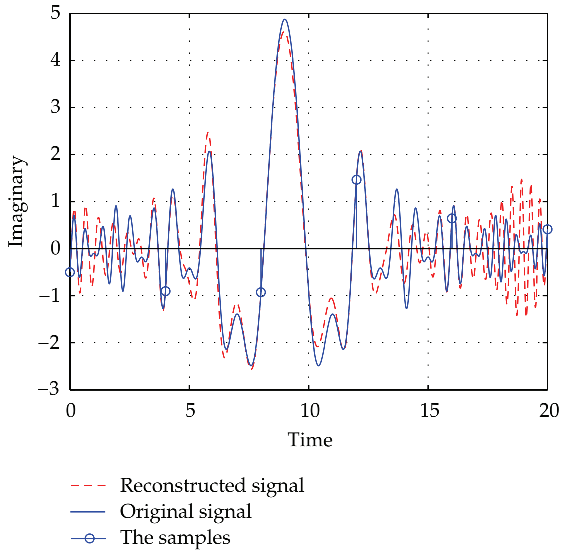

Figure 3: The imaginary part of the original signal, the reconstructed signal, and the samples.

we use LCT to analyze the signal, and the lower sampling rate may be found if we apply the results derived in this paper.

Acknowledgment

This work was partially supported by the National Natural Science Foundation of China (No. 60901058 and No. 61171195) and Beijing Natural Science Foundation (No. 1102029).

References

[1] A. I. Zayed, *Advances in Shannon's Sampling Theory*, CRC Press, Boca Raton, Fla, USA, 1993.

[2] F. Marvasti, *Nonuniform Sampling Theory and Practice*, Information Technology: Transmission, Processing and Storage, Kluwer Academic/Plenum Publishers, New York, NY, USA, 2001.

[3] M. Unser, "Sampling—50 years after Shannon," *Proceedings of the IEEE*, vol. 88, no. 4, pp. 569–587, 2000.

[4] M. Li, C. Cattani, and S. Y. Chen, "Viewing sea level by a one-dimensional random function with long memory," *Mathematical Problems in Engineering*, vol. 2011, Article ID 654284, 13 pages, 2011.

[5] E. G. Bakhoum and C. Toma, "Specific mathematical aspects of dynamics generated by coherence functions," *Mathematical Problems in Engineering*, vol. 2011, Article ID 436198, 10 pages, 2011.

[6] S. Y. Chen and Y. F. Li, "Determination of stripe edge blurring for depth sensing," *IEEE Sensors Journal*, vol. 11, no. 2, pp. 389–390, 2011.

[7] I. Daubechies, *Ten Lectures on Wavelets*, vol. 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 1992.

[8] B. Z. Li and T. Z. Xu, "Parseval relationship of samples in the fractional Fourier transform domain," *Journal of Applied Mathematics*, vol. 2012, Article ID 428142, 19 pages, 2012.

[9] Q. Zhang, "A note on operator sampling and fractional Fourier transform," *Mathematical Problems in Engineering*, vol. 2011, Article ID 303460, 9 pages, 2011.

[10] R. Tao, B. Deng, and Y. Wang, *Fractional Fourier Transform and Its Applications*, Tisinghua University Press, 2009.

[11] S. Y. Chen, H. Tong, and C. Cattani, “Markov models for image labeling,” *Mathematical Problems in Engineering*, vol. 2012, Article ID 814356, 18 pages, 2012.

[12] C. Cattani, “Fractional calculus and Shannon wavelets,” *Mathematical Problems in Engineering*, vol. 2012, Article ID 502812, 2012.

[13] S. C. Lim, C. H. Eab, K. H. Mak, M. Li, and S. Y. Chen, “Solving linear coupled fractional differential equations by direct operational method and some applications,” *Mathematical Problems in Engineering*, vol. 2012, Article ID 653939, 28 pages, 2012.

[14] K. B. Wolf, *Integral Transforms in Science and Engineering*, vol. 11 of *Mathematical Concepts and Methods in Science and Engineering*, Plenum Press, New York, NY, USA, 1979.

[15] K. K. Sharma and S. D. Joshi, “Uncertainty principle for real signals in the linear canonical transform domains,” *IEEE Transactions on Signal Processing*, vol. 56, no. 7 I, pp. 2677–2683, 2008.

[16] J. Zhao, R. Tao, Y. L. Li, and Y. Wang, “Uncertainty principles for linear canonical transform,” *IEEE Transactions on Signal Processing*, vol. 57, no. 7, pp. 2856–2858, 2009.

[17] A. Stern, “Uncertainty principles in linear canonical transform domains and some of their implications in optics,” *Journal of the Optical Society of America A*, vol. 25, no. 3, pp. 647–652, 2008.

[18] S. C. Pei and J. J. Ding, “Eigenfunctions of linear canonical transform,” *IEEE Transactions on Signal Processing*, vol. 50, no. 1, pp. 11–26, 2002.

[19] D. Wei, Q. Ran, Y. Li, J. Ma, and L. Tan, “A convolution and product theorem for the linear canonical transform,” *IEEE Signal Processing Letters*, vol. 16, no. 10, pp. 853–856, 2009.

[20] B. Z. Li, R. Tao, and Y. Wang, “Hilbert transform associated with the linear canonical transform,” *ACTA Armamentarii*, vol. 27, no. 5, pp. 827–830, 2006.

[21] Y. Fu and L. Li, “Generalized analytic signal associated with linear canonical transform,” *Optics Communications*, vol. 281, no. 6, pp. 1468–1472, 2008.

[22] K. K. Sharma and S. D. Joshi, “Signal separation using linear canonical and fractional Fourier transforms,” *Optics Communications*, vol. 265, no. 2, pp. 454–460, 2006.

[23] B. Barshan, M. A. Kutay, and H. M. Ozaktas, “Optimal filtering with linear canonical transformations,” *Optics Communications*, vol. 135, no. 1–3, pp. 32–36, 1997.

[24] B. Z. Li and T. Z. Xu, “Spectral analysis of sampled signals in the linear canonical transform domain,” *Mathematical Problems in Engineering*, vol. 2012, Article ID 536464, 11 pages, 2012.

[25] A. Koç, H. M. Ozaktas, C. Candan, and M. A. Kutay, “Digital computation of linear canonical transforms,” *IEEE Transactions on Signal Processing*, vol. 56, no. 6, pp. 2383–2394, 2008.

[26] J. J. Healy and J. T. Sheridan, “Sampling and discretization of the linear canonical transform,” *Signal Processing*, vol. 89, no. 4, pp. 641–648, 2009.

[27] R. Tao, B. Z. Li, Y. Wang, and G. K. Aggrey, “On sampling of band-limited signals associated with the linear canonical transform,” *IEEE Transactions on Signal Processing*, vol. 56, no. 11, pp. 5454–5464, 2008.

[28] B. Z. Li, R. Tao, and Y. Wang, “New sampling formulae related to linear canonical transform,” *Signal Processing*, vol. 87, no. 5, pp. 983–990, 2007.

[29] K. K. Sharma, “Approximate signal reconstruction using nonuniform samples in fractional fourier and linear canonical transform domains,” *IEEE Transactions on Signal Processing*, vol. 57, no. 11, pp. 4573–4578, 2009.

[30] D. Wei, Q. Ran, and Y. Li, “Multichannel sampling and reconstruction of bandlimited signals in the linear canonical transform domain,” *IET Signal Processing*, vol. 5, no. 8, pp. 717–727, 2011.

[31] Y. L. Liu, K. I. Kou, and I. T. Ho, “New sampling formulae for non-bandlimited signals associated with linear canonical transform and nonlinear Fourier atoms,” *Signal Processing*, vol. 90, no. 3, pp. 933–945, 2010.

[32] K. Yao and J. B. Thomas, “On some stability and interpolatory properties of nonuniform sampling expansions,” *IEEE Transactions on Circuit Theory*, vol. CT-14, pp. 404–408, 1967.


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