Research Article

Cluster Synchronization for Linearly Coupled Complex Networks with Identical and Nonidentical Nodes

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The cluster synchronization of linearly coupled complex networks with identical and nonidentical nodes is studied. Without assuming symmetry, we proved that these linearly coupled complex networks could achieve cluster synchronization under certain pinning control schemes. Sufficient conditions guaranteeing cluster synchronization for any initial values are derived by using Lyapunov function methods. Moreover, the adaptive feedback algorithms are proposed to adjust the control strength. Several numerical examples are given to illustrate our theoretical results.

1. Introduction

Recently, an increasing interest has been devoted to the study of complex networks. Among them, synchronization is the most interesting. In fact, synchronization of complex networks has been found to be a universal phenomenon in nature and has important potential applications in real-world dynamical systems. Great interests and attentions have been received for the synchronization of complex networks in many research and application fields including secure communication, seismology, parallel image processing, chemical reaction, and others [1–7].

There are many widely studied synchronization patterns, such as complete synchronization [8], lag synchronization [9], cluster synchronization [10], phase synchronization [11], and partial synchronization [12] Among them, the studies on cluster synchronization have received more and more attentions. The cluster synchronization requires that the coupled oscillators split into subgroups called clusters, such that the oscillators synchronize with one another in the same cluster, but there is no synchronization among different clusters,

which could describe the behaviors of the complex network in the real world. For instance, the metabolic, neural, or software networks containing some different function communities. Thus, it is a natural idea to consider the cluster synchronization of such community networks.

The complex network we considered in this paper is the linearly coupled ordinary differential equations (LCODEs). In fact, LCODEs are a large class of dynamical systems with continuous time and state, as well as discrete space, which are widely used to describe coupling oscillators. Nowadays, cluster synchronization of different kinds of LCODEs has been widely studied, and many results have already exist on the various properties of such problem. For instance, Ma et al. [13] constructed a novel coupling scheme with cooperative and competitive weight couplings that guarantees the cluster synchronization of any connected networks with identical nodes. The authors also derived a sufficient condition for the global stability of cluster synchronization. In [14], Wu et al. have discussed the problem of driving linearly coupled networks to an arbitrarily selected cluster synchronization pattern via pinning control. They introduced a single negative feedback controller for each cluster to pin the coupled system to the assigned cluster synchronization pattern for any initial values. However, in most cases, couplings between nodes are not the same even if the diffusive condition is still satisfied. Nodes usually receive instantaneous information as well as delayed information from their neighbors. Thus, it is a nature idea to study the synchronization of the networks with both delayed and nondelayed coupling. In this paper, we would investigate cluster synchronization of LCODEs with both delayed and nondelayed coupling under pinning control scheme. First, we assume all the node in LCODEs are identical. By utilizing the Lyapunov stability method, the global stability of cluster synchronization in networks is investigated, and several sufficient conditions for the global stability are given. Furthermore, we propose an adaptive feedback algorithms to adjust the control strength for LCODES.

Since in the real world, many networks contain some different function communities and the local dynamics between two function communities are different. For instances, in metabolic, neural, or software community networks, the individual nodes in each community can be viewed as the identical functional units, whereas the nodes in different communities are different since they have different functions [15]. One method to solve such problem is to consider the cluster synchronization of community networks with nonidentical nodes.There have been already some papers focused on sufficient conditions for the global stability of cluster synchronization of some related networks. A number of sufficient conditions were similarly obtained by Lu et al. [16] for the cluster synchronization of dynamical networks with community structure and nonidentical nodes in the presence of time delays by using a certain feedback control scheme. Lu et al. [17] studied the cluster synchronization of general bi-directed networks with nonidentical clusters and derived sufficient conditions for achieving local cluster synchronization of networks. The authors also discovered a relationship between the cluster synchronizability of a network and its intra-to-intercluster link ratio with the help of numerical examples. Recently, Wang et al. [18] considered the cluster synchronization of dynamical networks with community structure and nonidentical nodes and with identical local dynamics for all individual nodes in each community by using pinning control schemes. In this paper, we also investigate cluster synchronization of LCODEs with nonidentical nodes under pinning control scheme. By utilizing the Lyapunov-Krasovskii stability method, the global stability of cluster synchronization in networks is investigated, and several sufficient conditions for the global stability are given. Compared with [16–18], the complex network model we considered in this paper is more general. And moreover, the coupling matrices with and without time delay are asymmetric.

The paper is organized as follows. In Section 2, some necessary and useful definitions and lemmas are given. In Section 3, we study the global cluster synchronization of LCODEs with identical nodes and give a sufficient condition for it. Then, the adaptive feedback algorithms on control strength are proposed to achieve cluster synchronization in the complex network. In Section 4 cluster synchronization of LCODEs with nonidentical nodes is investigated and sufficient conditions are derived to achieve cluster synchronization. And the adaptive feedback algorithms on control strength are also proposed. In Section 5, numerical simulation are presented. We conclude the paper in Section 6.

2. Preliminaries

First, we introduce the mathematical definition of cluster synchronization.

Definition 2.1 (see [14]). Let $\{U_1, \ldots, U_m\}$ be a partition of the set $\{1, 2, \ldots, N\}$ into d nonempty subsets, that is, $U_l \neq \phi$ and $\bigcup_{l=1}^{m} = \{1, 2, \ldots, N\}$. For $i \in \{1, 2, \ldots, N\}$, let $\hat{i}$ denote the subscript of the subset in which the number i is, that is, $i \in U_{\hat{i}}$. A network with N identical oscillators is said to realize m-cluster synchronization with the partition $\{U_1, \ldots, U_m\}$ if, for any initial values, the state variables of the oscillators satisfy $\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0$ for $\hat{i} = \hat{j}$ and $\lim_{t \to \infty} \|x_i(t) - x_j(t)\| \neq 0$ for $\hat{i} \neq \hat{j}$.

For convenience of the statement to our main results, we now make some definitions for a class of functions and a class of matrices.

Definition 2.2 (see [19–21]). Suppose that $f(x, t)$ is a class of continuous functions $f : R^n \times [0, \infty) \to R^n$. Let $P = \text{diag}\{p_1, p_2, \ldots, p_n\}$ be a positive definite diagonal matrix and $\Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_n\}$ be a diagonal matrix. $f(x, t) \in \text{QUAD}(P, \Delta, \eta)$ if and only if

$$(x - y)^T P\big((f(x, t) - f(y, t)) - \Delta(x - y)\big) \leq -\eta(x - y)^T(x - y), \tag{2.1}$$

holds for some $\eta > 0$, where $x, y \in R^n$ and $t > 0$.

Definition 2.3 (see [14]). For $N \times N$ matrix

$$A = \begin{bmatrix} A_{11} & A_{11} & \ldots & A_{1d} \\ A_{21} & A_{22} & \ldots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \ldots & A_{dd} \end{bmatrix}, \tag{2.2}$$

where $A_{uv} \in R^{k_u \times k_v}$, $u, v = 1, 2, \ldots, d$. If each block A_{uv} is a zero-row-sum matrix, then we say that $A \in \mathbf{M_1(k)}$.

For an asymmetric matrix with zero-row-sums, we have the following.

Lemma 2.4 (see [19]). *Let Q and R be two symmetric matrices, and matrix S has suitable dimension. Then*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0, \tag{2.3}$$

if and only if both $R < 0$ and $Q - SR^{-1}S^T < 0$.

3. Cluster Synchronization of LCODEs with Identical Nodes

The complex network we considered in this section can be described as

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^{N} a_{ij} x_j(t) + c \sum_{j=1}^{N} b_{ij} x_j(t - \tau), \quad i = 1, 2, \ldots, N, \tag{3.1}$$

where N is the networks size, $x_i = (x_{1i}, x_{2i}, \ldots, x_{ni})^T \in R^n$ is the state vector of the ith oscillator, $f : R^n \times [0, \infty) \to R^n$ is a continues map, and $c > 0$ is the coupling strength. τ is the time delay. $A = (a_{ij}) \in R^{n \times n}$ and $B = (b_{ij}) \in R^{n \times n}$ are the coupling configuration matrix with zero-sum rows. It represents the topological structure of the network, in which $a_{ij} > 0$ if there is a connection from node j to node i ($i \neq j$) and is zero otherwise. B has the same properties. Here, A and B need not be symmetric since asymmetric topological structures are most common in the real world.

Without loss of generality, we set the partition of nodes $U_1 = \{1, 2, \ldots, k_1\}$, $U_2 = \{k_1 + 1, \ldots, k_1 + k_2\}$, ..., $U_m = \{k_1 + \cdots + k_{m-1} + 1, \ldots, k_1 + \cdots + k_{m-1} + k_m\}$, where $1 < m < N$, $1 < k_l < N$ and $\sum_{l=1}^{m} k_l = N$. Let $s_1(t), \ldots, s_m(t)$ be the m special solutions of the homogenous system $\dot{s}(t) = f(s(t), t)$, which satisfy $\lim_{t \to \infty} \|s_i(t) - s_j(t)\| \neq 0$ for $i \neq j$. By using similar pinning control method in [14], we let the controlled oscillator set J be $J = \{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_m\}$, which means we only put the control on the last node of each partition U_l. We use linear negative feedback controllers and the LCODES (3.1) become

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^{m} a_{ij} x_j(t) + c \sum_{j=1}^{N} b_{ij} x_j(t - \tau) - \xi_i(x_i(t) - s_i(t)), \quad i \in J,$$

$$\tag{3.2}$$

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^{m} a_{ij} x_j(t) + c \sum_{j=1}^{N} b_{ij} x_j(t - \tau), \quad i \notin J,$$

where $\xi_u > 0$ with $u \in J$ are the control strengths.

In this section, sufficient conditions are derived for the attainment of cluster synchronization for any initial value by control pinning, that is, by making

$$\lim_{t \to \infty} \|x_i(t) - s_{\bar{i}}(t)\| = 0 \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{3.3}$$

Theorem 3.1. *Suppose that the coupling matrices A and B in (3.2) satisfy $A \in \mathbf{M_1(k)}$ and $B \in \mathbf{M_1(k)}$. Let $P = \text{diag}\{p_1, p_2, \ldots, p_n\}$ be a positive definite diagonal matrix and $\Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_n\}$ be a diagonal matrix such that $f(x, t) \in QUAD(P, \Delta, \eta)$. Define $A^S = (A + A^T)/2$. For $k = 1, 2, \ldots, n$, if*

$$(\Lambda + p_k \delta_k) I + c p_k A^S - p_k \Xi + \frac{c^2}{4\Lambda} p_k^2 B B^T < 0, \tag{3.4}$$

where $\Xi = \text{diag}\{\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_N\}$ satisfies

$$\tilde{\xi}_i = \begin{cases} \xi_i, & i \in J \\ 0, & i \notin J. \end{cases} \tag{3.5}$$

Then, for any initial values, the solution $x_1(t)$, $x_2(t)$, ..., $x_N(t)$ of the system (3.1) under the control (3.2) can achieve cluster synchronization and satisfies (3.3).

Proof. We define $e_i(t) = x_i(t) - s_{\hat{i}}(t)$, where $i = 1, 2, \ldots, N$. Denote $e^k(t) = (e_{k1}(t), e_{k2}(t), \ldots, e_{kN}(t))^T$ for $k = 1, 2, \ldots n$. Since $A \in \mathbf{M_1(k)}$, which means that $\sum_{i \in U_l} a_{ij} = 0$ hold for all $i = 1, 2, \ldots, N$ and $l = 1, 2, \ldots, m$. It is readily seen that

$$\sum_{j=1}^{N} a_{ij} x_j = \sum_{l=1}^{m} \sum_{j \in U_l} a_{ij} x_j = \sum_{l=1}^{m} \sum_{j \in U_l} a_{ij} (x_j - s_l + s_l)$$

$$= \sum_{l=1}^{m} \sum_{j \in U_l} a_{ij} e_j + \sum_{l=1}^{m} \sum_{j \in U_l} a_{ij} s_l = \sum_{j=1}^{N} a_{ij} e_j. \tag{3.6}$$

And similarly, $\sum_{j=1}^{N} b_{ij} x_j = \sum_{j=1}^{N} b_{ij} e_j$. Thus, $e_i(t)$ for $i = 1, \ldots N$ satisfies the following differential equation:

$$\dot{e}_i(t) = f(x_i(t), t) - f(s_{\hat{i}}(t), t) + c \sum_{j=1}^{m} a_{ij} e_j(t) + c \sum_{j=1}^{N} b_{ij} e_j(t - \tau) - \xi_i e_i(t), \quad i \in J,$$

$$\dot{e}_i(t) = f(x_i(t), t) - f(s_{\hat{i}}(t), t) + c \sum_{j=1}^{m} a_{ij} e_j(t) + c \sum_{j=1}^{N} b_{ij} e_j(t - \tau), \quad i \notin J, \tag{3.7}$$

Choose a Lyapunov function as

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} \left(e_i(t)^T P e_i(t) + 2\Lambda \int_{t-\tau}^{t} e_i(\theta)^T e_i(\theta) d\theta \right). \tag{3.8}$$

Note that $f(x, t) \in \text{QUAD}(P, \Delta, \eta)$. We differentiate (3.8) along (3.7) and have

$$
\begin{aligned}
\frac{dV}{dt} &= \sum_{i=1}^{N} e_i(t)^T P \dot{e}_i(t) + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i^T(t-\tau) e_i(t-\tau) \\
&= \sum_{i=1}^{N} e_i(t)^T P \left[f(x_i(t), t) - f(s_{\hat{i}}(t), t) + c \sum_{j=1}^{N} a_{ij} e_j(t) + c \sum_{j=1}^{N} b_{ij} e_j(t-\tau) \right] \\
&\quad - \sum_{i \in J} e_i(t)^T P \xi_i e_i(t) + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i^T(t-\tau) e_i(t-\tau) \\
&\leq -\eta \sum_{i=1}^{N} e_i(t)^T e_i(t) + \sum_{i=1}^{N} e_i(t)^T P \Delta e_i(t) + c \sum_{i=1}^{N} e_i(t)^T P \sum_{j=1}^{N} a_{ij} e_j(t) \\
&\quad + c \sum_{i=1}^{N} e_i(t)^T P \sum_{j=1}^{N} b_{ij} e_j(t-\tau) - \sum_{i \in J} \xi_i e_i(t)^T P e_i(t) \\
&\quad + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i^T(t-\tau) e_i(t-\tau) \\
&= -\eta \sum_{i=1}^{N} e_i(t)^T e_i(t) + \sum_{k=1}^{n} p_k \delta_k e^k(t)^T e^k(t) + c \sum_{k=1}^{n} p_k e^k(t)^T A^S e^k(t) \\
&\quad - \sum_{k=1}^{n} p_k e^k(t)^T \Xi e^k(t) + c \sum_{k=1}^{n} p_k e^k(t)^T B e^k(t-\tau) \\
&\quad + \Lambda \sum_{k=1}^{n} e^k(t)^T e^k(t) - \Lambda \sum_{i=1}^{n} e^k(t-\tau)^T e^k(t-\tau) \\
&= -\eta \sum_{i=1}^{N} e_i(t)^T e_i(t) + \sum_{k=1}^{n} \left(e^k(t)^T, e^k(t-\tau)^T \right) \\
&\quad \times \begin{bmatrix} (\Lambda + p_k \delta_k) I + c p_k A^S - p_k \Xi & \dfrac{c}{2} p_k B \\ \dfrac{c}{2} p_k B^T & -\Lambda I \end{bmatrix} \left(e^k(t), e^k(t-\tau) \right),
\end{aligned}
\tag{3.9}
$$

where Ξ is defined in (3.5).

   If (3.4) is satisfied, from Lemma 2.4, it can be easily seen that $dV/dt < 0$, and thus (3.2) could achieve cluster synchronization. $\square$

   By using adaptive adjustments, we can find relatively small control strength to realize cluster synchronization. We regard the control strength of the network functions varying with time. Then, we could design the adaptive control strength. Then, we have the following result.

Theorem 3.2. *Suppose that the coupling matrix A and B in (3.2) satisfy $A \in \mathbf{M}_1(\mathbf{k})$ and $B \in \mathbf{M}_1(\mathbf{k})$. The control strength ξ_i in (3.2) is defined as*

$$\dot{\xi}_i(t) = \gamma_i e_i^T(t) P e_i(t) \quad \text{for} \ \ i \in J, \tag{3.10}$$

where γ_i is a positive constant. Let $P = \text{diag}\{p_1, p_2, \ldots, p_n\}$ be a positive definite diagonal matrix and $\Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_n\}$ be a diagonal matrix such that $f(x,t) \in QUAD(P, \Delta, \eta)$. For $k = 1, 2, \ldots, n$, if

$$(\Lambda + p_k \delta_k) I + c p_k A^S - p_k \Xi^* + \frac{c^2}{4\Lambda} p_k^2 B B^T < 0, \tag{3.11}$$

where $\Xi^* = \text{diag}\{\xi_1^*, \xi_2^*, \ldots, \xi_N^*\}$. ξ_i^* is a constant and satisfies $\xi_i^* = 0$ if $i \notin J$. Then, for any initial values, the solution $x_1(t)$, $x_2(t)$, $\ldots$, $x_N(t)$ of the system (3.1) under the control (3.2) can achieve cluster synchronization and satisfies (3.3).

Proof. Choose a Lyapunov function as

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} \left(e_i(t)^T P e_i(t) + 2\Lambda \int_{t-\tau}^{t} e_i(\theta)^T e_i(\theta) d\theta \right) + \sum_{i \in J} \frac{(\xi_i - \xi_i^*)^2}{2\gamma_i}. \tag{3.12}$$

Note that $f(x,t) \in QUAD(P, \Delta, \eta)$. We differentiate (3.12) along (3.7) and have

$$\begin{aligned}
\frac{dV}{dt} &= \sum_{i=1}^{N} e_i(t)^T P \dot{e}_i(t) + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i^T(t-\tau) e_i(t-\tau) \\
&\quad + \sum_{i \in J} (\xi_i - \xi_i^*) e_i^T(t) P e_i(t) \\
&= \sum_{i=1}^{N} e_i(t)^T P \left[f(x_i(t), t) - f(s_{\hat{i}}(t), t) + c \sum_{j=1}^{N} a_{ij} e_j(t) + c \sum_{j=1}^{N} b_{ij} e_j(t-\tau) \right] \\
&\quad - \sum_{i \in J} e_i(t)^T P \xi_i^* e_i(t) + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i^T(t-\tau) e_i(t-\tau).
\end{aligned} \tag{3.13}$$

By using similar calculations in (3.9), we have

$$\begin{aligned}
\frac{dV}{dt} &\le -\eta \sum_{i=1}^{N} e_i(t)^T e_i(t) + \sum_{k=1}^{n} \left(e^k(t)^T, e^k(t-\tau)^T \right) \\
&\quad \times \begin{bmatrix} (\Lambda + p_k \delta_k) I + c p_k A^S - p_k \Xi^* & \frac{c}{2} p_k B \\ \frac{c}{2} p_k B^T & -\Lambda I \end{bmatrix} \left(e^k(t), e^k(t-\tau) \right).
\end{aligned} \tag{3.14}$$

Noticing the inequalities in (3.11), we could obtain $dV(t)/dt < 0$. Thus, we get $x_i(t) \rightarrow s_{\hat{i}}(t)$, and $\dot{\xi}_i(t) \rightarrow 0$. By Cauchy convergence principle, $\xi_i(t)$ converges. $\qquad \square$

4. Cluster Synchronization of LCODEs with Nonidentical Nodes

The complex network considered in this section is

$$\dot{x}_i(t) = f_{\mu_i}(x_i(t), t) + c \sum_{j=1}^{N} a_{ij} x_j(t) + c \sum_{j=1}^{N} b_{ij} x_j(t - \tau), \quad i = 1, 2, \ldots, N, \tag{4.1}$$

where $x_i = (x_{1i}, x_{2i}, \ldots, x_{ni})^T \in R^n$ are the state variables of node i. The complex network (4.1) has N nodes and m communities with $N > m \geq 2$. If node i belongs to the jth community, then we let $\mu_i = j$. We denote by U_i the set of all nodes in the ith community and let $\tilde{U}_i^A$, which is the subset of U_i, be the index set of all nodes in the ith community having direct connections to other communities in A. And by the similar way, we can define $\tilde{U}_i^B$. The function $f_{\mu_i}(\cdot)$ describes the local dynamics of nodes in the μ_ith community, which is differentiable and capable of performing abundant dynamical behaviors. For any pair of indices i and j, if $\mu_i \neq \mu_j$, which means that node i and node j belong to different communities, then $f_{\mu_i} \neq f_{\mu_j}$. The constant $c > 0$ denotes the coupling strength. τ is the time delay in couplings. For $A = (a_{ij}) \in R^{n \times n}$ and $B = (b_{ij}) \in R^{n \times n}$ are the coupling configuration matrices with zero-sum rows, which represent the topological structure of the network. Take A for an example and B has the same properties. $a_{ij} > 0$ if there is a connection and is zero otherwise. Also, A and B need not be symmetric.

Let $u_i(t) \in R^n$, $i = 1, \ldots, N$, be the control inputs, then the controlled dynamical network with respect to (4.1) can be described by

$$\dot{x}_i(t) = f_{\mu_i}(x_i(t), t) + c \sum_{j=1}^{N} a_{ij} x_j(t) + c \sum_{j=1}^{N} b_{ij} x_j(t - \tau) + u_i(t), \quad i = 1, 2, \ldots, N. \tag{4.2}$$

We define the error variables by $e_i(t) = x_i(t) - s_{\mu_i}(t)$ for $i = 1, 2, \ldots, N$, where $s_{\mu_i} = (s_{1\mu_i}, s_{2\mu_i}, \ldots, s_{n\mu_i})^T \in R^n$ satisfies $\dot{s}_{\mu_i}(t) = f_{\mu_i}(s_{\mu_i}(t))$, which describes the identical local dynamics for the nodes in the μ_ith community. The N nodes are said to achieve cluster synchronization if

$$\lim_{t \to \infty} \|e_i(t)\| = 0, \quad i = 1, \ldots, N, \tag{4.3}$$

which means that the nodes within μ_ith community are fully synchronized to dynamic state s_{μ_i}, while nodes in separate communities behave independently. Define a set $M = (s_{\mu_1}, s_{\mu_2}, \ldots, s_{\mu_N}) \in R^{n \times N}$ as the cluster synchronization manifold for network (4.2). In fact, condition (4.3) implies that the manifold M is stable.

According to the above definition of error variables, we can write the corresponding error system with respect to (4.2) as

$$\dot{e}_i(t) = f_{\mu_i}(x_i(t)) - f_{\mu_i}(s_{\mu_i}(t)) + c \sum_{j=1}^{N} a_{ij} e_j(t) + c \sum_{j=1}^{N} a_{ij} s_{\mu_j}(t)$$

$$+ c \sum_{j=1}^{N} b_{ij} e_j(t - \tau) + c \sum_{j=1}^{N} b_{ij} s_{\mu_j}(t - \tau) + u_i(t), \quad i = 1, \ldots, N. \tag{4.4}$$

Since A and B are zero-row-sum matrices, we have

$$\sum_{j=1}^{N} a_{ij} s_{\mu_j}(t) = 0, \quad i \in U_i \setminus \tilde{U}_{\mu_i}^A,$$

$$\sum_{j=1}^{N} b_{ij} s_{\mu_j}(t) = 0, \quad i \in U_i \setminus \tilde{U}_{\mu_i}^B. \tag{4.5}$$

Let d_i stands for the feedback control strength. We design a local feedback control as

$$u_i(t) = \begin{cases} l - cd_i^A e_i(t) - c\sum_{j=1}^{N} a_{ij} s_{\mu_j}(t) - cd_i^B e_i(t-\tau) - c\sum_{j=1}^{N} b_{ij} s_{\mu_j}(t-\tau), & i \in \tilde{U}_{\mu_i}^A \bigcup \tilde{U}_{\mu_i}^B \\ 0, & \text{otherwise,} \end{cases} \tag{4.6}$$

where $d_i^A = d_i > 0$ for $i \in \tilde{U}_{\mu_i}^A$ and $d_i^B = d_i > 0$ for $i \in \tilde{U}_{\mu_i}^B$, which means we put control on the nodes that have communications with nodes in other different clusters. By intuition, the terms $cd_i^A e_i(t)$ and $cd_i^B e_i(t-\tau)$ in (4.6) is used to synchronize all nodes in the same cluster, while the remainder terms $c\sum_{j=1}^{N} a_{ij} s_{\mu_j}(t)$ and $c\sum_{j=1}^{N} b_{ij} s_{\mu_j}(t-\tau)$ in the controller is to weaken the mutual influences among clusters at the intersection nodes. It is easy to verify that the manifold M is an invariant manifold for the network (4.2).

Theorem 4.1. *Let $P = \text{diag}\{p_1, p_2, \ldots, p_n\}$ be a positive definite diagonal matrix and $\Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_n\}$ be a diagonal matrix such that $f_{\mu_i}(x,t) \in QUAD(P, \Delta, \eta)$ for $1 \leq i \leq N$. For $k = 1, 2, \ldots, n$, if*

$$(\Lambda + p_k \delta_k)I + cp_k\left(A^S - D_A\right) + \frac{c^2}{4\Lambda}p_k^2(B - D_B)(B - D_B)^T < 0, \tag{4.7}$$

where $D_A = \text{diag}\{d_1^A, d_2^A, \ldots, d_N^A\}$ and $D_B = \text{diag}\{d_1^B, d_2^B, \ldots, d_N^B\}$. Λ is a given positive constant. Then, for any initial values, the solution $x_1(t)$, $x_1(t)$, ..., $x_N(t)$ of the system (4.1) under the control (4.6) can achieve cluster synchronization and satisfies (4.3).

Proof. Denote $e^k(t) = (e_{k1}(t), e_{k2}(t), \ldots, e_{kN}(t))^T$, $k = 1, 2, \ldots, n$. Define a Lyapunov-Krasovskii function as

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} \left\{ e_i(t)^T P e_i(t) + 2\Lambda \int_{t-\tau}^{t} e_i(\theta)^T e_i(\theta) d\theta \right\}. \tag{4.8}$$

Differentiating (4.8) along the solution of error system (4.4) under the control (4.6) gives

$$
\begin{aligned}
\frac{dV(t)}{dt} &= \sum_{i=1}^{N} e_i(t)^T P \dot{e}_i(t) + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i(t-\tau)^T e_i(t-\tau) \\
&= \sum_{i=1}^{N} e_i(t)^T P \left(f_{\mu_i}(x_i(t)) - f_{\mu_i}(s_{\mu_i}(t)) + c \sum_{j=1}^{N} a_{ij} e_j(t) \right) \\
&\quad - \sum_{i=1}^{N} c d_i^A e_i(t)^T P e_i(t) + \sum_{i=1}^{N} e_i(t)^T P c \sum_{j=1}^{N} b_{ij} e_j(t-\tau) - \sum_{i=1}^{N} c d_i^B e_i(t)^T P e_j(t-\tau) \\
&\quad + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i(t-\tau)^T e_i(t-\tau) \\
&= \sum_{i=1}^{N} e_i(t)^T P(f_{\mu_i}(x_i(t)) - f_{\mu_i}(s_{\mu_i}(t)) - \Delta e_i(t)) + \sum_{i=1}^{N} e_i(t)^T P \Delta e_i(t) \\
&\quad + \sum_{i=1}^{N} e_i(t)^T P c \sum_{j=1}^{N} a_{ij} e_j(t) - \sum_{i=1}^{N} c d_i^A e_i(t)^T P e_i(t) + \sum_{i=1}^{N} e_i(t)^T P c \sum_{j=1}^{N} b_{ij} e_j(t-\tau) \\
&\quad - \sum_{i=1}^{N} c d_i^B e_i(t)^T P e_j(t-\tau) + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i(t-\tau)^T e_i(t-\tau) \\
&\leq -\eta \sum_{i=1}^{N} e_i(t)^T e_i(t) + \sum_{k=1}^{n} p_k \delta_k e^k(t)^T e^k(t) + c \sum_{k=1}^{n} p_k e^k(t)^T A^S e^k(t) \qquad (4.9) \\
&\quad - c \sum_{k=1}^{n} p_k e^k(t)^T D_A e^k(t) + c \sum_{k=1}^{n} p_k e^k(t)^T (B - D_B) e^k(t-\tau) \\
&\quad + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) - \Lambda \sum_{i=1}^{N} e_i(t-\tau)^T e_i(t-\tau) \\
&= -\eta \sum_{i=1}^{N} e_i(t)^T e_i(t) + \sum_{k=1}^{n} p_k e^k(t)^T \left(\delta_k I_N + c \left(A^S - D_A \right) \right) e^k(t) \\
&\quad + c \sum_{k=1}^{n} p_k e^k(t)^T (B - D_B) e^k(t-\tau) + \Lambda \sum_{i=1}^{N} e_i(t)^T e_i(t) \\
&\quad - \Lambda \sum_{i=1}^{N} e_i(t-\tau)^T e_i(t-\tau) \\
&= -\eta \sum_{i=1}^{N} e_i(t)^T e_i(t) + \sum_{k=1}^{n} \left(e^k(t)^T, e^k(t-\tau)^T \right) \\
&\quad \times \begin{bmatrix} (\Lambda + p_k \delta_k) I + c p_k (A^S - D_A) & \frac{c}{2} p_k (B - D_B) \\ \frac{c}{2} p_k (B - D_B)^T & -\Lambda I \end{bmatrix} \left(e^k(t), e^k(t-\tau) \right).
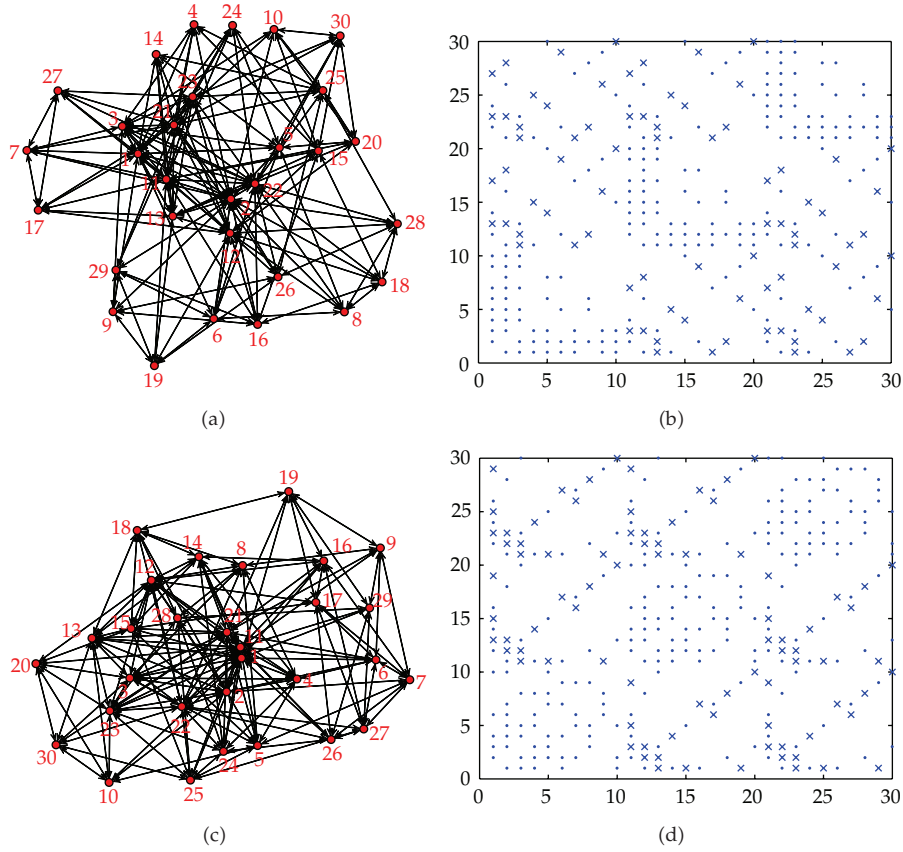\end{aligned}
$$

(a)

(b)

(c)

(d)

Figure 1: The controlled complex network (3.2). (a) the topological structure of matrix A; (b) The sketch of matrix A: the point (i, j) is denoted as "·" if $a_{ij} > 0$ and the point (i, j) is denoted as "×" if $a_{ij} < 0$; (c) the topological structure of matrix B; (d) the sketch of matrix B: the point (i, j) is denoted as "·" if $b_{ij} > 0$ and the point (i, j) is denoted as "×" if $b_{ij} < 0$.

Noticing the inequalities of (4.7), from Lemma 2.4, we obtain

$$\frac{dV(t)}{dt} \leq -\eta \sum_{i=1}^{N} e_i(t)^T e_i(t) < 0. \tag{4.10}$$

Thus, we get $x_i(t) \rightarrow s_{\mu_i}(t)$. □

5. Numerical Simulation

In this section, we give numerical simulations to verify the theorems obtained in Section 3 and Section 4. Consider the unified system,

$$f(x, t, \alpha) = \begin{cases} (25\alpha + 10)(x_2 - x_1), \\ (28 - 35\alpha)x_1 - x_1 x_3 + (29\alpha - 1)x_2, \\ x_1 x_2 - \dfrac{8 + \alpha}{3} x_3. \end{cases} \tag{5.1}$$
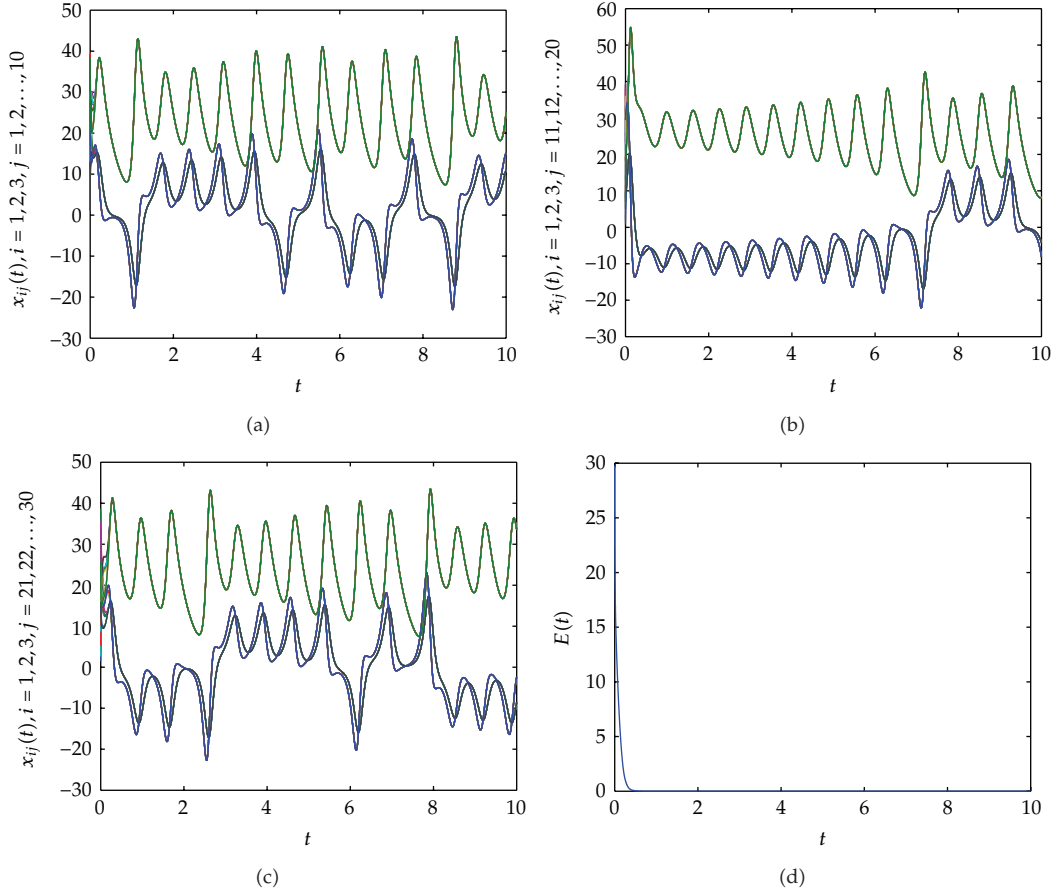
(a)

(b)

(c)

(d)

Figure 2: The controlled complex network (3.2). (a, b, c) The behavior of $x(t)$ in 3 communities. (d) The error $E(t)$.

The system is chaotic for all $\alpha \in [0, 1]$. If $\alpha = 0$, the system (5.1) is Lorenz's attractor. The ultimate bound and positively invariant set for system (5.1) is given in [22]. For $0 \leq \alpha < 1/29$, $x_2^2 + (x_3 - 28 + 35\alpha)^2 \leq C^2$ and $x_1^2 \leq C^2$, where

$$C = \frac{(28 - 35\alpha)(8 + \alpha)}{2\sqrt{3(5 + 88\alpha)(1 - 29\alpha)}}. \tag{5.2}$$

Thus, we have $|x_1| \leq C$, $|x_2| \leq C$ and $|x_3| \leq C + 28 - 35\alpha$. Let $P = \text{diag}\{p_1, p_2, p_3\}$ and $x - s = (e_1, e_2, e_3)^T$, we obtain

$$
\begin{aligned}
(x - s)^T P(f(x, t, \alpha) - f(s, t, \alpha)) \\
\leq -(25\alpha + 10)p_1 e_1^2 + (29\alpha - 1)p_2 e_2^2 - \frac{8 + \alpha}{3} p_3 e_3^2 \\
+ ((25\alpha + 10)p_1 + (28 - 35\alpha)p_2 + p_2(C + 28 - 35\alpha))|e_1 e_2| \\
+ p_3 C |e_1 e_3| + (p_3 - p_2)x_1 e_2 e_3 + (p_2 - p_3)e_1 e_2 e_3.
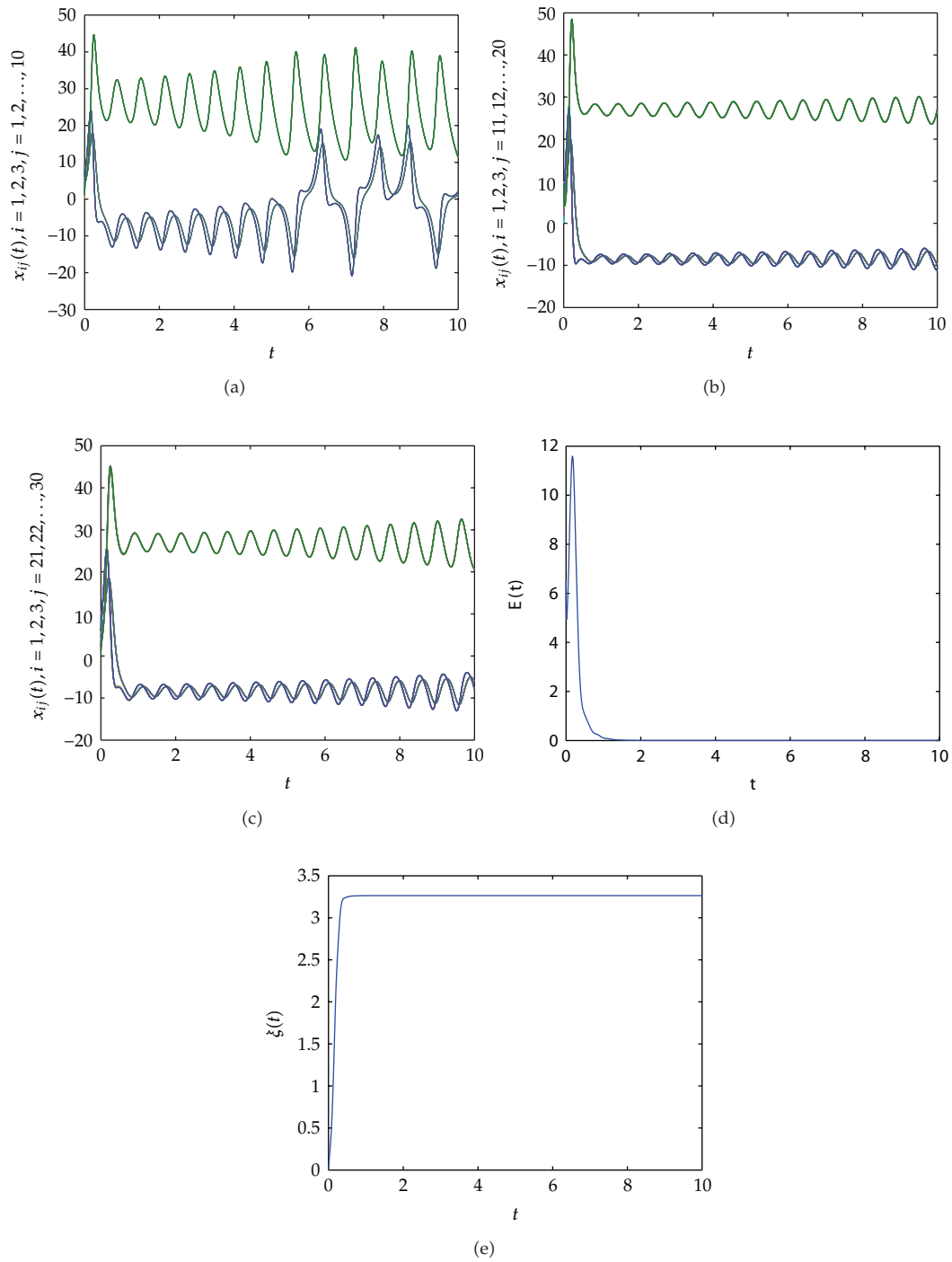\end{aligned}
\tag{5.3}
$$

Figure 3: The controlled complex network (3.2). (a,b,c) The behavior of $x(t)$ in 3 communities. (d) The error $E(t)$. (e) The adaptive control strength $\xi(t)$.
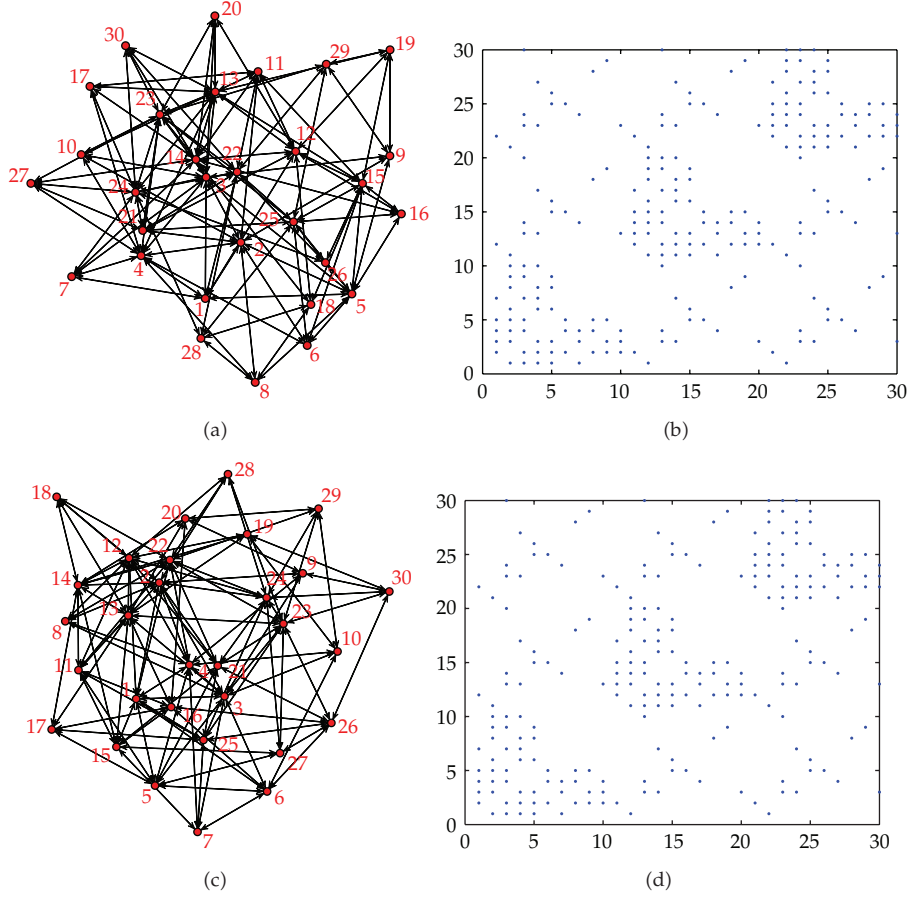
(a)

(b)

(c)

(d)

Figure 4: The controlled complex network (3.2). (a) The topological structure of matrix A; (b) the sketch of matrix A: the point (i, j) is denoted as "$\cdot$" if $a_{ij} > 0$; (c) the topological structure of matrix B; (d) the sketch of matrix B: the point (i, j) is denoted as "$\cdot$" if $b_{ij} > 0$.

Let $p_2 = p_3$, we have

$$
\begin{aligned}
(x - s)^T P & (f(x, t, \alpha) - f(s, t, \alpha)) \\
& \leq \left(-(25\alpha + 10)p_1 + \frac{p_3 C}{2}\vartheta + \frac{(25\alpha + 10)p_1 + (28 - 35\alpha)p_2 + p_2(C + 28 - 35\alpha)}{2}\nu \right)e_1^2 \\
& \quad + \left((29\alpha - 1)p_2 + \frac{(25\alpha + 10)p_1 + (28 - 35\alpha)p_2 + p_2(C + 28 - 35\alpha)}{2\nu} \right)e_2^2 \\
& \quad + \left(-\frac{8 + \alpha}{3}p_3 + \frac{p_3 C}{2\vartheta} \right)e_3^2 \\
& \leq (p_1\delta_1 - \eta)e_1^2 + (p_2\delta_2 - \eta)e_2^2 + (p_3\delta_3 - \eta)e_3^2.
\end{aligned}
\tag{5.4}
$$

Set $P = \text{diag}\{5, 1, 1\}$, $\Delta = \text{diag}\{23, 36, 17\}$, $\varepsilon = 2$ and $\vartheta = 1$. Let $\alpha = 0$, $\alpha = 0.01$ and $\alpha = 0.03$ respectively in (5.1) since the nodes are nonidentical in different communities.
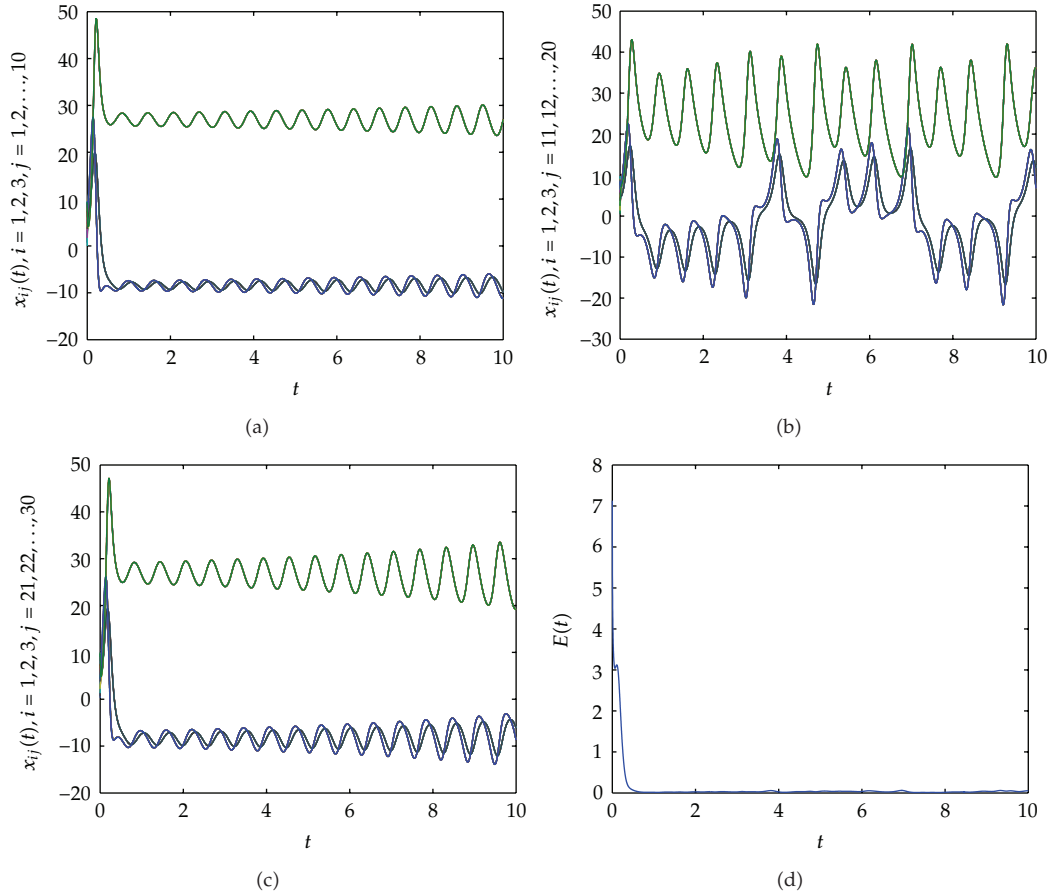
(a)

(b)

(c)

(d)

Figure 5: The controlled complex network (3.2). (a, b, c) The behavior of $x(t)$ in 3 communities. (d) The error $E(t)$.

If $\alpha = 0$, we have $\eta_1 = 3.2074$ and $f(x, 0) \in \text{QUAD}(P, \Delta, \eta_1)$. If $\alpha = 0.01$, we have $\eta_2 = 2.0243$ and $f(x, 0.01) \in \text{QUAD}(P, \Delta, \eta_2)$. If $\alpha = 0.03$, we have $\eta_3 = 0.1547$ and $f(x, 0.02) \in \text{QUAD}(P, \Delta, \eta_3)$.

5.1. Simulation to Cluster Synchronization with Identical Nodes

In this simulation, we consider a network with 30 nodes and 3 communities. It is too high and we do not show it out. We show the topology structure in Figure 1.

Let $s_1(t)$, $s_2(t)$, $s_3(t)$ be the solution of the uncoupling system $\dot{s}_i(t) = f(s_i(t), t)$ with initial values $s_1(0) = [1, 2, 3]^T$, $s_2(0) = [4, 5, 6]^T$, and $s_3(0) = [7, 8, 9]^T$. Define $E(t) = (1/30) \sum_{i=1}^{30} \|x_i(t) - s_{\bar{i}}(t)\|^2$. If $\lim_{t \to \infty} E(t) = 0$, the complex network achieves cluster synchronization.

For the controlled dynamic network (4.6). Let $\tau = 0.1$, $\xi_i = 275$, $i \in J = \{7, 8, 9, 10, 11, 12, 13, 28, 29, 30\}$, $c = 5$, $\Lambda = 0.0934$ in the Theorem 3.1. It can be easily verified that such parameters fits (4.7).

Figures 2(a), 2(b), and 2(c) give the behavior of $x(t)$, as well as Figure 2(d) which shows how $E(t)$ evolve in pinning united chaotic attractor complex network with initial values chosen randomly in the interval $[-10, 10]$.

For the controlled dynamic network (4.6) with adaptive control strength $d_i(t)$, let $\Xi^* = \Xi$ and keep other parameters the same as selected above, which makes (3.11) correct.

Figure 3 shows the behavior of $x(t)$ and how $E(t)$ evolve in pinning Lorenz chaotic attractor complex network with adaptive control strength and the initial values are chosen randomly in the interval $[-10, 10]$.

5.2. Simulation to Cluster Synchronization with Nonidentical Nodes

In this simulation, we consider a network with 30 nodes and 3 communities. It is too high and we do not show it out. We give the topology structure Figure 4.

Let $s_1(t)$, $s_2(t)$, $s_3(t)$ be the solution of the uncoupling system $\dot{s}_i(t) = f(s_i(t), t)$ with initial values $s_1(0) = [1, 2, 3]^T$, $s_2(0) = [4, 5, 6]^T$, and $s_3(0) = [7, 8, 9]^T$. Define $E(t) = (1/30) \sum_{i=1}^{30} \|x_i(t) - s_{\bar{i}}(t)\|^2$. If $\lim_{t \to \infty} E(t) = 0$, the complex network achieves cluster synchronization.

For the controlled dynamic network (4.6). Let $\tau = 0.1$, $D_A = 10.2173I$, $D_A = 0.0171I$, $c = 6$, and $\Lambda = 0.0071$ in the Theorem 4.1. It can be easily verified that such parameters fits (4.7).

Figures 5(a), 5(b), and 5(c) give the behavior of $x(t)$, as well as Figure 5(d) which shows how $E(t)$ evolve in pinning united chaotic attractor complex network with initial values chosen randomly in the interval $[-10, 10]$.

6. Conclusion

In the paper, we have investigated the cluster synchronization on pinning control of LCODEs with identical and nonidentical nodes. We give a sufficient condition to make the complex network achieve cluster synchronization. Moreover, adaptive feedback control techniques are used to adjust control strength. Finally, some numerical examples are given, which is essential to verify our theoretical analysis.

Acknowledgments

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