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Research Article

Two-Valued Control for a Second-Order Plant with Additive External Disturbance

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In this work a two-valued state feedback control for a plant of second order with known constant coefficients and an additive bounded disturbance is designed. In this controller the control signal can take only two possible values. The controller design is based on Lyapunov-like function method, achieving the convergence of the tracking error to a user-defined residual set. A boundedness condition for the user-defined reference signal is defined, which is necessary to allow out-put tracking. The developed scheme avoids large commutation rate of the control input. The controller design and stability analysis have important contributions with respect to closely related controllers based on the direct Lyapunov method, namely, (i) conditions to guarantee the expected convergence of the tracking error are established. These conditions are imposed on the reference signal and the extreme values of the control input. The stability analysis is developed by means of the Lyapunov-like function method and the Barbalat's Lemma and includes (ii) the bounded nature of the Lyapunov function, (iii) the monotonic convergence of the Lyapunov function to a residual set, and (iv) the asymptotic convergence of the tracking error to a residual set of user-defined size.

1. Introduction

The design of two-valued feedback controllers has attracted important research [1–8]. It is intended for two-valued actuators, which lead to a control input that takes only two possible preassigned finite values. Thus, the control input is discontinuous with respect to time [1, 9]. The two-valued actuators have the following advantages with respect to proportional ones [1, 10, 11]: they are simple, relatively cheaper, lead to fast output response, and overcomes the issue of actuator static gain. The electrical kiln [5], hydrogel valves [12], the compressor [6, 13], the satellite antenna [2, page 25] are some systems with two-value actuators.

A basic method for two-valued control is the relay feedback with fixed hysteresis band. The aim of the hysteresis is to avoid high commutation rate of the control input, what is known as input chattering [10, 13] and may lead to high power consumption and wear of mechanical components (cf. [5, 6]). The width of the hysteresis band determines the commutation rate of the control signal and the size of the residual set to which the output error converges. The width has to be chosen to obtain a trade-off between commutation rate and size of the residual set, because one fact improves at the expense of the other [6]. An improper designed relay feedback may lead to overshoot, large amplitude oscillation of the output, or large settling time (cf. [10, 13]). These can be overcomed by two-valued control based on the direct Lyapunov method, as shown in [1]. In that paper the authors consider a single input single output (SISO) linear plant in controllable form, with time varying coefficients and additive disturbances. They ensure the convergence of the tracking error to a residual set whose size is user defined. They show that large commutation rate is avoided if the initial values of the tracking error and its time derivative are such that the initial value of the Lyapunov function is inside a target region of user defined size.

In the case of input output stable plants with fixed preset values of the control input extremes, a two-valued input implies the following: (i) the output remains inside some bounded region, regardless the controller, (ii) a user defined trajectory with periodic behavior and large frequency or large magnitude cannot be reached by the system output. The suitability of the frequency and magnitude of the user defined trajectory depends on the extreme values of the control input, and the plant model coefficients, as it will show in this work. Nevertheless, as far as we know, there is not a condition that indicates such suitability, in the literature on finite-valued control based on the direct Lyapunov method. In this work such condition is established. The controller design and stability analysis have important contributions with respect to closely related controllers based on the direct Lyapunov method, namely, (i) conditions to guarantee the expected convergence of the tracking error are established. These conditions are imposed on the reference signal and the extreme values of the control input. The stability analysis is developed by means of the Lyapunov-like function method and the Barbalat's Lemma and includes (ii) the bounded nature of the Lyapunov function, (iii) the monotonic convergence of the Lyapunov function to a residual set, and (iv) the asymptotic convergence of the tracking error to a residual set of user defined size.

The work is only valid for a second-order plant with constant coefficients and an unknown external disturbance with known upper bounds. Upper bounds can be known from previous modelling tasks. A controller for the plant is developed, the controller is based on the direct Lyapunov method, ensuring the convergence of the tracking error to a residual set whose size are user defined. The work is organized as follows. In Section 2 the plant model and the goal of the control design is detailed. In Section 3 the condition that indicates if the desired output is suitable to achieve tracking is established. In addition, a preliminary rough control law, which achieves the convergence of the tracking error to a residual set whose size is user defined is formulated. Nevertheless, it may lead to large commutation rate of the control input. This will remedy in subsequent sections. In Section 4 the control law to avoid large commutation rate of the control input is formulated. In Sections 5 and 6 the boundedness of the closed loop signals and the convergence of the tracking error are proven. In Section 7 numerical simulation is presented. Finally, in Sections 8 and 9 discussion and conclusions are presented.

2. Problem Statement

The plant, the reference model, and the state goal of the control design are detailed in this section. Consider the following second-order plant (systems analyzed with this model can be found in [14–16])

$$\ddot{y} = -a_1\dot{y} - a_o y + bu + d, \tag{2.1}$$

$$u \in \{u_{\rm mn}, u_{\rm mx}\}, \quad u_{\rm mx} > u_{\rm mn}, \tag{2.2}$$

where $y(t) \in \mathbb{R}$ is the system output, $u(t) \in \mathbb{R}$ is the input, a_o, a_1, b are plant coefficients, being b the control gain, and d is an uncertain term that may result from modelling error or an external disturbance. Let us consider the following assumptions.

(Ai) The coefficients a_o, a_1, b are constant, known, and positive. (Aii) The signals y, $\dot{y}$ are available for measurement. (Aiii) The values of $u_{\rm mn}$, $u_{\rm mx}$ are constant, user defined, satisfy $u_{\rm mx} > u_{\rm mn}$, and are not restricted to positive values. (Aiv) The uncertainty d is time varying and satisfies either

$$\text{(i) } |d| \leq \mu_o, \mu_o > 0, \text{ where } \mu_{\rm o} \text{ is a known positive constant, or (ii) } d = \mu_o, \mu_o = 0. \tag{2.3}$$

Now, the control goal can be established. Let

$$e(t) = y(t) - y_d(t), \tag{2.4}$$

$$\ddot{y}_d = -a_{m1}\dot{y}_d - a_{mo}y_d + a_{mo}r, \tag{2.5}$$

$$\Omega_e = \{e \in \mathbb{R} : |e| \leq C_{\rm be}\}, \tag{2.6}$$

where a_{m1}, a_{mo} are positive constant of the user choice, the command signal r is bounded and user defined, y_d is the desired output or reference signal, $C_{\rm be}$ is a positive constant, user defined. It is important that the initial conditions $y_d(t_o)$, $\dot{y}_d(t_o)$ be chosen such that the initial value of the Lyapunov function be inside the target set, to avoid large commutation rate (see [1]). This requirement is included in the control scheme. The aim of the reference model (2.5) is to provide an adequate nature of y_d and $\dot{y}_d$, such that the control input can induce tracking while large input commutation rate is avoided.

The objective of the control design is to formulate a control law for the control input u, provided the plant model (2.1), subject to assumptions (Ai) to (Aiv), such that (Gi) the tracking error $e(t)$ converges asymptotically to the residual set Ω_e, (Gii) large commutation rate of the control input is avoided. Other goal of the control design is to develop a condition that indicates if a given desired output is suitable to achieve tracking.

3. A Preliminary Rough Control Law

In this section, a preliminary control law for the control input u, provided by the plant (2.1), subject to assumptions (Ai) to (Aiv) is developed. The main contribution is that the tracking error asymptotically converges to Ω_e, being Ω_e defined in (2.6). The Lyapunov-like function method, which is commonly used to design robust controllers for plants with continuous control inputs is used. A preliminary control law is developed. This control does not prevent large commutation rate but will be improved in subsequent sections.

Subtracting $\ddot{y}_d$ from both sides of (2.1)

$$\begin{aligned}\ddot{y} - \ddot{y}_d &= -a_1\dot{y} - a_o y + bu - \ddot{y}_d + d,\\ \ddot{y} - \ddot{y}_d &= -a_1(\dot{y} - \dot{y}_d) - a_o(y - y_d)\\ &\quad - a_1\dot{y}_d - a_o y_d + bu - \ddot{y}_d + d,\end{aligned} \tag{3.1}$$

since the first- and second-time derivatives of the tracking error are $\dot{e} = \dot{y} - \dot{y}_d$, $\ddot{e} = \ddot{y} - \ddot{y}_d$, it is obtained:

$$\begin{aligned}\ddot{e} &= -a_1\dot{e} - a_o e + bu - \ddot{y}_d - a_1\dot{y}_d - a_o y_d + d\\ &= -a_1\dot{e} - a_o e + bv,\end{aligned} \tag{3.2}$$

$$v = u - \frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} + \frac{d}{b}. \tag{3.3}$$

The term v is introduced with two objectives. The first one is for notational simplicity, and the second one is to simplify the design of a control law that overcomes the effect of y_d and its time derivatives. Equation (3.2) can be rewritten as

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -a_1x_2 - a_o x_1 + bv,\end{aligned} \tag{3.4}$$

$$x_1 = e, \qquad x_2 = \dot{e}. \tag{3.5}$$

Consider the following Lyapunov function:

$$V(x(t)) = \frac{1}{2a_o}S^2 + \frac{1}{2}x_1^2, \tag{3.6}$$

$$x(t) = \left[x_1(t) \quad x_2(t)\right]^{\top}, \tag{3.7}$$

$$S(x(t)) = a_1x_1 + x_2. \tag{3.8}$$

The time derivative of V along trajectory (3.4) is

$$\begin{aligned}\dot{V} &= \frac{1}{a_o}S\dot{S} + x_1\dot{x}_1\\ &= \frac{1}{a_o}S(a_1x_2 + \dot{x}_2) + x_1x_2\\ &= -a_1x_1^2 + \frac{b}{a_o}Sv\\ &= -c_1a_1x_1^2 - (1 - c_1)a_1x_1^2 + \frac{b}{a_o}Sv,\end{aligned} \tag{3.9}$$

where v is defined in (3.3), whereas c_1 is constant, user defined, and satisfies $c_1 \in [0,\ 1]$. Constant c_1 is introduced in (3.9) with the objective to change the commutation rate. A complete expression of the improved controller is given in (4.9) and it shows explicitly the usefulness of this constant. The above equation suggests that if the control law for u is properly defined, then

$$
\begin{aligned}
&-(1-c_1)a_1x_1^2 + \left(\frac{b}{a_o}\right)Sv \le 0 \\
&\Longrightarrow \dot{V} \le -c_1a_1x_1^2.
\end{aligned}
\tag{3.10}
$$

The condition $\dot{V} \le -c_1a_1x_1^2$ implies that the tracking error converges asymptotically to a small value. To find a rough control law that achieves this, the term Sv can be written as

$$
\begin{aligned}
Sv &= -S\left(-u + \frac{\ddot{y}_d + a_1\dot{y}_d + a_oy_d}{b}\right) + \frac{d}{b}S \\
&= -|S|\left(-u\,\mathrm{sgn}(S) + \frac{\ddot{y}_d + a_1\dot{y}_d + a_oy_d}{b}\,\mathrm{sgn}(S)\right) + \frac{d}{b}S.
\end{aligned}
\tag{3.11}
$$

From assumption (Aiv) it follows that $(d/b)S \le (\mu_o/b)|S|$. Substituting this into (3.11), it is obtained:

$$
\begin{aligned}
Sv &\le -|S|\left(-u\,\mathrm{sgn}(S) + \frac{\ddot{y}_d + a_1\dot{y}_d + a_oy_d}{b}\,\mathrm{sgn}(S)\right) + \frac{\mu_o}{b}|S|, \\
Sv &\le -|S|\left(-u\,\mathrm{sgn}(S) + \frac{\ddot{y}_d + a_1\dot{y}_d + a_oy_d}{b}\,\mathrm{sgn}(S) - \frac{\mu_o}{b}\right).
\end{aligned}
\tag{3.12}
$$

If the term in parenthesis is positive, then $Sv \le 0$. It can be achieved with the following rule:

$$
u = \begin{cases} \overline{u} - \delta\,\mathrm{sgn}(S) & \text{if } S \ne 0 \\ \text{does not change} & \text{if } S = 0, \end{cases}
\tag{3.13}
$$

where

$$
\overline{u} = \left(\frac{1}{2}\right)(u_{\mathrm{mn}} + u_{\mathrm{mx}}), \qquad \delta = \left(\frac{1}{2}\right)(u_{\mathrm{mx}} - u_{\mathrm{mn}}),
\tag{3.14}
$$

substituting (3.13) into (3.12), it is obtained

$$
Sv \le -|S|\left[\delta + \ \mathrm{sgn}(S)\left(\frac{\ddot{y}_d + a_1\dot{y}_d + a_oy_d}{b} - \overline{u}\right) - \frac{\mu_o}{b}\right],
\tag{3.15}
$$

the following property is needed.

Proposition 3.1. *If*

$$u_{\mathrm{mn}} + \frac{\mu_o}{b} \leq \frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} \leq u_{\mathrm{mx}} - \frac{\mu_o}{b} \quad \forall t \geq t_o, \tag{3.16}$$

then

$$\delta + \mathrm{sgn}(S)\left(\frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \overline{u}\right) - \frac{\mu_o}{b} \geq 0 \quad \forall t \geq t_o, \tag{3.17}$$

the proof is presented in Appendix A.

Remark 3.2. A given desired output y_d with excessive magnitude or excessive frequency would not fulfill (3.16). Nevertheless, since it is assumed that the coefficients a_o, a_1, b, u_{mn}, u_{mx} are constant, and the plant model (2.1) is usually predefined, condition (3.16) can be fulfilled by modifying the values of a_{mo}, a_{m1}, and r of the reference model (2.5). This can be carried out by means of simulation.

Remark 3.3. Condition (3.16) implies that the value of u_{mn} is low enough and the value of u_{mx} is high enough such that the control input can drive the output y towards the desired output y_{d}.

Substituting (3.17) in (3.15), it follows $Sv \leq 0$ for $S \neq 0$, and from (3.9) it follows that $\dot{V} \leq -a_1 x_1^2 \leq -c_1 a_1 x_1^2$. If $S = 0$, thus $Sv = 0$ and from (3.9) it follows that $\dot{V} \leq -c_1 a_1 x_1^2$. Therefore, $\dot{V} \leq -c_1 a_1 x_1^2$ if rule (3.13) is used. These results are summarized in the following Theorem.

Theorem 3.4. *Consider the plant model* (2.1) *subject to assumptions (Ai) to (Aiv), the Lyapunov function* $V(x(t))$ *defined in* (3.6) *and the signal S defined in* (3.8). *If condition* (3.16) *is fulfilled, and rule* (3.13) *is used, then*

$$\dot{V} \leq -c_1 a_1 x_1^2. \tag{3.18}$$

Therefore, the control law (3.13) implies the convergence of the tracking error x_1 to a small value.

Remark 3.5. The control rule (3.13) operates as follows. For $t = t_o$, if $S(x(t_o)) = 0$ the control input may take any on the values u_{mn}, u_{mx}. If $S(x(t_o)) \neq 0$, the control input is defined by $u = \overline{u} - \delta\ \mathrm{sgn}(S(x(t_o)))$ for $t = t_o$. *The* control input u retains its initial value until the signal $\mathrm{sgn}(S(x(t)))$ changes its value with respect to $\mathrm{sgn}(S(t_o))$. Then, u changes according to the rule $u = \overline{u} - \delta\,\mathrm{sgn}(S)$. The input u retains such value until the value of $\mathrm{sgn}(S)$ changes again, so that u changes according to $u = \overline{u} - \delta\,\mathrm{sgn}(S)$. This is repeated successively. If $S = 0$ in some instant time, the input u does not change its value.

Remark 3.6. According to [2, 7], the use of discontinuous control law may lead to (i) loss of trajectory unicity, (ii) sliding motion of trajectories along the discontinuity surface, what may imply chattering (see [2, pages 282-283]), and (iii) input chattering, which is an undesired large commutation rate component in the control input (see [2, page 292]). Large commutation rate may lead to high power consumption and wear of mechanical components (cf. [5, 6]). A rigorous design of a direct Lyapunov method should include the following tasks (cf. [7]): (i) ensure that trajectory unicity is preserved, (ii) develop the Filippov's construction for the case that sliding motion occurs, in order to avoid chattering. In the case under study, there may be sliding motion of the states x_1, x_2 along the surface $S = 0$. In [1] sliding motion for a closely related control scheme is illustrated. Thus, the control scheme (3.13) may lead to undesired large commutation rate in the input u when S takes on small values, so that goal (Gii) is not fulfilled. If large commutation rate is not a problem, sliding mode control could be used as an alternative approach [17–19].

In next section, the control law given by (3.13) is improved.

4. The Final Control Law

In previous section it was formulated a control law that achieves adequate convergence of the tracking error but leads to large commutation rate. In this section, the convergence of the quadratic function V to a small residual set and adequate initial values of the Lyapunov function V are considered and the large commutation rate is overcome.

In [1, 20] authors show that the convergence of Lyapunov function to a target manifold leads to the convergence of tracking error to a residual set of user defined size. From (3.6) it follows that if V converges to some small residual set of adequate size, then the tracking error e converges to the residual set Ω_e defined in (2.6).

Proposition 4.1. *Consider the function V defined in* (3.6), *the tracking error e defined in* (2.4) *and the set Ω_e defined in* (2.6). *Let*

$$\Omega_v = \{V(x(t)) \in \mathbb{R} : V(x(t)) \leq C_{\mathrm{bv}}\}, \tag{4.1}$$

$$C_{\mathrm{bv}} = \left(\frac{1}{2}\right) C_{be}^2 \max\left\{1, \frac{a_1^2}{a_o}\right\}. \tag{4.2}$$

The convergence of V and e are related as follows: If V converges to Ω_v, then e converges to Ω_e, where C_{be} is a positive constant defined by the user. The proof is presented in Appendix B.

Therefore, it is necessary to formulate a control law for u that ensures:

$$\dot{V} \leq -c_1 a_1 x_1^2 \quad \text{for } V \geq C_{\mathrm{bv}}. \tag{4.3}$$

so as to achieve the expected convergence of the tracking error. Indeed, if the above condition is ensured, then: (i) V converges asymptotically to Ω_v, where $\Omega_v = \{V(x) \in \mathbb{R} : V(x) \leq C_{\mathrm{bv}}\}$, (ii) if in addition V reaches Ω_v for some instant, V remains inside thereafter, (iii) the tracking error e converges asymptotically to Ω_e. Control scheme (3.13) can achieve condition (4.3) as mentioned in Theorem 3.4. does not involve the case $V < C_{\mathrm{bv}}$, then the control input u can take arbitrary values for $V < C_{\mathrm{bv}}$ without disrupting the convergence of e to Ω_e. As in [1], it is advisable to stop the commutation of u when $V < C_{\text{bv}}$, in order to avoid large commutation rate:

$$u = \begin{cases} \overline{u} - \delta \operatorname{sgn}(S) & \text{if } V \geq C_{\text{bv}}, S \neq 0, \\ \text{stops commutation} & \text{otherwise.} \end{cases} \tag{4.4}$$

This control rule implies that large commutation rate is prevented for $V \leq C_{\text{bv}}$, but not for $V > C_{\text{bv}}$. Indeed, there may be sliding motion of the trajectories along $S = 0$ if $V > C_{\text{bv}}$, as occurs in [1]. A possible remedy is to impose a boundary layer around $S = 0$ for $V > C_{\text{bv}}$. From (4.3) it follows that commutation can be stopped when the condition $\dot{V} \leq -c_1 a_1 x_1^2$ is satisfied for $V \geq C_{\text{bv}}$ under arbitrary values of the control input u. The restrictions over x_1 and x_2 ensuring the above condition will be determined at the following. Expression (3.9) can be rewritten as

$$\dot{V} \leq -c_1 a_1 x_1^2 - (1 - c_1) a_1 x_1^2 + \left(\frac{b}{a_o}\right) |S||v|. \tag{4.5}$$

The requirement (3.16) implies that $|v|$ is bounded by a constant, as the following proposition shows.

Proposition 4.2. *The signal V defined in* (3.3) *satisfies*

$$|v| \leq u_{\text{mx}} - u_{\text{mn}}. \tag{4.6}$$

The proof is presented in Appendix C.

Substituting (4.6) into (4.5) yields

$$\dot{V} \leq -c_1 a_1 x_1^2 - (1 - c_1) a_1 x_1^2 + \left(\frac{b}{a_o}\right) |S||u_{\text{mx}} - u_{\text{mn}}|. \tag{4.7}$$

Therefore,

$$\dot{V} \leq -c_1 a_1 x_1^2 \quad \text{for } V \geq C_{\text{bv}}, \quad \text{if } -(1 - c_1) a_1 x_1^2 + \left(\frac{b}{a_o}\right) |S||u_{\text{mx}} - u_{\text{mn}}| \leq 0, \tag{4.8}$$

where $-(1 - c_1) a_1 x_1^2 + (b/a_o)|S||u_{\text{mx}} - u_{\text{mn}}| \leq 0$ defines a boundary layer around $S = 0$ in the $x_1 - x_2$ state space, for $V > C_{\text{bv}}$. The control law can take advantage of the above expression. Equation (4.8) indicates that it is possible to turn off commutation of the control input u while obtaining $\dot{V} \leq -c_1 a_1 x_1^2$ for $V \geq C_{\text{bv}}$ if x_1, x_2 satisfy the condition therein. The control law for u is then formulated as follows: (i) u follows the rule $u = \overline{u} - \delta \operatorname{sgn}(S)$ for the case that $V \geq C_{\text{bv}}$

and condition in (4.8) is not fulfilled, (ii) u follows the rule $u = \overline{u} - \delta\,\mathrm{sgn}(S)$ for the case that $V = C_{\mathrm{bv}}$ and $S \neq 0$, and (iii) u stops commutation (stops cm) otherwise (ow). Equivalently:

$$u = \begin{cases} \overline{u} - \delta\,\mathrm{sgn}(S) & \text{if } V \geq C_{\mathrm{bv}} \text{ and } -(1-c_1)\; a_1 x_1^2 + \left(\dfrac{b}{a_o}\right)|S|(u_{\mathrm{mx}} - u_{\mathrm{mn}}) > 0, \\ \overline{u} - \delta\,\mathrm{sgn}(S) & \text{if } V = C_{\mathrm{bv}} \text{ and } S \neq 0, \\ \text{stops cm} & \text{otherwise,} \end{cases} \tag{4.9}$$

and for $t = t_o$:

(i) if $S(x(t_o)) = 0,\ u$ can take any of the values u_{mn} or u_{mx},

(ii) if $V(x(t_o)) < C_{\mathrm{bv}},\ u$ can take any on the values $u_{\mathrm{mn}},\ u_{\mathrm{mx}}$,

(iii) if $S(x(t_o)) \neq 0$ and $V(x(t_o)) = C_{\mathrm{bv}},\ u$ takes on the value $u = \overline{u} - \delta\,\mathrm{sgn}(S(x(t_o)))$,

(iv) if $V(x(t_o)) > C_{\mathrm{bv}}$ and $-(1-c_1)a_1 x_1^2 + \left(\dfrac{b}{a_o}\right)|S|(u_{\mathrm{mx}} - u_{\mathrm{mn}}) > 0$,

u takes on the value $u = \overline{u} - \delta\,\mathrm{sgn}(S(x(t_o)))$

(4.10)

the signals necessary for the computation of u are: $\overline{u}$ and δ (3.14), C_{bv} (4.2), S (3.8), e (3.5), y_d provided by (2.1), c_1 is a user defined positive constant.

Remark 4.3. The control law (4.9) operates as follows. For $t = t_o$, u follows (4.10). The control input u retains its initial value until some of the conditions in (4.9) is fulfilled. At that instant time, the control input u follows the rule $u = \overline{u} - \delta\ \mathrm{sgn}(S)$, inducing the decrease of V. The input u retains such value until some of the conditions in (4.9) are fulfilled again. This procedure is repeated in the same way. Condition (3.16) should be fulfilled.

Remark 4.4. Equation (4.9) indicates that the control signal u commutes as less as possible. When the commutation stops, the control signal keeps the value acquired during previous commutation mode. The commutation rate of the control input u does not reach excessive values, because if S becomes zero, then u stops commutation.

Remark 4.5. The constant C_{be} cannot be zero, as we explain at the following. From (4.9) it follows that the control input u commutes when $V = C_{\mathrm{bv}}$ and $S \neq 0$. A value $C_{\mathrm{be}} = 0$ would imply $C_{\mathrm{bv}} = 0$, as it follows from the definition (4.2). Therefore, the control input u would commute when $V = 0$ and $S \neq 0$. Such condition is not possible, according to definition (3.6). If C_{be} is overly small, then C_{bv} is also small, as it follows from the definition (4.2). Therefore, the time that V takes to reach $V = C_{\mathrm{bv}}$ *is* small, implying a larger commutation rate, according to (4.9).

The discussion and simulation examples shown in [1] indicate that if condition $V(x(t_o)) \leq C_{\mathrm{bv}}$ is fulfilled, then $V(x(t)) \leq C_{\mathrm{bv}}$ for all $t \geq t_o$ and large commutation rate is avoided. Indeed, from definition (3.6) it follows that the condition $V(x(t_o)) \leq C_{\mathrm{bv}}$ is fulfilled if $e(t_o)$, $\dot{e}(t_o)$ have adequate magnitude, or equivalently, if the distance between $y_d(t_o)$ and $y(t_o)$, and distance between $\dot{y}_d(t_o)$ and $\dot{y}(t_o)$ are adequate. Thus, the following control strategy is chosen:

$$u = \begin{cases} \overline{u} - \delta \operatorname{sgn}(S) & \text{if } V = C_{\text{bv}}, \neq 0, \\ \text{stops cm} & \text{otherwise.} \end{cases} \tag{4.11}$$

$y_d(t_o)$, $\dot{y}_d(t_o)$ are chosen such that $V(x(t_o)) \in \Omega_v$, $\Omega_v = \{V(x(t)) \in \mathbb{R} : V(x(t)) \leq C_{\text{bv}}\}$. For $t = t_o$, the signal control is computed as:

$$\begin{aligned} &\text{(i) if } V(x(t_o)) < C_{\text{bv}}, \quad u \text{ can take any on the values } u_{\text{mn}},\ u_{\text{mx}}, \\ &\text{(ii) if } S(x(t_o)) \neq 0, \quad V(x(t_o)) = C_{\text{bv}}, \quad u \text{ takes on the value } u = \overline{u} - \delta \operatorname{sgn}(S(x(t_o))), \\ &\text{(iii) if } S(x(t_o)) = 0, \quad V(x(t_o)) = C_{\text{bv}}, \quad u \text{ can take any on the values } u_{\text{mn}},\ u_{\text{mx}}, \end{aligned} \tag{4.12}$$

Remark 4.6. The control law (4.11) operates as follows. For $t = t_o$, u follows (4.12). The control input u retains its initial value until $V = C_{\text{bv}}$ *and* $S \neq 0$. At that instant time, the control input u follows the rule $u = \overline{u} - \delta \operatorname{sgn}(S)$, inducing the decrease of V. The input u retains such value until $V = C_{\text{bv}}$ and $S \neq 0$ is fulfilled again. This procedure is repeated in the same way. If $V = C_{\text{bv}}$ and $S = 0$, then u *does* not change. Condition (3.16) should be fulfilled. Notice that the constant c_1 is not necessary to formulate the control law (4.11).

Remark 4.7. Equation (4.11) indicates that the control signal u commutes as less as possible. When the commutation stops, the control signal keeps the value acquired during previous commutation mode. The commutation rate of the control input u does not reach excessive values, because if S becomes zero, then u stops commutation.

4.1. Implementation Issues

In experimental implementation could be difficult to detect the exact moment when $V = C_{bv}$, so that it could be difficult to use the control scheme (4.11). Then, with the aim to apply the control strategy to one system, it is possibility to use the control scheme given by (4.9) instead of (4.11), because (4.9) includes the case $V > C_{\text{bv}}$. Other possibility is to use a threshold δ_V in (4.11) as follows:

$$u = \begin{cases} \overline{u} - \delta \operatorname{sgn}(S) & \text{if } V \in [C_{\text{bv}} - \delta_V, C_{\text{bv}}], S \neq 0, \\ \text{stops cm} & \text{otherwise,} \end{cases} \tag{4.13}$$

where δ_V is a positive constant that satisfies $\delta_V < C_{\text{bv}}$, and $y_d(t_o)$ and $\dot{y}_d(t_o)$ are chosen such that $V(x(t_o)) \in \Omega_v$, $\Omega_v = \{V(x(t)) \in \mathbb{R} : V(x(t)) \leq C_{\text{bv}}\}$.

5. Boundedness Analysis

In this section we analyze the boundedness properties of the closed loop signals. As in [21], the notation $(\cdot) \in L_\infty$ is used. This notation indicates that $(\cdot)$ is bounded.

Theorem 5.1 (boundedness of the closed loop signals). *Consider the plant model* (2.1)*, subject to assumptions (Ai) to (Aiv), the tracking error e, the desired output y_d and the function S provided by* (2.4), (2.5), (3.8)*, respectively. If condition* (3.16) *is fulfilled and the controller* (4.9) *is applied, then the signals x_1, x_2, S remain bounded.*

Proof. From (4.8) it follows that

$$\dot{V} \leq -c_1 a_1 x_1^2 \quad \text{if } V > C_{\text{bv}}, \quad -(1-c_1) a_1 x_1^2 + \left(\frac{b}{a_o}\right) |S| |u_{\text{mx}} - u_{\text{mn}}| \leq 0. \tag{5.1}$$

From (4.9) it follows that

$$\begin{aligned} &u = \overline{u} - \delta \operatorname{sgn}(S), \\ &\text{if } V > C_{\text{bv}}, \quad -(1-c_1) a_1 x_1^2 + \left(\frac{b}{a_o}\right) |S| |u_{\text{mx}} - u_{\text{mn}}| > 0, \\ &\text{or } V = C_{\text{bv}}, \quad S \neq 0. \end{aligned} \tag{5.2}$$

Therefore, it follows from Theorem 3.4 that

$$\begin{aligned} &\dot{V} \leq -c_1 a_1 x_1^2, \\ &\text{if } V > C_{\text{bv}}, \quad -(1-c_1) a_1 x_1^2 + \left(\frac{b}{a_o}\right) |S| |u_{\text{mx}} - u_{\text{mn}}| > 0, \\ &\text{or } V = C_{\text{bv}}, \quad S \neq 0. \end{aligned} \tag{5.3}$$

and controller (4.9) is applied. From (4.7) it follows that

$$\begin{aligned} &\dot{V} \leq -c_1 a_1 x_1^2, \\ &\text{if } V > C_{\text{bv}}, \quad -(1-c_1) a_1 x_1^2 + \left(\frac{b}{a_o}\right) |S| |u_{\text{mx}} - u_{\text{mn}}| \leq 0, \\ &\textit{or } V = C_{\text{bv}}, \quad S = 0. \end{aligned} \tag{5.4}$$

From (5.3), (5.4) it follows that

$$\dot{V} \leq -c_1 a_1 x_1^2 \quad \text{if } V \geq C_{\text{bv}}. \tag{5.5}$$

Since the above expression is not valid for $V < C_{\mathrm{bv}}$, it does not lead to a straightforward proof of the boundedness and convergence of V. In order to show that the Lyapunov function V is bounded, a Lyapunov-like function $f_a = f_a(V)$ that satisfies

$$\begin{aligned}
&\text{(i)}\ f_a \geq 0 \quad \forall t \geq t_o,\\
&\text{(ii)}\ V \leq c_a + c_b f_a^{c_c}(V) \quad \forall t \geq t_o,\\
&\text{(iii)}\ \dot{f}_a \leq 0 \quad \forall t \geq t_o,
\end{aligned} \tag{5.6}$$

will be used, being c_a, c_b, c_c positive constants. If f_a satisfies the above three conditions, then $f_a \in L_\infty$ and consequently $V \in L_\infty$. One example of such function is

$$f_a(V) = \begin{cases} \left(\dfrac{1}{2}\right)(V - C_{\mathrm{bv}})^2 & \text{if } V \geq C_{\mathrm{bv}},\\ 0 & \text{otherwise.}\end{cases} \tag{5.7}$$

The reader is referenced to [2, page 309], [22, 23] for closely related functions. Its time derivative is:

$$\begin{gathered}
\dot{f}_a = \frac{\partial f_a}{\partial V}\dot{V},\\
\frac{\partial f_a}{\partial V} = \begin{cases} V - C_{\mathrm{bv}} & \text{if } V \geq C_{\mathrm{bv}},\\ 0 & \text{otherwise,}\end{cases}
\end{gathered} \tag{5.8}$$

$$\Longrightarrow \dot{f}_a = \begin{cases} (V - C_{\mathrm{bv}}) & \text{if } V \geq C_{\mathrm{bv}},\\ 0 & \text{otherwise,}\end{cases} \tag{5.9}$$

since $V - C_{\mathrm{bv}}$ is positive or zero for $V \geq C_{\mathrm{bv}}$, it can be multiplied by (5.12) without changing the order of the inequality:

$$(V - C_{\mathrm{bv}})\dot{V} \leq -a_1 x_1^2 (V - C_{\mathrm{bv}}) \quad \text{if } V \geq C_{\mathrm{bv}}, \tag{5.10}$$

substituting it into (5.9), it is obtained:

$$\begin{cases} \dot{f}_a \leq -a_1 x_1^2 (V - C_{\mathrm{bv}}) \leq 0 & \text{if } V \geq C_{\mathrm{bv}},\\ \dot{f}_a = 0 & \text{otherwise.}\end{cases} \tag{5.11}$$

Then, f_a satisfies properties (5.6), as discussed at the following. From the definition (5.7) it follows that $f_a \geq 0$, $V \leq C_{\mathrm{bv}} + \sqrt{2 f_a}$, so that properties (5.6)(i) and (5.6)(ii) are satisfied. From (5.11) it follows that property (5.6)(iii) is true, so that $f_a \in L_\infty$. Since (5.6)(ii) is true, then $V \in L_\infty$. From (3.6) it follows that $x_1 \in L_\infty$, $S \in L_\infty$, and from (3.8) it follows that $x_2 \in L_\infty$. This completes the proof. □

If the control law (4.11) is applied to the plant (2.1), and the initial value of the function V is located inside the target region, then V remains inside the target region thereafter, as is proven at the following.

Theorem 5.2 (boundedness of the Lyapunov function for $V \le V(x(t_o))$). *Consider the plant model* (2.1)*, subject to assumptions (Ai) to (Aiv), the tracking error e, the desired output y_d and the function S provided by* (2.4)*,* (2.5)*, and* (3.8)*, respectively. If condition* (3.16) *is fulfilled, the controller* (4.11) *is applied and $V(x(t_o)) \le C_{\mathrm{bv}}$, then $V(x(t)) \le C_{bv}$ for all $t \ge t_o$, $x_1 \in L_\infty$, $x_2 \in L_\infty$.*

Proof. The nature of $\dot{V}$ for $V = C_{\mathrm{bv}}$ must be examined considering each of the cases $S = 0$ and $S \neq 0$ separately. From Theorem 3.4 it follows that if the control law (3.13) is used, $S \neq 0$ and $V = C_{\mathrm{bv}}$, then $\dot{V} \le -a_1 x_1^2$. To show that this expression is also valid for the case when $S = 0$ and $V = C_{\mathrm{bv}}$, (3.9) is used. From (3.9) it follows that $\dot{V} \le -a_1 x_1^2$ if $S = 0$, regardless the value of V. Consequently, if the control law (4.11) is used, $S = 0$ and $V = C_{\mathrm{bv}}$, then $\dot{V} \le -a_1 x_1^2$. So far, it has been shown that if the control law (4.11) is used and $V = C_{\mathrm{bv}}$, then

$$\dot{V} \le -c_1 a_1 x_1^2, \tag{5.12}$$

regardless the value of S. This implies that if $V(x(t_1)) = C_{\mathrm{bv}}$ for any $t_1 \ge t_o$, then $V(x(t_1+\delta_t)) \le V(x(t_1))$ for a small value of δ_t. Consequently, $V(x(t_2)) \le V(x(t_1))$ for all $t_2 \ge t_1$. Moreover, if $V(x(t_o)) \le C_{\mathrm{bv}}$, then $V(x(t)) \le C_{\mathrm{bv}}$ for all $t \ge t_o$. From (3.6) it follows that $x_1 \in L_\infty$, $S \in L_\infty$, and from (3.8) it follows that $x_2 \in L_\infty$. This completes the proof. □

6. Convergence of the Tracking Error

In this section it is proven that if the controller (4.11) is applied to the plant model (2.1), the tracking error $e(t)$ converges to a residual set $\Omega_e = \{e \in \mathbb{R} : |e| \le C_{\mathrm{be}}\}$.

Equation (5.11) will be arranged into a single expression. Using

$$f_b = \begin{cases} V - C_{\mathrm{bv}} & \text{if } V \ge C_{\mathrm{bv}}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Equation (5.11) can be rewritten as

$$\begin{cases} \dot{f}_a \le -a_1 x_1^2 f_b \le 0 & \text{if } V \ge C_{\mathrm{bv}}, \\ \dot{f}_a = 0 = -a_1 x_1^2 f_b, & \text{otherwise} \end{cases} \Longrightarrow \dot{f}_a \le -a_1 x_1^2 f_b. \tag{6.2}$$

A difficulty is that df_b/dt is not continuous, as $\partial f_b/\partial V$ is discontinuous at $V = C_{\mathrm{bv}}$. Consequently, the Barbalat's Lemma can not be applied on f_b. One remedy is to express (6.2) in terms of a new function whose first derivative with respect to V is continuous. One instance of that function is

$$f_c = \begin{cases} \left(\sqrt{V} - \sqrt{C_{\mathrm{bv}}}\right)^2 & \text{if } V \ge C_{\mathrm{bv}}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

The reader is referenced to [2, page 309], [22, 24, 25] for closely related functions. To express (6.2) in terms of f_c, the following property is needed.

Proposition 6.1. *The function* (6.3) *satisfies*

$$f_b \geq f_c, \tag{6.4}$$

the proof is presented in Appendix D.

Substituting (6.4) into (6.2), it is obtained

$$\dot{f}_a \leq -a_1 x_1^2 f_b \leq -a_1 x_1^2 f_c \leq 0. \tag{6.5}$$

Arranging and integrating, as in [26, 27], it is obtained

$$\begin{gathered} a_1 \int_{to}^{t} x_1^2 f_c d\tau \leq f_a(V_o) - f_a(V), \\ V_o = V(x(t_o)), \\ f_a(V) + a_1 \int_{to}^{t} x_1^2 f_c d\tau \leq f_a(V_o), \end{gathered} \tag{6.6}$$

since $f_c \geq 0$, then $x_1^2 f_c \in L_1$. As in [21], $(\cdot) \in L_1$ is used to indicate that $\int_{t_o}^{t} |(\cdot)| d\tau$ is bounded.

Proposition 6.2. *The term* $x_1^2 f_c$ *satisfies:* $x_1^2 f_c \in L_\infty$, $d(x_1^2 f_c)/dt \in L_\infty$. *The proof is presented in Appendix E.*

In view of the above proposition and applying the Barbalat's Lemma (cf. [21, page 76], [26–28]), it is obtained

$$\begin{gathered} \lim_{t \to \infty} x_1^2 f_c = 0 \\ \Longrightarrow \lim_{t \to \infty} x_1^2 = 0 \quad \text{or} \lim_{t \to \infty} f_c = 0. \end{gathered} \tag{6.7}$$

If f_c is the one that converges towards zero, it follows from (6.3) that V converges to Ω_v, where $\Omega_v = \{V(x) \in \mathbb{R} : V(x) \leq C_{\text{bv}}\}$. From Proposition 4.1 it follows that e converges to Ω_e, where $\Omega_e = \{e \in \mathbb{R} : |e| \leq C_{\text{be}}\}$. The results can be summarized as follows.

Theorem 6.3 (convergence of the tracking error). *Consider the plant model* (2.1), *subject to assumptions (Ai) to (Aiv), the tracking error* e, *the desired output* y_d *and the function* S *provided by* (2.4), (2.5), *and* (3.8), *respectively. If condition* (3.16) *is fulfilled and the controller* (4.11) *is applied, then* e *converges to* Ω_e, *where* $\Omega_e = \{e \in \mathbb{R} : |e| \leq C_{be}\}$, *being* C_{be} *a positive user defined constant.*

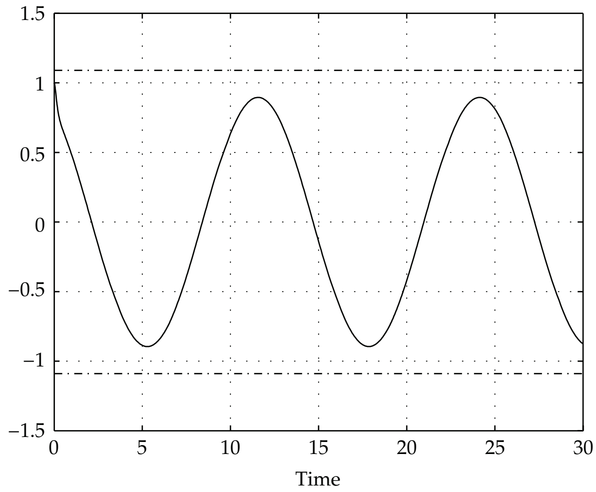

Figure 1: Example 1, top horizontal line: $u_{\mathrm{mx}}-\mu_o/b$, bottom horizontal line: $u_{\mathrm{mn}}+\mu_o/b$, thin line: $(1/b)(\ddot{y}_d+a_1\dot{y}_d+a_o y_d)$.

7. Simulation Example

The aim of the following example is to show that the controller (4.11) achieves the benefits mentioned in Theorem 6.3. To that end, assumptions (Ai) to (Aiv) of Section 2 and condition (3.16) have to fulfill. Considerations of Section 4 are taken into account. Consider the plant:

$$\begin{aligned}
&\ddot{y}=-\dot{y}-y+u+d, \quad y(t_o)=0.97,\ \dot{y}(t_o)=0,\\
&u\in\{-1.2,\ 1.2\},\\
&d=0.1\left(1+0.1\sin\left(\frac{2\pi}{3}t\right)\right),\\
&|d|\leq 0.11 \Longrightarrow \mu_o=0.11,
\end{aligned} \tag{7.1}$$

which is an example of the plant (2.1) with $a_1=1$, $a_o=1$, $b=1$, $u_{\mathrm{mn}}=-1.2$, $u_{\mathrm{mx}}=1.2$, $\mu_o=0.11$. Therefore, assumptions (Ai) to (Aiv) stated in Section 2 are fulfilled. Since $a_o>(1/4)a_1^2$, the linear part of the plant is underdamped. The aim is that y converges towards the value $r=\cos(0.5t)$ with an accuracy of 0.05. Thus, set $C_{\mathrm{be}}=0.05$ and y_d is defined by means of the second order system:

$$y_d=\frac{\lambda_r^2}{(p+\lambda_r)^2}r, \tag{7.2}$$

which is a special case of (2.1). $V(x(t_o))\leq C_{\mathrm{bv}}$ is fulfilled choosing $y_d(t_o)=1$ and $\dot{y}_d(t_o)=0$. By means of simulation it is possible to check that y_d, $\dot{y}_d$, $\ddot{y}_d$, and $\lambda_r=12$ satisfy condition (3.16), as shown in Figure 1. The chosen values of λ_r, $y_d(t_o)$, $\dot{y}_d(t_o)$ imply that $y_d\approx r$ for all $t\geq t_o$. Since $a_1=1$, the signal S defined in (3.8) is then $S=a_1e+\dot{e}=e+\dot{e}$. The closed loop behavior of y, e, u is shown in Figure 2, whereas the state plane is shown in

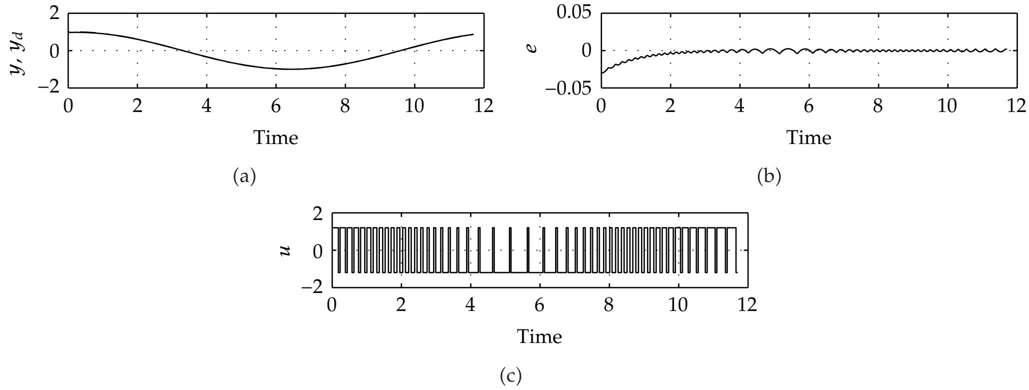

Figure 2: Example 1, (a) output y (continuous line) and desired output y_d (dashed line); (b) tracking error $e(t)$; (c) control input u.

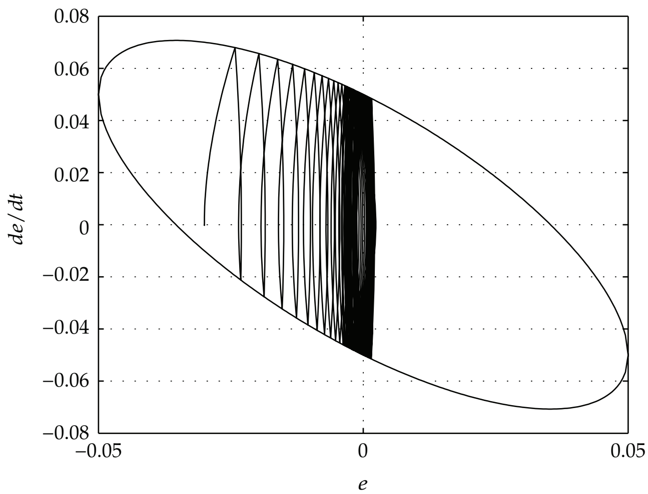

Figure 3: Example 1, the solid line represents $V = C_{\text{bv}}$.

Figure 3. The state plane confirms that $V(x(t_o)) \leq C_{\text{bv}}$, so that $V(x(t)) \leq C_{\text{bv}}$ for all $t \geq t_o$ and u only commutes when $V = C_{\text{bv}}$. The figure of the control input u and the state plane indicate that large commutation rate is avoided. This confirms the importance of choosing $y_d(t_o)$, $\dot{y}_d(t_o)$ such that $V(x(t_o)) \leq C_{\text{bv}}$.

8. Discussion

The discussion presented here is only valid for the plant model (2.1), which is a model second order with known constant coefficients and an additive disturbance. The time derivative of the function V defined in (3.6) results in a negative semidefinite term, so that it is necessary to use Lyapunov-like function method to prove the convergence of the tracking error. Hereafter, some conclusions based on both stability proofs and numerical simulations are stated. In the classical relay feedback method [10, 13] the commutation of the control input is function of the tracking error, but it does not take into account the time derivative of the tracking error.

The developed condition (3.16) indicates wether a given desired output y_d is suitable to achieve the convergence of the tracking error e to a residual set of user defined size, that is, Ω_e, for the plant (2.1). For any instance of the plant (2.1), one may modify the coefficients of the reference model (2.5) to satisfy condition (3.16), by means of trial and error. This procedure does not involve the output y, nor the input u, and is previous to the implementation of the controller.

It is important to set the initial values of the desired output and its time derivative such that $V(x(t_o)) \leq C_{\mathrm{bv}}$. This implies that the Lyapunov function V is always located inside the target region $\Omega_v = \{V \in \mathbb{R} : V \leq C_{\mathrm{bv}}\}$, so that it avoids convergence towards this region, and consequently large commutation rate of the control input are avoided.

The contribution of the scheme with respect to classical relay feedback control based on hysteresis is to ensure the convergence of the tracking error $e(t)$ to a residual set whose size is user defined. The main contribution with respect to closely related control based on the direct Lyapunov method is that the condition that the desired output y_d has to fulfill in order to achieve tracking is defined.

9. Conclusions

The controller achieves the convergence of the tracking error to a residual set of user defined size if the desired output satisfies the formulated condition. This condition uses the upper bound of the additive disturbance, as in a basic nonadaptive robust controller. It allows us to develop a rigorous proof of the tracking error convergence to a residual set that is user defined, by means of the Lyapunov-like function. If the initial values of the desired output and its time derivative are properly defined, the Lyapunov function is located inside a target region at initial time and thereafter. In this case, the control input only commutes when the Lyapunov function reaches the boundary.

Appendices

A. Proof of Proposition 3.1

From (3.16) it follows that

$$\overline{u} - \delta + \frac{\mu_o}{b} \leq \frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} \leq \overline{u} + \delta - \frac{\mu_o}{b}, \tag{A.1}$$

subtracting $\overline{u}$:

$$-\delta + \frac{\mu_o}{b} \leq \frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \overline{u} \leq \delta - \frac{\mu_o}{b}. \tag{A.2}$$

Thus,

$$\delta - \left(\frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \overline{u}\right) - \frac{\mu_o}{b} \geq 0,$$
$$\delta + \left(\frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \overline{u}\right) - \frac{\mu_o}{b} \geq 0. \tag{A.3}$$

Thus,

$$\delta + \operatorname{sgn}(S)\left(\frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \overline{u}\right) - \frac{\mu_o}{b} \geq 0. \tag{A.4}$$

This completes the proof.

B. Proof of Proposition 4.1

Proposition 4.1 will be proven by finding the value of a positive constant C_b^* such that:

$$\text{If } V \text{ converges to } \Omega_v, \quad \Omega_v = \{V \in \mathbb{R} : V \leq C_b^*\},$$
$$\text{then } e \text{ converges to } \Omega_e, \quad \Omega_e = \{e \in \mathbb{R} : |e| \leq C_{\text{be}}\}. \tag{B.1}$$

The value of C_b^* will be fond out by means of two different ways. On the one hand, it follows from (3.6) and the definition of Ω_v in (B.1) that

$$\frac{1}{2a_o}S^2 \leq V, \Longrightarrow |S| \leq \sqrt{2a_o V}$$
$$\Longrightarrow \text{if } V \text{ converges to } \Omega_v, \text{then } S \text{ converges to } \Omega_s,$$
$$\Omega_s = \left\{S : |S| \leq \sqrt{2a_o C_b^*}\right\}. \tag{B.2}$$

From the definitions (3.8), (3.5), S can be expressed as a linear function of e, $\dot{e}$: $S = a_1 e + \dot{e}$. Thus, e in terms of S is given by

$$e = \frac{1}{p + a_1}S. \tag{B.3}$$

In view of this equation and according to [2, page 279-280], and [29], it follows:

$$\text{If } S \text{ converges asymptotically to } \Omega_s,$$
$$\text{then } e \text{ converges asymptotically to } \Omega_e, \tag{B.4}$$

where

$$\Omega_s = \left\{S : |S| \leq \sqrt{2a_o C_b^*}\right\}, \qquad \Omega_e = \left\{e : |e| \leq \left(\frac{1}{a_1}\right)\sqrt{2a_o C_b^*}\right\}. \tag{B.5}$$

This and (B.2) imply:

$$\begin{gathered}\text{if } V \text{ converges to } \Omega_v, \text{then } e \text{ converges to } \Omega_e, \\ \Omega_e = \{e : |e| \le c_a\}, \qquad c_a = \left(\frac{1}{a_1}\right)\sqrt{2a_o C_b^*}.\end{gathered} \tag{B.6}$$

See [2, page 279-280] and [29] for closely related results. The value of C_b^* that leads to $c_a = C_{\text{be}}$ has been found, thus (B.1) is satisfied. This value is $C_b^* = a_1^2 C_{\text{be}}^2/(2a_o)$, which is a first value of C_b^*. On the other hand, it follows from (3.6) that

$$\left(\frac{1}{2}\right)e^2 \le V \Longrightarrow |e| \le \sqrt{2V}. \tag{B.7}$$

This and the definition of Ω_v in (B.1) imply

$$\begin{gathered}\text{if } V \text{ converges to } \Omega_v, \text{then } e \text{ converges to } \Omega_e, \\ \Omega_e = \{e : |e| \le c_c\}, \qquad c_c = \sqrt{2C_b^*}.\end{gathered} \tag{B.8}$$

Now, the value of C_b^* that leads to $c_c = C_{\text{be}}$ has been found out so that (B.1) is satisfied. This value is $C_b^* = (1/2)C_{\text{be}}^2$, which is a second value of C_b^*. Since both values of C_b^* are valid, then C_{bv} can be defined as the maximum:

$$C_{\text{bv}} = \max\left\{\frac{a_1^2 C_{\text{be}}^2}{2a_o}, \frac{C_{\text{be}}^2}{2}\right\} = \frac{C_{\text{be}}^2}{2}\max\left\{\frac{a_1^2}{a_o}, 1\right\} \tag{B.9}$$

This completes the proof.

C. Proof of Proposition 4.2

From (2.2) and (3.16) it follows that

$$-u \le -u_{\text{mn}}, \tag{C.1}$$

$$-u \ge -u_{\text{mx}}, \tag{C.2}$$

$$\frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} + \frac{\mu_o}{b} \le u_{\text{mx}}, \tag{C.3}$$

$$\frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \frac{\mu_o}{b} \ge u_{\text{mn}}. \tag{C.4}$$

From (C.3), (C.4), (2.3) it follows that

$$\frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \frac{d}{b} \leq \frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} + \frac{\mu_o}{b} \leq u_{\text{mx}}, \tag{C.5}$$

$$\frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \frac{d}{b} \geq \frac{\ddot{y}_d + a_1\dot{y}_d + a_o y_d}{b} - \frac{\mu_o}{b} \geq u_{\text{mn}}, \tag{C.6}$$

combining (C.1) with (C.5) and (C.2) with (C.6), and using definition (3.3), yields

$$\begin{aligned} -v &= \frac{\ddot{y} + a_1\dot{y}_d + a_o y_d}{b} - \frac{d}{b} - u \leq u_{\text{mx}} - u_{\text{mn}}, \\ -v &= \frac{\ddot{y} + a_1\dot{y}_d + a_o y_d}{b} - \frac{d}{b} - u \geq u_{\text{mn}} - u_{\text{mx}}, \end{aligned} \tag{C.7}$$

combining the above equations yields:

$$\begin{aligned} &-\left(u_{\text{mx}} - u_{\text{mn}}\right) \leq -v \leq u_{\text{mx}} - u_{\text{mn}}, \\ &\Longrightarrow |-v| \leq u_{\text{mx}} - u_{\text{mn}}, \\ &|v| \leq u_{\text{mx}} - u_{\text{mn}}. \end{aligned} \tag{C.8}$$

This completes the proof.

D. Proof of Proposition 6.1

The squared term of (6.3) in terms of V can be expressed as

$$\left(\sqrt{V} - \sqrt{C_{\text{bv}}}\right)^2 = V - 2\sqrt{C_{\text{bv}}}\sqrt{V} + C_{\text{bv}}. \tag{D.1}$$

If $V > C_{\text{bv}}$, then $\sqrt{V} > \sqrt{C_{\text{bv}}}$, and $-\sqrt{V} < -\sqrt{C_{\text{bv}}}$. Using this in (D.1), it is obtained

$$\begin{aligned} \left(\sqrt{V} - \sqrt{C_{\text{bv}}}\right)^2 &< V - 2C_{\text{bv}}^{1/2}C_{\text{bv}}^{1/2} + C_{\text{bv}} \quad \text{if } V > C_{\text{bv}}. \\ \left(\sqrt{V} - \sqrt{C_{\text{bv}}}\right)^2 &< V - C_{\text{bv}} \quad \text{if } V > C_{\text{bv}}. \end{aligned} \tag{D.2}$$

From this property and definitions (6.3), (6.1), it is obtained $f_c \leq f_b$. This completes the proof.

E. Proof of Proposition 6.2

From the definition (6.3) and $V \in L_\infty$, it follows $f_c \in L_\infty$. Since $x_1 \in L_\infty$, then $x_1^2 f_c \in L_\infty$. This completes the proof of the first part of Proposition 6.2. The time derivative of $x_1^2 f_c$ is

$$\frac{dx_1^2 f_c}{dt} = 2x_1\dot{x}_1 f_c + x_1^2 \dot{f}_c, \tag{E.1}$$

$$\dot{f}_c = \frac{\partial f_c}{\partial V}\dot{V}. \tag{E.2}$$

Equation (6.3) is used to compute $\partial f_c / \partial V$:

$$\frac{\partial f_c}{\partial V} = \begin{cases} \dfrac{\sqrt{V} - \sqrt{C_{\text{bv}}}}{\sqrt{V}} & \text{if } V \geq C_{\text{bv}}, \\ 0 & \text{otherwise.} \end{cases} \tag{E.3}$$

Since $V \in L_\infty$, then the above equation indicates that $(\partial f_c / \partial V) \in L_\infty$. Since $x_1 \in L_\infty$, $x_2 \in L_\infty$, $S \in L_\infty$, then (3.9) indicates that $\dot{V} \in L_\infty$. Thus, (E.2) indicates that $\dot{f}_c \in L_\infty$, whereas (E.1) indicates that $d(x_1^2 f_c)/dt \in L_\infty$. This completes the proof.

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