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Research Article

New Solutions for (1+1)-Dimensional and (2+1)-Dimensional Ito Equations

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Using the extended F-expansion method based on computerized symbolic computation technique, we find several new solutions of (1+1)-dimensional and (2+1)-dimensional Ito equations. These solutions contain hyperbolic and triangular solutions. It is shown that the power of the extended F-expansion method is its ease of use to determine shock or solitary type of solutions. In addition, as an illustrative sample, the properties for the extended F-expansion solutions of the Ito equations are shown with some figures.

1. Introduction

The nonlinear wave phenomena can be observed in various scientific fields, such as plasma physics, optical fibers, fluid dynamics, and chemical physics. The nonlinear wave phenomena can be obtained in solutions of nonlinear evolution equations (NEEs). The study of NLEEs appear everywhere in applied mathematics and theoretical physics including engineering sciences and biological sciences. These NLEEEs play a key role in describing key scientific phenomena. For example, the nonlinear Schrödinger's equation describes the dynamics of propagation of solitons through optical fibers. The Korteweg-de Vries equation models the shallow water wave dynamics near ocean shore and beaches. Additionally, the Schrödinger-Hirota equation describes the dispersive soliton propagation through optical fibers. These are just a few examples in the whole wide world of NLEEs and their applications, (see, for instance, [1–4]). While the above mentioned NLEEs are scalar NLEEs, there is a large number of NLEEs that are coupled. Some of them are two-coupled NLEEs such as the Gear-Grimshaw equation [2], while there are several others that are three-coupled NLEEs. An example of a three-coupled NLEE is the Wu-Zhang equation [4]. These coupled NLEEs are also studied in various areas of theoretical physics as well.

The exact solutions of these NEEs play an important role in the understanding of nonlinear phenomena. In the past decades, many methods were developed for finding exact solutions of NEEs such as the inverse scattering method [5, 6], improved projective Riccati equations method [7, 8], Cole-Hopf transformation method [9], exp-function method [10–16], bifurcation theory method [17], (G'/G)-expansion method [18, 19], homotopy perturbation method [20], tanh function method [20–24], and Jacobi and Weierstrass elliptic function method [25, 26]. Although Porubov et al. [27–29] have obtained some exact periodic solutions to some nonlinear wave equations, they use the Weierstrass elliptic function and involve complicated deducing. A Jacobi elliptic function (JEF) expansion method, which is straightforward and effective, was proposed for constructing periodic wave solutions for some nonlinear evolution equations. The essential idea of this method is similar to the tanh method by replacing the tanh function with some JEFs such as sn, cn, and dn. For example, the Jacobi periodic solution in terms of sn may be obtained by applying the sn-function expansion. Many similarl repetitious calculations have to be done to search for the Jacobi doubly periodic wave solutions in terms of cn and dn [30].

Recently, F-expansion method [31–34] was proposed to obtain periodic wave solutions of NLEEs, which can be thought of as a concentration of JEF expansion since F here stands for every function of JEFs. The objectives of this work are twofold. First, we seek to extend others works to establish new exact solutions of distinct physical structures for the nonlinear equations (1.1) and (1.2). The extended F-expansion (EFE) method will be used to achieve the first goal. The second goal is to show that the power of the EFE method is its ease of use to determine shock or solitary type of solutions. In this paper, we study two well-known PDEs, namely, generalized (1+1)-dimensional and generalized (2+1)-dimensional Ito equations. Many studies are concerning the (1+1)-dimensional Ito equation and the (2+1)-dimensional Ito equation [35–42].

The history of the KdV equation started with experiments by John Scott Russell in 1834, followed by theoretical investigations by Lord Rayleigh and Joseph Boussinesq around 1870, and, finally, Korteweg and de Vries in 1895 [39]. The KdV equation was not studied much after this until Zabusky and Kruskal (1965) [40] discovered numerically that its solutions seemed to decompose at large times into a collection of "solitons": well-separated solitary waves. Ito [41, 42] obtained the well-known generalized (1+1)-dimensional and generalized (2+1)-dimensional Ito equations by generalization of the bilinear KdV equation as

$$u_{tt} + u_{xxxt} + 3(u_x u_t + u u_{xt}) + 3u_{xx} \int_{\infty}^{x} u_t dx' = 0, \tag{1.1}$$

$$u_{tt} + u_{xxxt} + 3(2u_x u_t + u u_{xt}) + 3u_{xx} \int_{\infty}^{x} u_t dx' + \alpha u_{yt} + \beta u_{xt} = 0. \tag{1.2}$$

Also Sawada-Kotera-Ito (SK-Ito) seventh-order equation is the special case of the generalized seventh-order KdV equation as

$$\begin{aligned} u_t + 252u^2 u_x + 63u_{3x} + 378u u_x u_{2x} + 126u^2 u_{3x} \\ + 63u_{2x} u_{3x} + 42u_x u_{4x} + 21u u_{5x} + u_{7x} = 0. \end{aligned} \tag{1.3}$$

SK-Ito equation is characterized by the presence of three dispersive terms u_x, u_{3x}, and u_{7x}, respectively. SK-Ito seventh-order equation is completely integrable and admits of

conservation laws [43]. Moreover, the Ito-type coupled KdV (ItcKdV) equation [44], written in the following form:

$$u_t + \alpha u u_x + \beta v v_x + \gamma u_{xxx} = 0; \qquad v_t + \beta (uv)_x = 0, \tag{1.4}$$

if we take the special values $\alpha = -6$, $\beta = -2$, and $\gamma = -1$. Equation (1.4) describes the interaction process of two internal long waves which has infinitely many conserved quantities [45, 46].

In this paper, we extend the EFE method with symbolic computation to (1.1) and (1.2) for constructing their interesting Jacobi doubly periodic wave solutions. It is shown that soliton solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions. In addition, the algorithm that we use here is also a computerized method, in which we are generating an algebraic system.

2. Extended F-Expansion Method

In this section, we introduce a simple description of the EFE method, for a given partial differential equation as

$$G(u, u_x, u_y, u_z, u_{xy}, \ldots) = 0. \tag{2.1}$$

We like to know whether travelling waves (or stationary waves) are solutions of (2.1). The first step is to unite the independent variables x, y, and t into one particular variable through the new variable as

$$\zeta = x + y - vt, \qquad u(x, y, t) = U(\zeta), \tag{2.2}$$

where v is wave speed, and reduce (2.1) to an ordinary differential equation (ODE) as

$$G(U, U', U'', U''', \ldots) = 0. \tag{2.3}$$

Our main goal is to derive exact or at least approximate solutions, if possible, for this ODE. For this purpose, let us simply use U as the expansion in the form

$$u(x, y, t) = U(\zeta) = \sum_{i=0}^{N} a_i F^i + \sum_{i=1}^{N} a_{-i} F^{-i}, \tag{2.4}$$

where

$$F' = \sqrt{A + BF^2 + CF^4}, \tag{2.5}$$

Table 1: Relation between values of (A, B, C) and corresponding F.

A	B	C	$F(\zeta)$
1	$-1-m^2$	m^2	$\mathrm{sn}(\zeta)$ or $\mathrm{cd}(\zeta)=\mathrm{cn}(\zeta)/\mathrm{dn}(\zeta)$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{cn}(\zeta)$
m^2-1	$2-m^2$	-1	$\mathrm{dn}(\zeta)$
m^2	$-1-m^2$	1	$\mathrm{ns}(\zeta)(1/\mathrm{sn}(\zeta))$ or $\mathrm{dc}(\zeta)=\mathrm{dn}(\zeta)/\mathrm{cn}(\zeta)$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}(\zeta)=1/\mathrm{cn}(\zeta)$
-1	$2-m^2$	m^2-1	$\mathrm{nd}(\zeta)=1/\mathrm{dn}(\zeta)$
1	$2-m^2$	$1-m^2$	$\mathrm{sc}(\zeta)=\mathrm{sn}(\zeta)/\mathrm{cn}(\zeta)$
1	$2m^2-1$	$-m^2(-1-m^2)$	$\mathrm{sd}(\zeta)=\mathrm{sn}(\zeta)/\mathrm{dn}(\zeta)$
$1-m^2$	$2-m^2$	1	$\mathrm{cs}(\zeta)=\mathrm{cn}(\zeta)/\mathrm{sn}(\zeta)$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{ds}(\zeta)=\mathrm{dn}(\zeta)/\mathrm{sn}(\zeta)$
$1/4$	$(1-2m^2)/2$	$1/4$	$\mathrm{ns}(\zeta)+\mathrm{cs}(\zeta)$
$(1-m^2)/4$	$(1+m^2)/2$	$(1-m^2)/2$	$\mathrm{nc}(\zeta)+\mathrm{sc}(\zeta)$
$1/4$	$(m^2-2)/2$	$m^2/4$	$\mathrm{ns}(\zeta)+\mathrm{ds}(\zeta)$
$m^2/4$	$(m^2-2)/2$	$m^2/4$	$\mathrm{sn}(\zeta)+i\mathrm{cs}(\zeta)$

the highest degree of $(d^pU/d\zeta^p)$, is taken as

$$O\left(\frac{d^pU}{d\zeta^p}\right)=N+p,\quad p=1,2,3,\ldots, \tag{2.6}$$

$$O\left(U^q\frac{d^pU}{d\zeta^p}\right)=(q+1)N+p,\quad q=0,1,2,\ldots,\ p=1,2,3,\ldots, \tag{2.7}$$

where A, B, and C are constants, and N in (2.3) is a positive integer that can be determined by balancing the nonlinear term(s) and the highest order derivatives. Normally N is a positive integer, so that an analytic solution in closed form may be obtained. Substituting (2.1)–(2.5) into (2.3) and comparing the coefficients of each power of $F(\zeta)$ in both sides, we will get an overdetermined system of nonlinear algebraic equations with respect to $v, a_0, a_1, \ldots$. We will solve the over-determined system of nonlinear algebraic equations by use of Mathematica. The relations between values of A, B, C, and corresponding JEF solution $F(\zeta)$ of (2.4) are given in Table 1. Substituting the values of A, B, C, and the corresponding JEF solution $F(\zeta)$ chosen from Table 1 into the general form of solution, then an ideal periodic wave solution expressed by JEF can be obtained.

$\mathrm{sn}(\zeta)$, $\mathrm{cn}(\zeta)$, and $\mathrm{dn}(\zeta)$ are the JE sine function, JE cosine function, and the JEF of the third kind, respectively. And

$$\mathrm{cn}^2(\zeta)=1-\mathrm{sn}^2(\zeta),\qquad \mathrm{dn}^2(\zeta)=1-m^2\mathrm{sn}^2(\zeta), \tag{2.8}$$

with the modulus m $(0<m<1)$.

When $m\to 1$. the Jacobi functions degenerate to the hyperbolic functions, that is,

$$\mathrm{sn}\zeta\to\tanh\zeta,\qquad \mathrm{cn}\zeta\to\mathrm{sech}\zeta,\qquad \mathrm{dn}\zeta\to\mathrm{sech}\zeta, \tag{2.9}$$

when $m \to 0$, the Jacobi functions degenerate to the triangular functions, that is,

$$\mathrm{sn}\zeta \to \sin\zeta, \qquad \mathrm{cn}\zeta \to \cos\zeta, \qquad \mathrm{dn} \to 1. \tag{2.10}$$

3. Generalized (1+1)-Dimensional Ito Equation

We first consider the generalized (1+1)-dimensional Ito equation (1.1) as follows:

$$u_{tt} + u_{xxxt} + 3(u_x u_t + u u_{xt}) + 3u_{xx}\int_{\infty}^{x} u_t dx' = 0, \tag{3.1}$$

if we use the transformation $u = v_x$, it carries (3.1) into

$$v_{xtt} + v_{xxxxt} + 3(v_{xx}v_{xt} + v_x v_{xxt}) + 3v_{xxx}v_t = 0, \tag{3.2}$$

if we use $\zeta = x - \nu t$ transforms (3.2) into the ODE, we have

$$-\nu V''' + V^{(v)} - 3\nu\left(V''V'' + V'V'''\right) - 3\nu V'''V' = 0, \tag{3.3}$$

where by integrating once we obtain, upon setting the constant of integration to zero,

$$V''' + 3\left(V'\right)^2 - \nu V' = 0, \tag{3.4}$$

if we use the transformation $W = V'$, then (3.4) can be written as follows:

$$W'' + 3W^2 - \nu W = 0. \tag{3.5}$$

Balancing the term W'' with the term W^2 we obtain $N = 2$ then

$$W(\zeta) = a_0 + a_1\varphi + a_{-1}\varphi^{-1} + a_2\varphi^2 + a_{-2}\varphi^{-2}, \qquad \varphi' = \sqrt{A + B\varphi^2 + C\varphi^4}. \tag{3.6}$$

Substituting (3.6) into (3.5) and comparing the coefficients of each power of φ in both sides, we will get an over-determined system of nonlinear algebraic equations with respect to ν, a_i, $i = 0, 1, -1, -2, 2$. Solving the over-determined system of nonlinear algebraic equations by use of Mathematica, we obtain three groups of constants

(1)

$$\begin{gathered} a_{-1} = a_{-1} = 0, \qquad a_0 = -\frac{2B}{3}, \qquad a_2 = -2C, \\ a_{-2} = -2A, \qquad \nu = -\frac{2\left(B^2 + 12AC\right)}{B}, \end{gathered} \tag{3.7}$$

(2)

$$a_{-1} = a_2 = a_{-1} = 0, \qquad a_0 = -\frac{2B}{3}, \qquad a_{-2} = -2A, \qquad \nu = -\frac{2\left(B^2 - 3AC\right)}{B}, \tag{3.8}$$

(3)

$$a_{-1} = a_{-2} = a_{-1} = 0, \qquad a_0 = -\frac{2B}{3}, \qquad a_2 = -2C, \qquad \nu = -\frac{2\left(B^2 - 3AC\right)}{B}. \tag{3.9}$$

The solutions of (3.1) are

$$\begin{aligned} u_1 = &-\frac{2\left(1+m^2\right)}{3} - 2m^2\mathrm{sn}^2\left(x - \frac{2\left(\left(1+m^2\right)^2 + 12m^2\right)}{1+m^2}t\right) \\ &- 2\mathrm{ns}^2\left(x - \frac{2\left(\left(1+m^2\right)^2 + 12m^2\right)}{1+m^2}t\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_2 = &-\frac{2\left(1+m^2\right)}{3} - 2m^2\mathrm{cd}^2\left(x - \frac{2\left(\left(1+m^2\right)^2 + 12m^2\right)}{1+m^2}t\right) \\ &- 2\mathrm{dc}^2\left(x - \frac{2\left(\left(1+m^2\right)^2 + 12m^2\right)}{1+m^2}t\right), \end{aligned} \tag{3.11}$$

$$\begin{aligned} u_3 = &-\frac{2\left(2m^2-1\right)}{3} + 2m^2\mathrm{cn}^2\left(x + \frac{2\left(12m^2\left(m^2-1\right) + \left(2m^2-1\right)^2\right)}{2m^2-1}t\right) \\ &- 2\left(1-m^2\right)\mathrm{nc}^2\left(x + \frac{2\left(12m^2\left(m^2-1\right) + \left(2m^2-1\right)^2\right)}{2m^2-1}t\right), \end{aligned} \tag{3.12}$$

$$\begin{aligned} u_4 = &-\frac{2\left(2-m^2\right)}{3} + 2\mathrm{dn}^2\left(x - \frac{2\left(\left(2-m^2\right)^2 - 12\left(-1+m^2\right)\right)}{2-m^2}t\right) \\ &- 2\left(m^2-1\right)\mathrm{nd}^2\left(x - \frac{2\left(\left(2-m^2\right)^2 - 12\left(-1+m^2\right)\right)}{2-m^2}t\right), \end{aligned} \tag{3.13}$$

$$u_5 = -\frac{2\left(1+m^2\right)}{3} - 2\mathrm{ns}^2\left(x - \frac{2\left(\left(1+m^2\right)^2 - 3m^2\right)}{1+m^2}t\right), \tag{3.14}$$

$$u_6 = -\frac{2(2-m^2)}{3} - 2\left(1-m^2\right)\mathrm{sc}^2\left(x + \frac{2\left(12(1-m^2)+(2-m^2)^2\right)}{2-m^2}t\right) - 2\mathrm{cs}^2\left(x + \frac{2\left(12(1-m^2)+(2-m^2)^2\right)}{2-m^2}t\right), \tag{3.15}$$

$$u_7 = \frac{2(2m^2-1)}{3} + 2m^2\left(1+m^2\right)\mathrm{sd}^2\left(x + \frac{2\left(-12m^2(1-m^2)+(-1+2m^2)^2\right)}{2m^2-1}t\right) - 2\mathrm{ds}^2\left(x + \frac{2\left(-12m^2(1-m^2)+(-1+2m^2)^2\right)}{2m^2-1}t\right), \tag{3.16}$$

$$u_8 = \frac{2(1-2m^2)}{6} - \frac{1}{2}\left(\mathrm{ns}\left(x + \frac{2\left(0.75+(0.5-m^2)^2\right)}{0.5-m^2}t\right) + \mathrm{cs}\left(x + \frac{2\left(0.75+(0.5-m^2)^2\right)}{0.5-m^2}t\right)\right)^2 - \frac{1}{2}\left(\mathrm{ns}\left(x + \frac{2\left(0.75+(0.5-m^2)^2\right)}{0.5-m^2}t\right) + \mathrm{cs}\left(x + \frac{2\left(0.75+(0.5-m^2)^2\right)}{0.5-m^2}t\right)\right)^{-2}, \tag{3.17}$$

$$u_9 = -\frac{1+m^2}{3} + \frac{1-m^2}{2} \times \left(\mathrm{nc}\left(x + \frac{2\left(12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2\right)}{0.5m^2+0.5}t\right) + \mathrm{sc}\left(x + \frac{2\left(12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2\right)}{0.5m^2+0.5}t\right)\right)^2 + \frac{1-m^2}{2}\left(\mathrm{nc}\left(x + \frac{2\left(12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2\right)}{0.5m^2+0.5}t\right) + \mathrm{sc}\left(x + \frac{2\left(12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2\right)}{0.5m^2+0.5}t\right)\right)^{-2}, \tag{3.18}$$

$$
\begin{aligned}
u_{10} = & -\frac{m^2-2}{3} \\
& -\frac{m^2}{2}\left(\mathrm{ns}\left(x+\frac{2\left(0.75m^2+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}t\right)\right. \\
& \qquad \left.+\mathrm{ds}\left(x+\frac{2\left(0.75m^2+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}t\right)\right)^2 \\
& -\frac{1}{2}\left(\mathrm{ns}\left(x+\frac{2\left(0.75m^2+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}t\right)\right. \\
& \qquad \left.+\mathrm{ds}\left(x+\frac{2\left(0.75m^2+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}t\right)\right)^{-2},
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
u_{11} = & \frac{m^2-2}{3} \\
& -\frac{m^2}{2}\left(\mathrm{sn}\left(x+\frac{2\left(0.75m^4+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}t\right)\right. \\
& \qquad \left.+i\mathrm{cs}\left(x+\frac{2\left(0.75m^4+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}t\right)\right)^2 \\
& +\frac{m^2}{2}\left(\mathrm{sn}\left(x+\frac{2\left(0.75m^4+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}t\right)\right. \\
& \qquad \left.+i\mathrm{cs}\left(x+\frac{2\left(0.75m^4+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}t\right)\right)^{-2},
\end{aligned}
\tag{3.20}
$$

$$
u_{12} = -\frac{2\left(1+m^2\right)}{3} - 2\mathrm{dc}^2\left(x-\frac{2\left(\left(1+m^2\right)^2-3m^2\right)}{1+m^2}t\right), \tag{3.21}
$$

$$
u_{13} = -\frac{2\left(2m^2-1\right)}{3} - 2\left(1-m^2\right)\mathrm{nc}^2\left(x+\frac{2\left(-3m^2\left(m^2-1\right)+\left(2m^2-1\right)^2\right)}{2m^2-1}t\right), \tag{3.22}
$$

$$
u_{14} = -\frac{2\left(2-m^2\right)}{3} - 2\left(m^2-1\right)\mathrm{nd}^2\left(x-\frac{2\left(\left(2-m^2\right)^2+3\left(-1+m^2\right)\right)}{2-m^2}t\right), \tag{3.23}
$$

$$
u_{15} = -\frac{2\left(2-m^2\right)}{3} - 2\mathrm{cs}^2\left(x+\frac{2\left(\left(2-m^2\right)^2-3\left(1-m^2\right)\right)}{2-m^2}t\right), \tag{3.24}
$$

$$u_{16} = \frac{2(2m^2-1)}{3} - 2\mathrm{ds}^2\left(x + \frac{2\left(3m^2(1-m^2) + (-1+2m^2)^2\right)}{2m^2-1}t\right), \tag{3.25}$$

$$u_{17} = \frac{2(1-2m^2)}{6} - \frac{1}{2}\left(\mathrm{ns}\left(x + \frac{2\left((0.5-m^2)^2 - 3/16\right)}{0.5-m^2}t\right) + \mathrm{cs}\left(x + \frac{2\left((0.5-m^2)^2 - (3/16)\right)}{0.5-m^2}t\right)\right)^{-2}, \tag{3.26}$$

$$u_{18} = -\frac{1+m^2}{3} + \frac{1-m^2}{2} \times \left(\mathrm{nc}\left(x + \frac{2\left((0.5+0.5m^2)^2 - 3(0.5-0.5m^2)(0.25-0.25m^2)\right)}{0.5m^2+0.5}t\right) + \mathrm{sc}\left(x + \frac{2\left((0.5+0.5m^2)^2 - 3(0.5-0.5m^2)(0.25-0.25m^2)\right)}{0.5m^2+0.5}t\right)\right)^{-2}, \tag{3.27}$$

$$u_{19} = -\frac{m^2-2}{3} - \frac{1}{2}\left(\mathrm{ns}\left(x + \frac{2\left((0.5m^2-1)^2 - 3/16\right)}{0.5m^2-1}t\right) + \mathrm{ds}\left(x + \frac{2\left((0.5m^2-1)^2 - 3/16\right)}{0.5m^2-1}t\right)\right)^{-2}, \tag{3.28}$$

$$u_{20} = \frac{m^2-2}{3} + \frac{m^2}{2}\left(\mathrm{sn}\left(x + \frac{2\left((0.5m^2-1)^2 - m^4/16\right)}{0.5m^2-1}t\right) + i\mathrm{cs}\left(x + \frac{2\left((0.5m^2-1)^2 - m^4/16\right)}{0.5m^2-1}t\right)\right)^{-2}, \tag{3.29}$$

$$u_{21} = -\frac{2(1+m^2)}{3} - 2m^2\mathrm{sn}^2\left(x - \frac{2\left((1+m^2)^2 - 3m^2\right)}{1+m^2}t\right), \tag{3.30}$$

$$u_{22} = -\frac{2(1+m^2)}{3} - 2m^2\mathrm{cd}^2\left(x - \frac{2\left((1+m^2)^2 - 3m^2\right)}{1+m^2}t\right), \tag{3.31}$$

$$u_{23} = -\frac{2(2m^2-1)}{3} + 2m^2\mathrm{cn}^2\left(x + \frac{2\left((2m^2-1)^2 - 3m^2(m^2-1)\right)}{2m^2-1}t\right), \tag{3.32}$$

$$u_{24} = -\frac{2(2-m^2)}{3} + 2\mathrm{dn}^2\left(x - \frac{2\left((2-m^2)^2 + 3(-1+m^2)\right)}{2-m^2}t\right), \tag{3.33}$$

$$u_{25} = -\frac{2(2-m^2)}{3} - 2\left(1-m^2\right)\mathrm{sc}^2\left(x + \frac{2\left((2-m^2)^2 - 3(1-m^2)\right)}{2-m^2}t\right), \tag{3.34}$$

$$u_{26} = \frac{2(2m^2-1)}{3} + 2m^2\left(1+m^2\right)\mathrm{sd}^2\left(x + \frac{2\left(3m^2(1-m^2) + (-1+2m^2)^2\right)}{2m^2-1}t\right), \tag{3.35}$$

$$u_{27} = \frac{2(1-2m^2)}{6} - \frac{1}{2}\left(\mathrm{ns}\left(x + \frac{2\left((0.5-m^2)^2 - 3/16\right)}{0.5-m^2}t\right) + \mathrm{cs}\left(x + \frac{2\left((0.5-m^2)^2 - 3/16\right)}{0.5-m^2}t\right)\right)^2, \tag{3.36}$$

$$u_{28} = -\frac{1+m^2}{3} + \frac{1-m^2}{2} \times \left(\mathrm{nc}\left(x + \frac{2\left((0.5+0.5m^2)^2 - 3(0.5-0.5m^2)(0.25-0.25m^2)\right)}{0.5m^2+0.5}t\right) + \mathrm{sc}\left(x + \frac{2\left((0.5+0.5m^2)^2 - 3(0.5-0.5m^2)(0.25-0.25m^2)\right)}{0.5m^2+0.5}t\right)\right)^2, \tag{3.37}$$

$$u_{29} = -\frac{m^2-2}{3} - \frac{m^2}{2}\left(\mathrm{ns}\left(x + \frac{2\left((0.5m^2-1)^2 - m^2/16\right)}{0.5m^2-1}t\right) + \mathrm{ds}\left(x + \frac{2\left((0.5m^2-1)^2 - m^2/16\right)}{0.5m^2-1}t\right)\right)^2, \tag{3.38}$$

$$u_{30} = \frac{m^2-2}{3} - \frac{m^2}{2}\left(\mathrm{sn}\left(x + \frac{2\left((0.5m^2-1)^2 - m^4/16\right)}{0.5m^2-1}t\right) + i\mathrm{cs}\left(x + \frac{2\left((0.5m^2-1)^2 - (m^4/16)\right)}{0.5m^2-1}t\right)\right)^2. \tag{3.39}$$

3.1. Soliton Solutions

Some solitary wave solutions can be obtained, if the modulus m approaches to 1 in (3.10)–(3.39) as follows:

$$
\begin{aligned}
u_{31} &= -\frac{4}{3} - 2\tanh^2(x-16t) - 2\coth^2(x-16t),\\
u_{32} &= -\frac{2}{3} + 2\operatorname{sech}^2(x+2t),\\
u_{33} &= -\frac{2}{3} + 2\operatorname{sech}^2(x-2t),\\
u_{34} &= -\frac{4}{3} - 2\coth^2(x-t),\\
u_{35} &= -\frac{2}{3} - 2\operatorname{csch}^2(x+2t),\\
u_{36} &= \frac{2}{3} + 4\sinh^2(x+2t) - 2\operatorname{csch}^2(x+2t),\\
u_{37} &= \frac{-1}{3} - \frac{1}{2}(\coth(x-4t) + \operatorname{csch}(x-4t))^2 - \frac{1}{2}(\coth(x-4t) + \operatorname{csch}(x-4t))^{-2},\\
u_{38} &= \frac{-1}{3} - \frac{1}{2}(\tanh(x-4t) + i\operatorname{csch}(x-4t))^2 + \frac{1}{2}(\tanh(x-4t) + i\operatorname{csch}(x-4t))^{-2},\\
u_{39} &= \frac{-1}{3} - \frac{1}{2}\left(\coth\left(x-\frac{1}{4}t\right) + \operatorname{csch}\left(x-\frac{1}{4}t\right)\right)^{-2},\\
u_{40} &= \frac{-1}{3} + \frac{1}{2}\left(\tanh\left(x-\frac{1}{4}t\right) + i\operatorname{csch}\left(x-\frac{1}{4}t\right)\right)^{-2},\\
u_{41} &= -\frac{4}{3} - 2\tanh^2(x-t),\\
u_{42} &= \frac{2}{3} + 4\sinh^2(x+2t),\\
u_{43} &= \frac{-1}{3} - \frac{1}{2}\left(\coth\left(x-\frac{1}{4}t\right) + \operatorname{csch}\left(x-\frac{1}{4}t\right)\right)^{2},\\
u_{44} &= \frac{1}{3} - \frac{1}{2}(\coth(x+2t) + \operatorname{csch}(x+2t))^2,\\
u_{45} &= \frac{-1}{3} - \frac{1}{2}\left(\tanh\left(x-\frac{3}{4}t\right) + i\operatorname{csch}\left(x-\frac{3}{4}t\right)\right)^{2}.
\end{aligned}
\tag{3.40}
$$

3.2. Triangular Periodic Solutions

Some trigonometric function solutions can be obtained, if the modulus m approaches to zero in (3.10)–(3.39) as follwos:

$$
\begin{aligned}
u_{46} &= -\frac{2}{3} - 2\csc^2(x-2t),\\
u_{47} &= -\frac{2}{3} - 2\sec^2(x-2t),\\
u_{48} &= \frac{2}{3} - 2\sec^2(x-2t),
\end{aligned}
$$

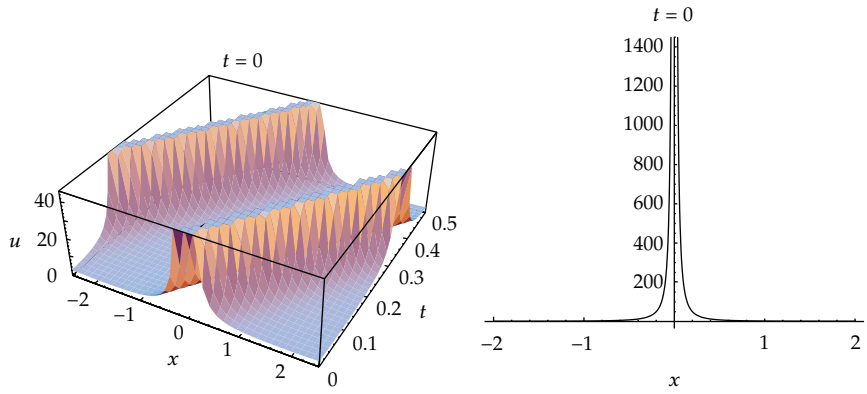
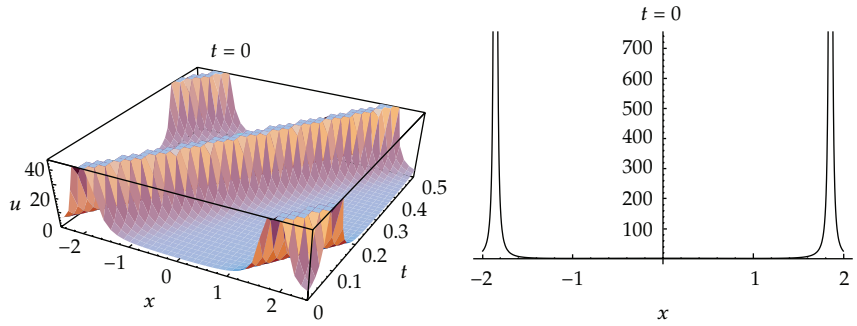


Figure 1: The modulus of solitary wave solution u_1 (3.10) where $m = 0.5$.

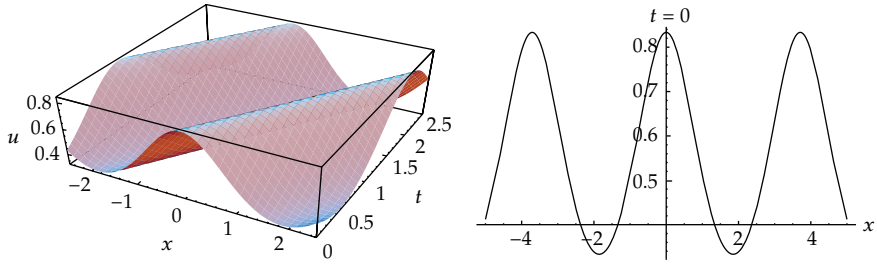
$$
\begin{aligned}
u_{49} &= -\frac{4}{3} - 2\tan^2(x+16t) - 2\cot^2(x+16t),\\
u_{50} &= \frac{1}{3} - \frac{1}{2}(\csc(x+4t)+\cot(x+4t))^2 - \frac{1}{2}(\csc(x+4t)+\cot(x+4t))^{-2},\\
u_{51} &= -\frac{1}{3} + \frac{1}{2}(\sec(x+7t)+\tan(x+7t))^2 + \frac{1}{2}(\sec(x+7t)+\tan(x+7t))^{-2},\\
u_{52} &= \frac{2}{3} - \sin^2(x-2t),\\
u_{53} &= -\frac{4}{3} - 2\cot^2(x+t),\\
u_{54} &= \frac{1}{3} - \frac{1}{2}\left(\csc\left(x+\frac{1}{4}t\right)+\cot\left(x+\frac{1}{4}t\right)\right)^{-2},\\
u_{55} &= -\frac{1}{3} + \frac{1}{2}\left(\sec\left(x-\frac{1}{2}t\right)+\tan\left(x-\frac{1}{2}t\right)\right)^{-2},\\
u_{56} &= \frac{2}{3} - \frac{1}{2}\sin^2\left(x+\frac{3}{8}t\right),\\
u_{57} &= -\frac{4}{3} - 2\tan^2(x+t),\\
u_{58} &= \frac{1}{3} - \frac{1}{2}\left(\csc\left(x+\frac{1}{4}t\right)+\cot\left(x+\frac{1}{4}t\right)\right)^{2},\\
u_{59} &= -\frac{1}{3} + \frac{1}{2}\left(\sec\left(x-\frac{1}{2}t\right)+\tan\left(x-\frac{1}{2}t\right)\right)^{2}.
\end{aligned}
\tag{3.41}
$$

The modulus of solitary wave solutions u_1, u_2, u_{21}, and u_{23} is displayed in Figures 1, 2, 3, and 4, respectively, with values of parameters listed in their captions.



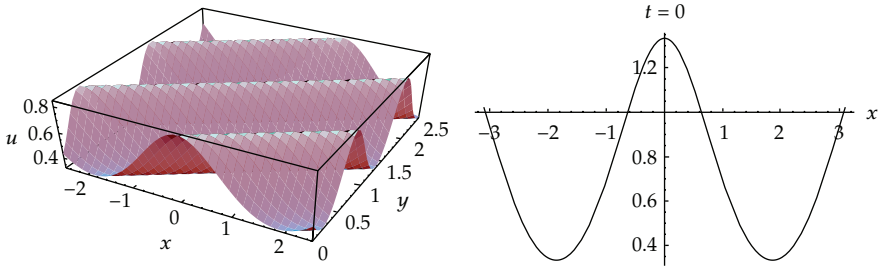
(a) (b)

Figure 2: The modulus of solitary wave solution u_2 (3.11) where $m = 0.5$.



(a) (b)

Figure 3: The modulus of solitary wave solution u_{21} (3.30) where $m = 0.5$.



(a) (b)

Figure 4: The modulus of solitary wave solution u_{23} (3.32) where $m = 0.5$.

4. Generalized (2+1)-Dimensional Ito Equation

In this section we consider the generalized (2+1)-dimensional Ito equation (1.2) as follows:

$$u_{tt} + u_{xxxt} + 3(2u_x u_t + u u_{xt}) + 3u_{xx} \int_{\infty}^{x} u_t dx' + \alpha u_{yt} + \beta u_{xt} = 0, \tag{4.1}$$

if we use the transformation $u = v_x$, it carries (4.1) into

$$v_{xtt} + v_{xxxxt} + 3(2v_{xx}v_{xt} + v_x v_{xxt}) + 3v_{xxx}v_t + \alpha v_{xyt} + \beta v_{xxt} = 0, \tag{4.2}$$

if we use $\zeta = x + y - vt$ carries (4.2) into the ODE, we have

$$(v - \alpha - \beta)V''' - V^{(v)} - 3\left((V')^2\right)'' = 0, \tag{4.3}$$

where by integrating twice we obtain, upon setting the constant of integration to zero,

$$(v - \alpha - \beta)V' - V''' - 3(V')^2 = 0, \tag{4.4}$$

if we use the transformation $W = V'$, it carries (4.4) into

$$(v - \alpha - \beta)W - W'' - 3W^2 = 0. \tag{4.5}$$

Balancing the term W'' with the term W^2, we obtain $N = 2$, then

$$W(\zeta) = a_0 + a_1\psi + a_{-1}\psi^{-1} + a_2\psi^2 + a_{-2}\psi^{-2}, \qquad \psi' = \sqrt{A + B\psi^2 + C\psi^4}. \tag{4.6}$$

Proceeding as in the previous case, we obtain
(1)

$$\begin{aligned} &a_{-1} = a_{-1} = 0, \qquad a_0 = -\frac{2B}{3}, \qquad a_2 = -2C, \\ &a_{-2} = -2A, \qquad v = \alpha + \beta - \frac{2(B^2 + 12AC)}{B}, \end{aligned} \tag{4.7}$$

(2)

$$a_{-1} = a_2 = a_{-1} = 0, \qquad a_0 = -\frac{2B}{3}, \qquad a_{-2} = -2A, \qquad v = \alpha + \beta - \frac{2(B^2 - 3AC)}{B}, \tag{4.8}$$

(3)

$$a_{-1} = a_{-2} = a_{-1} = 0, \qquad a_0 = -\frac{2B}{3}, \qquad a_2 = -2C, \qquad v = \alpha + \beta - \frac{2(B^2 - 3AC)}{B}. \tag{4.9}$$

The solutions of (4.1) are

$$u_1 = -\frac{2(1+m^2)}{3} - 2m^2\mathrm{sn}^2\left(x + \left(\alpha+\beta - \frac{2\left((1+m^2)^2+12m^2\right)}{1+m^2}\right)t\right)$$
$$-2\mathrm{ns}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((1+m^2)^2+12m^2\right)}{1+m^2}\right)t\right),$$

$$u_2 = -\frac{2(1+m^2)}{3} - 2m^2\mathrm{cd}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((1+m^2)^2+12m^2\right)}{1+m^2}\right)t\right)$$
$$-2\mathrm{dc}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((1+m^2)^2+12m^2\right)}{1+m^2}\right)t\right),$$

$$u_3 = -\frac{2(2m^2-1)}{3} + 2m^2\mathrm{cn}^2\left(x+y+\left(\alpha+\beta+\frac{2\left(12m^2(m^2-1)+(2m^2-1)^2\right)}{2m^2-1}\right)t\right)$$
$$-2\left(1-m^2\right)\mathrm{nc}^2\left(x+y+\left(\alpha+\beta+\frac{2\left(12m^2(m^2-1)+(2m^2-1)^2\right)}{2m^2-1}\right)t\right),$$

$$u_4 = -\frac{2(2-m^2)}{3} + 2\mathrm{dn}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((2-m^2)^2-12(-1+m^2)\right)}{2-m^2}\right)t\right)$$
$$-2\left(m^2-1\right)\mathrm{nd}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((2-m^2)^2-12(-1+m^2)\right)}{2-m^2}\right)t\right),$$

$$u_5 = -\frac{2(1+m^2)}{3} - 2\mathrm{ns}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((1+m^2)^2-3m^2\right)}{1+m^2}\right)t\right),$$

$$u_6 = -\frac{2(2-m^2)}{3} - 2\left(1-m^2\right)\mathrm{sc}^2\left(x+y+\left(\alpha+\beta+\frac{2\left(12(1-m^2)+(2-m^2)^2\right)}{2-m^2}\right)t\right)$$
$$-2\mathrm{cs}^2\left(x+y+\left(\alpha+\beta+\frac{2\left(12(1-m^2)+(2-m^2)^2\right)}{2-m^2}\right)t\right),$$

$$u_7 = \frac{2(2m^2-1)}{3} + 2m^2\left(1+m^2\right)\mathrm{sd}^2\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(-12m^2(1-m^2)+(-1+2m^2)^2\right)}{2m^2-1}\right)t\right)\right) - 2\mathrm{ds}^2\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(-12m^2(1-m^2)+(-1+2m^2)^2\right)}{2m^2-1}\right)t\right)\right),$$

$$u_8 = \frac{2(1-2m^2)}{6} - \frac{1}{2}\left(\mathrm{ns}\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75+(0.5-m^2)^2\right)}{0.5-m^2}\right)t\right)\right) + \mathrm{cs}\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75+(0.5-m^2)^2\right)}{0.5-m^2}\right)t\right)\right)\right)^2 - \frac{1}{2}\left(\mathrm{ns}\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75+(0.5-m^2)^2\right)}{0.5-m^2}\right)t\right)\right) + \mathrm{cs}\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75+(0.5-m^2)^2\right)}{0.5-m^2}\right)t\right)\right)\right)^{-2},$$

$$u_9 = -\frac{1+m^2}{3} + \frac{1-m^2}{2} \times \left(\mathrm{nc}\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2\right)}{0.5m^2+0.5}\right)t\right)\right) + \mathrm{sc}\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2\right)}{0.5m^2+0.5}\right)t\right)\right)\right)^2 + \frac{1-m^2}{2} \times \left(\mathrm{nc}\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2\right)}{0.5m^2+0.5}\right)t\right)\right) + \mathrm{sc}\left(\left(x+y+\left(\alpha+\beta+\frac{2\left(12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2\right)}{0.5m^2+0.5}\right)t\right)\right)\right)^{-2},$$

$$u_{10} = -\frac{m^2-2}{3} - \frac{m^2}{2}\left(\mathrm{ns}\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75m^2+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}\right)t\right)\right.$$
$$\left.+\mathrm{ds}\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75m^2+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}\right)t\right)\right)^2$$
$$-\frac{1}{2}\left(\mathrm{ns}\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75m^2+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}\right)t\right)\right.$$
$$\left.+\mathrm{ds}\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75m^2+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}\right)t\right)\right)^{-2},$$

$$u_{11} = \frac{m^2-2}{3} - \frac{m^2}{2}\left(\mathrm{sn}\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75m^4+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}\right)t\right)\right.$$
$$\left.+i\mathrm{cs}\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75m^4+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}\right)t\right)\right)^2$$
$$+\frac{m^2}{2}\left(\mathrm{sn}\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75m^4+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}\right)t\right)\right.$$
$$\left.+i\mathrm{cs}\left(x+y+\left(\alpha+\beta+\frac{2\left(0.75m^4+\left(-1+0.5m^2\right)^2\right)}{0.5m^2-1}\right)t\right)\right)^{-2},$$

$$u_{12} = -\frac{2\left(1+m^2\right)}{3} - 2\mathrm{dc}^2\left(x+y+\left(\alpha+\beta-\frac{2\left(\left(1+m^2\right)^2-3m^2\right)}{1+m^2}\right)t\right),$$

$$u_{13} = -\frac{2\left(2m^2-1\right)}{3} - 2\left(1-m^2\right)\mathrm{nc}^2\left(x+y+\left(\alpha+\beta+\frac{2\left(-3m^2\left(m^2-1\right)+\left(2m^2-1\right)^2\right)}{2m^2-1}\right)t\right),$$

$$u_{14} = -\frac{2\left(2-m^2\right)}{3} - 2\left(m^2-1\right)\mathrm{nd}^2\left(x+y+\left(\alpha+\beta-\frac{2\left(\left(2-m^2\right)^2+3\left(-1+m^2\right)\right)}{2-m^2}\right)t\right),$$

$$u_{15} = -\frac{2\left(2-m^2\right)}{3} - 2\mathrm{cs}^2\left(x+y+\left(\alpha+\beta+\frac{2\left(\left(2-m^2\right)^2-3\left(1-m^2\right)\right)}{2-m^2}\right)t\right),$$

$$u_{16} = \frac{2(2m^2-1)}{3} - 2\mathrm{ds}^2\left(x+y+\left(\alpha+\beta+\frac{2\left(3m^2(1-m^2)+(-1+2m^2)^2\right)}{2m^2-1}\right)t\right),$$

$$u_{17} = \frac{2(1-2m^2)}{6} - \frac{1}{2}\left(\mathrm{ns}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5-m^2)^2-3/16\right)}{0.5-m^2}\right)t\right)\right.$$
$$\left.+\mathrm{cs}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5-m^2)^2-3/16\right)}{0.5-m^2}\right)t\right)\right)^{-2},$$

$$u_{18} = -\frac{1+m^2}{3}+\frac{1-m^2}{2}$$
$$\times\left(\mathrm{nc}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5+0.5m^2)^2-3(0.5-0.5m^2)(0.25-0.25m^2)\right)}{0.5m^2+0.5}\right)t\right)\right.$$
$$\left.+\mathrm{sc}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5+0.5m^2)^2-3(0.5-0.5m^2)(0.25-0.25m^2)\right)}{0.5m^2+0.5}\right)t\right)\right)^{-2},$$

$$u_{19} = -\frac{m^2-2}{3}$$
$$-\frac{1}{2}\left(\mathrm{ns}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5m^2-1)^2-3/16\right)}{0.5m^2-1}\right)t\right)\right.$$
$$\left.+\mathrm{ds}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5m^2-1)^2-3/16\right)}{0.5m^2-1}\right)t\right)\right)^{-2},$$

$$u_{20} = \frac{m^2-2}{3}+\frac{m^2}{2}$$
$$\times\left(\mathrm{sn}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5m^2-1)^2-m^4/16\right)}{0.5m^2-1}\right)t\right)\right.$$
$$\left.+i\mathrm{cs}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5m^2-1)^2-m^4/16\right)}{0.5m^2-1}\right)t\right)\right)^{-2},$$

$$u_{21} = -\frac{2(1+m^2)}{3} - 2m^2\mathrm{sn}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((1+m^2)^2-3m^2\right)}{1+m^2}\right)t\right),$$

$$u_{22} = -\frac{2(1+m^2)}{3} - 2m^2\mathrm{cd}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((1+m^2)^2-3m^2\right)}{1+m^2}\right)t\right),$$

$$u_{23} = -\frac{2(2m^2-1)}{3} + 2m^2\mathrm{cn}^2\left(x+y+\left(\alpha+\beta+\frac{2\left((2m^2-1)^2-3m^2(m^2-1)\right)}{2m^2-1}\right)t\right),$$

$$u_{24} = -\frac{2(2-m^2)}{3} + 2\mathrm{dn}^2\left(x+y+\left(\alpha+\beta-\frac{2\left((2-m^2)^2+3(-1+m^2)\right)}{2-m^2}\right)t\right),$$

$$u_{25} = -\frac{2(2-m^2)}{3} - 2\left(1-m^2\right)\mathrm{sc}^2\left(x+y+\left(\alpha+\beta+\frac{2\left((2-m^2)^2-3(1-m^2)\right)}{2-m^2}\right)t\right),$$

$$u_{26} = \frac{2(2m^2-1)}{3} + 2m^2\left(1+m^2\right)\mathrm{sd}^2\left(x+y+\left(\alpha+\beta+\frac{2\left(3m^2(1-m^2)+(-1+2m^2)^2\right)}{2m^2-1}\right)t\right),$$

$$u_{27} = \frac{2(1-2m^2)}{6} - \frac{1}{2}\left(\mathrm{ns}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5-m^2)^2-3/16\right)}{0.5-m^2}\right)t\right) + \mathrm{cs}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5-m^2)^2-3/16\right)}{0.5-m^2}\right)t\right)\right)^2,$$

$$u_{28} = -\frac{1+m^2}{3} + \frac{1-m^2}{2} \times \left(\mathrm{nc}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5+0.5m^2)^2-3(0.5-0.5m^2)(0.25-0.25m^2)\right)}{0.5m^2+0.5}\right)t\right) + \mathrm{sc}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5+0.5m^2)^2-3(0.5-0.5m^2)(0.25-0.25m^2)\right)}{0.5m^2+0.5}\right)t\right)\right)^2,$$

$$u_{29} = -\frac{m^2-2}{3} - \frac{m^2}{2} \times \left(\mathrm{ns}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5m^2-1)^2-m^2/16\right)}{0.5m^2-1}\right)t\right) + \mathrm{ds}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5m^2-1)^2-m^2/16\right)}{0.5m^2-1}\right)t\right)\right)^2,$$

$$u_{30} = \frac{m^2-2}{3} - \frac{m^2}{2} \times \left(\mathrm{sn}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5m^2-1)^2 - m^4/16 \right)}{0.5m^2-1} \right) t \right) + i\mathrm{cs}\left(x+y+\left(\alpha+\beta+\frac{2\left((0.5m^2-1)^2 - m^4/16 \right)}{0.5m^2-1} \right) t \right) \right)^2. \tag{4.10}$$

4.1. Soliton Solutions

Some solitary wave solutions can be obtained, if the modulus m approaches to 1 in (4.10) as follows:

$$u_{31} = -\frac{4}{3} - 2\tanh^2(x+y+(\alpha+\beta-16)t) - 2\coth^2(x+y+(\alpha+\beta-16)t),$$

$$u_{32} = -\frac{2}{3} + 2\mathrm{sech}^2(x+y+(\alpha+\beta+2)t),$$

$$u_{33} = -\frac{2}{3} + 2\mathrm{sech}^2(x+y+(\alpha+\beta-2)t),$$

$$u_{34} = -\frac{4}{3} - 2\coth^2(x+y+(\alpha+\beta-1)t),$$

$$u_{35} = -\frac{2}{3} - 2\mathrm{csch}^2(x+y+(\alpha+\beta+2)t),$$

$$u_{36} = \frac{2}{3} + 4\sinh^2(x+y+(\alpha+\beta+2)t) - 2\mathrm{csch}^2(x+y+(\alpha+\beta+2)t),$$

$$u_{37} = \frac{-1}{3} - \frac{1}{2}\left(\coth(x+y+(\alpha+\beta-4)t) + \mathrm{csch}(x+y+(\alpha+\beta-4)t)\right)^2 - \frac{1}{2}\left(\coth(x+y+(\alpha+\beta-4)t) + \mathrm{csch}(x+y+(\alpha+\beta-4)t)\right)^{-2},$$

$$u_{38} = \frac{-1}{3} - \frac{1}{2}\left(\tanh(x+y+(\alpha+\beta-4)t) + i\mathrm{csch}(x+y+(\alpha+\beta-4)t)\right)^2 + \frac{1}{2}\left(\tanh(x+y+(\alpha+\beta-4)t) + i\mathrm{csch}(x+y+(\alpha+\beta-4)t)\right)^{-2},$$

$$u_{39} = \frac{-1}{3} - \frac{1}{2}\left(\coth\left(x+y+\left(\alpha+\beta-\frac{1}{4}\right)t\right) + \mathrm{csch}\left(x+y+\left(\alpha+\beta-\frac{1}{4}\right)t\right)\right)^{-2},$$

$$u_{40} = \frac{-1}{3} + \frac{1}{2}\left(\tanh\left(x+y+\left(\alpha+\beta-\frac{1}{4}\right)t\right) + i\mathrm{csch}\left(x+y+\left(\alpha+\beta-\frac{1}{4}\right)t\right)\right)^{-2},$$

$$u_{41} = -\frac{4}{3} - 2\tanh^2(x+y+(\alpha+\beta-1)t),$$

$$u_{42} = \frac{2}{3} + 4\sinh^2(x + y + (\alpha + \beta + 2)t),$$

$$u_{43} = \frac{-1}{3} - \frac{1}{2}\left(\coth\left(x + y + \left(\alpha + \beta - \frac{1}{4}\right)t\right) + \operatorname{csch}\left(x + y + \left(\alpha + \beta - \frac{1}{4}\right)t\right)\right)^2,$$

$$u_{44} = \frac{1}{3} - \frac{1}{2}(\coth(x + y + (\alpha + \beta + 2)t) + \operatorname{csch}(x + y + (\alpha + \beta + 2)t))^2,$$

$$u_{45} = \frac{-1}{3} - \frac{1}{2}\left(\tanh\left(x + y + \left(\alpha + \beta - \frac{3}{4}\right)t\right) + i\operatorname{csch}\left(x + y + \left(\alpha + \beta - \frac{3}{4}\right)t\right)\right)^2. \tag{4.11}$$

4.2. Triangular Periodic Solutions

Some trigonometric function solutions can be obtained, if the modulus m approaches to zero in (4.10) as follows:

$$u_{46} = -\frac{2}{3} - 2\csc^2(x + y + (\alpha + \beta - 2)t),$$

$$u_{47} = -\frac{2}{3} - 2\sec^2(x + y + (\alpha + \beta - 2)t),$$

$$u_{48} = \frac{2}{3} - 2\sec^2(x + y + (\alpha + \beta - 2)t),$$

$$u_{49} = -\frac{4}{3} - 2\tan^2(x + y + (\alpha + \beta + 16)t) - 2\cot^2(x + y + (\alpha + \beta + 16)t),$$

$$u_{50} = \frac{1}{3} - \frac{1}{2}(\csc(x + y + (\alpha + \beta + 4)t) + \cot(x + y + (\alpha + \beta + 4)t))^2$$
$$- \frac{1}{2}(\csc(x + y + (\alpha + \beta + 4)t) + \cot(x + y + (\alpha + \beta + 4)t))^{-2},$$

$$u_{51} = -\frac{1}{3} + \frac{1}{2}(\sec(x + y + (\alpha + \beta + 7)t) + \tan(x + y + (\alpha + \beta + 7)t))^2$$
$$+ \frac{1}{2}(\sec(x + y + (\alpha + \beta + 7)t) + \tan(x + y + (\alpha + \beta + 7)t))^{-2},$$

$$u_{52} = \frac{2}{3} - \sin^2(x + y + (\alpha + \beta - 2)t),$$

$$u_{53} = -\frac{4}{3} - 2\cot^2(x + y + (\alpha + \beta + 1)t),$$

$$u_{54} = \frac{1}{3} - \frac{1}{2}\left(\csc\left(x + y + \left(\alpha + \beta + \frac{1}{4}\right)t\right) + \cot\left(x + y + \left(\alpha + \beta + \frac{1}{4}\right)t\right)\right)^{-2},$$

$$u_{55} = -\frac{1}{3} + \frac{1}{2}\left(\sec\left(x + y + \left(\alpha + \beta - \frac{1}{2}\right)t\right) + \tan\left(x + y + \left(\alpha + \beta - \frac{1}{2}\right)t\right)\right)^{-2},$$

$$u_{56} = \frac{2}{3} - \frac{1}{2}\sin^2\left(x + y + \left(\alpha + \beta + \frac{3}{8}\right)t\right),$$

$$u_{57} = -\frac{4}{3} - 2\tan^2(x + y + (\alpha + \beta + 1)t),$$

$$u_{58} = \frac{1}{3} - \frac{1}{2}\left(\csc\left(x + y + \left(\alpha + \beta + \frac{1}{4}\right)t\right) + \cot\left(x + y + \left(\alpha + \beta + \frac{1}{4}\right)t\right)\right)^2,$$

$$u_{59} = -\frac{1}{3} + \frac{1}{2}\left(\sec\left(x + y + \left(\alpha + \beta - \frac{1}{2}\right)t\right) + \tan\left(x + y + \left(\alpha + \beta - \frac{1}{2}\right)t\right)\right)^2. \tag{4.12}$$

If we take $m \to 1$, in the two Sections 3 and 4, we obtain the solutions degenerated by the hyperbolic extended hyperbolic functions methods (tanh, coth, sinh, sech,..., etc.) (see, for example [42]). Moreover, when $m \to 0$, the solutions obtained by triangular and extended triangular functions methods (tan, sine, cosine, sec,..., etc.) are found as disused in Sections 3.1, 3.2, 4.1, and 4.2.

5. Conclusion

By introducing appropriate transformations and using extended F-expansion method, we have been able to obtain, in a unified way with the aid of symbolic computation system-mathematica, a series of solutions including single and the combined Jacobi elliptic function. Also, extended F-expansion method showed that soliton solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions. When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions and give the solutions by the extended hyperbolic functions methods. When $m \to 0$, the Jacobi functions degenerate to the triangular functions and give the solutions by extended triangular functions methods. In fact, the disadvantage of extended F-expansion method is the existence of complex solutions which are listed here just as solutions.

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