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Research Article

Chaos Generated by Switching Fractional Systems

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We, for the first time, investigate the basic behaviours of a chaotic switching fractional system via both theoretical and numerical ways. To deeply understand the mechanism of the chaos generation, we also analyse the parameterization of the switching fractional system and the dynamics of the system's trajectory. Then we try to write down some detailed rules for designing chaotic or chaos-like systems by switching fractional systems, which can be used in the future application. At last, for the first time, we proposed a new switching fractional system, which can generate three attractors with the positive largest Lyapunov exponent.

1. Introduction

Over the last two decades, since chaos has been demonstrated that it can be useful and well controlled [1–3], increasing interests have been found in many areas such as secure communication [4], data encryption [5], nonlinear optimization [6], synchronization [7], and biology [8] (for more areas, see [3]). Therefore, generating chaos, especially generating chaos via simple physical devices, has been paid extensive and massive attention [9–14]. Just like the n-scroll Chua's circuit [15], switching piecewise-linear function can easily generate various chaos dynamic behaviours. Especially the literatures [16–18] and give great details to generate chaos via switching systems.

On the other hand, fractional calculus is a mathematical branch which has more than 300 years of history but just been interested recently in physics [19], chemistry, biology, and engineering [20]. Since fractional-order calculus can be treated as an expanding concept of integer-order calculus, there are many chaos systems that are expanded from integer-order ones: the fractional-order Chua system [21, 22], the fractional-order Duffing system [23], the fractional jerk model [24], the fractional-order Lorenz system [25], the fractional-order Chen system [26], the fractional-order Lü system [27], the fractional-order Rössler system [28], the fractional-order Arneodo system [29], the fractional-order Newton-Leipnik system [30], and the fractional-order Genesio-Tesi system [31]. And also, the fractional-order Ikeda delay system [32], non-integer-order cellular neural networks [33], and the fractional-order systems proposed in [34, 35] all can generate chaos.

Generating a new chaotic or chaos-like system is always on the core of the chaos research with great theoretical and applied meanings. In addition a new chaotic system usually gets more complex dynamic behaviours being less recognized by people who does not catch up with the nonlinear dynamics. Thus, a new chaos system may be popular used in chaotic secure communication and encryption. Not like lots of the existing switching systems literatures [15–18], there are few literatures about switching fractional systems, even generating a fractional chaos easily is an increasing topic of many applied and theoretical fields. Also, as far as we know, there have not been basic dynamical behaviours given to any chaotic or chaos-like switching fractional systems ever. A significant work of switching fractional systems is proposed by S. Mohammad and H. Mohammad in [36]. The paper discussed the switching rules and how to choose the parameters to generate chaos and the rule of designing switching function have been discussed. However, the detailed relationship between the parameters and the behaviours of the chaos has not been discussed, not even being part of some analysis of the basic behaviours of the systems. In the present paper, we try to research the relationship between the parameters of the system and the behaviours of the chaos in both analytic and numerical ways. And we even epitomized some more detailed rules of generating chaotic or chaos-like dynamic behaviours via switching fractional systems then Mohammad did. Under these rules, we can generate chaotic or chaos-like dynamic behaviours easily via simple switching fractional systems. At last, we propose a new chaotic or chaos-like switching fractional system and count out its Lyapunov exponent.

The paper is organized as follows. Section 2 introduces basic definitions and some theories and lemmas, which are useful in following sections. In Section 3, we analyse an existing chaos generator and do numerical simulations to research quantificationally the relationship between the parameter and the dynamic behaviours. And we propose a new switching fractional systems, which can generate chaotic or chaos-like dynamic behaviours. Finally Section 4 concludes the paper.

2. Background of Fractional Calculus

2.1. Basic Definitions

There exits three main definitions of fractional-order derivatives. They are Grünwald-Letnikov fractional derivatives (G-L):

$$ {}_aD_t^{\alpha} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-\alpha} \sum\nolimits_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t-rh) \quad (\alpha > 0), \tag{2.1} $$

Riemann-Liouville fractional derivatives (R-L):

$$ {}_{\alpha}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau)d\tau \quad (m-1 \le \alpha < m,\ \alpha > 0), \tag{2.2} $$

and Caputo's fractional derivative:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}} \quad (n-1 \le \alpha < n,\ \alpha > 0). \tag{2.3} $$

And since they can transform to each other, our use of fractional derivatives can be free for all of these three definitions.

2.2. Some Theories and Lemmas

For the requirement in the next part, we list several theories and lemmas. All of them come from [37].

We first define $\gamma(\varepsilon, \varphi)$ [36]. $\gamma(\varepsilon, \varphi)$ $(\varepsilon > 0,\ 0 < \varphi < \pi)$ denotes the contour consisting of the following three parts:

(i) $\arg \tau = \varphi,\ |\tau| \ge \varepsilon$,

(ii) $-\varphi \le \arg \tau \le \varphi,\ |\tau| = \varepsilon$,

(iii) $\arg \tau = -\varphi,\ |\tau| \ge \varepsilon$.

More details of $\gamma(\varepsilon, \varphi)$ can be found in literature [36].

Lemma 2.1. *If $\alpha < 2$, $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$ and $\varepsilon > 0$ is arbitrary, for arbitrary complex z the following expansion holds:*

$$ \frac{1}{\Gamma(z)} = \frac{1}{2\pi\alpha i} \int_{\gamma(\varepsilon,\mu)} \exp\left(\varsigma^{1/\alpha}\right) \varsigma^{(1-z-\alpha)/\alpha} d\varsigma, \tag{2.4} $$

here $\Gamma(z)$ is Euler's gamma function.

Lemma 2.2. *If $\alpha > 0$, $\beta > 0$, one obtains*

$$ \int_0^z E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1} dt = z^\beta E_{\alpha,\beta+1}(\lambda z^\alpha). \tag{2.5} $$

Theorem 2.3. *If $0 < \alpha < 2$, β is an arbitrary complex number, and μ is an arbitrary real number such that*

$$ \frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}, \tag{2.6} $$

then for an arbitrary integer $p \ge 1$ the following expansion holds:

$$ \begin{gathered} E_{\alpha,\beta}(z) = -\sum\nolimits_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + I_p(z), \\ |z| \longrightarrow \infty, \quad \mu \le \left|\arg(z)\right| \le \pi, \end{gathered} \tag{2.7} $$

where

$$I_p(z) = \frac{1}{2\pi\alpha i z^p} \int_{\gamma(1,\phi)} \exp\left(\varsigma^{1/\alpha}\right) \varsigma^{(1-\beta)/\alpha+p} d\varsigma. \tag{2.8}$$

Using Lemma 2.1 and Theorem 2.3, if $0 < \alpha < 2$, $|z| \to \infty$, $\pi\alpha/2 < |\arg(z)| \leq \pi$, we can quickly obtain the following:

$$E_{\alpha,\alpha}(z) = \frac{z^{-1}}{\Gamma(-\alpha)}, \tag{2.9}$$

$$E_{\alpha,\alpha+1}(z) = \left(\frac{1}{\Gamma(1-\alpha)} - 1\right) z^{-1}, \tag{2.10}$$

which will be used in the next section of our paper.

Consider the following initial value problem for a nonhomogeneous fraction differential equation under nonzero initial conditions:

$$\begin{aligned} &{}_0D_t^{\alpha} y(t) - \lambda y(t) = h(t) \quad (t > 0), \\ &\left[{}_0D_t^{\alpha-k} y(t)\right]_{t=0} = b_k \quad (k = 1, 2, \ldots, n), \end{aligned} \quad n - 1 < \alpha < n, \tag{2.11}$$

then we obtain the following solution:

$$y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda (t-\tau)^{\alpha}\right) h(\tau) d\tau. \tag{2.12}$$

3. Chaos Generation

3.1. Analysis of an Existing Chaos Generator

First, we discuss the chaos generated by the switching fractional system S_1 and S_2 with the switching function (3.2):

$$\begin{aligned} S_1 &: \begin{cases} D^{\alpha_{11}} x_1 = a_1 x_1 + b_1 y_1, \\ D^{\alpha_{12}} y_1 = -b_1 x_1 + a_1 y_1, \\ D^{\alpha_{13}} z_1 = -c_1 z_1, \end{cases} \\ S_2 &: \begin{cases} D^{\alpha_{21}} x_2 = a_2 x_2 + b_2 y_2, \\ D^{\alpha_{22}} y_2 = -b_2 x_2 + a_2 y_2, \\ D^{\alpha_{23}} z_2 = -c_2 z_2 + p, \end{cases} \end{aligned} \tag{3.1}$$

where $0 < \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23} < 1$, $a_1, a_2, b_1, b_2, c_1, c_2, p \neq 0$,

$$g(x, y, z) = x^2 + y^2 + z^2 - 1. \tag{3.2}$$

This kind of chaos was proposed by the literature [36]. That paper has discussed how to design the switching rule to generate chaos, how to choose the parameters to generate chaos, and how to design the switching function to generate chaos. Here we want to discuss the dynamic behaviours of the chaos more carefully and research the relationship between the parameters of the system and the dynamic behaviours of the chaos more carefully and quantificationally.

Here we design the switching rule as follows: when S_1 is active, the system will switch to S_2 at the time $g(x_1(t), y_1(t), z_1(t)) \geq 0$ with the initial condition of S_2 being $(x_1(t), y_1(t), z_1(t))$. Similarly, when S_2 is active, the system will switch to S_1 at the time $g(x_2(t), y_2(t), z_2(t)) < 0$ with the initial condition of S_1 being $(x_2(t), y_2(t), z_2(t))$.

And we take S_2 to be asymptotically stable and take S_1 to be unstable. Then we get the restricted conditions of the parameters: $c_2 > 0$, $|b_2| > a_2 \max\{\tan(\alpha_{21}\pi/2), \tan(\alpha_{22}\pi/2)\}$, $|p/c_2| < 1$, and $c_1 < 0$ or $|b_1| < a_1 \tan(\alpha_{11}\pi/2)$ or $|b_1| < a_1 \tan(\alpha_{12}\pi/2)$, only one of the establishments of the last three conditions is enough. Here we choose $|b_1| < a_1 \min\{\tan(\alpha_{11}\pi/2), \tan(\alpha_{12}\pi/2)\}$ and make $c_1 > 0$, so that, under the discussion in paper [36], the switching fractional system will perform chaos.

Then we want to discuss the dynamic behaviours of the chaos more carefully. The switching function (3.2) and the switching rules divide the whole space into two regions, which we denote $\Sigma = \{(x, y, z) \mid x^2 + y^2 + z^2 - 1 \leq 0\}$ and $\overline{\Sigma} = \{(x, y, z) \mid x^2 + y^2 + z^2 - 1 > 0\}$, respectively. When the system is in $\overline{\Sigma}$, S_2 is active. Since S_2 is asymptotically stable, S_2 will converge to its fixed point, which will be discussed in the following. Either the system orbits reach or does not reach the fixed point, because $g(0, 0, p/c_2) < 0$, as we take above, the system orbits will go through the plane $x^2 + y^2 + z^2 = 1$ and then switch to Σ. When the system is in Σ, S_1 is active. Since S_1 is unstable, the system orbits will diverge and go through the plane $x^2 + y^2 + z^2 = 1$ and then switch to $\overline{\Sigma}$. We will discuss S_1 and S_2 in the following analytically, respectively.

We first rewrite S_2 as follows:

$$S_2 : D^{\alpha}\boldsymbol{\nu} = \mathbf{A}\boldsymbol{\nu} + \mathbf{U}, \tag{3.3}$$

where

$$\boldsymbol{\nu} = (x_2, y_2, z_2), \qquad \mathbf{A} = \begin{bmatrix} a_2 & b_2 & 0 \\ -b_2 & a_2 & 0 \\ 0 & 0 & -c_2 \end{bmatrix}, \qquad \mathbf{U} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{bmatrix}. \tag{3.4}$$

And we can obtain the eigenvalues: $\lambda_{21,22} = a_2 \pm ib_2$, $\lambda_{23} = -c_2$. Under the constricted conditions of the parameters, we can obtain

$$\frac{\alpha_{ij}\pi}{2} < \left|\arg(\lambda_{ij})\right| \quad (i = 2, j = 1, 2, 3). \tag{3.5}$$

Since the eigenvalues are different, we obtain a transformation matrix:

$$\mathbf{T} = \begin{bmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{2i} & \frac{1}{2} & 0 \\ \frac{-1}{2i} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.6}$$

so that we obtain

$$D^{\alpha}\boldsymbol{\nu}' = \boldsymbol{\Lambda}\boldsymbol{\nu}' + \mathbf{U}, \tag{3.7}$$

where

$$\boldsymbol{\nu}' = \mathbf{T}\boldsymbol{\nu}, \tag{3.8}$$
$$\boldsymbol{\Lambda} = \mathbf{TAT}^{-1} = \operatorname{diag}(\lambda_{21}, \lambda_{22}, \lambda_{23}). \tag{3.9}$$

Because of (3.5), we can solve the transformed fractional differential equations (3.7) in the solution (2.12) with (2.9), so that we obtain

$$x_2' = b_{21}\frac{\lambda_{21}t^{-1}}{\Gamma(-\alpha_{21})}, \tag{3.10}$$

$$b_{21} = \left[{}_0D_t^{\alpha_{21}-1}\, x_2(t)\right]_{t=0}, \tag{3.11}$$

$$y_2' = b_{22}\frac{\lambda_{22}t^{-1}}{\Gamma(-\alpha_{22})}, \tag{3.12}$$

$$b_{22} = \left[{}_0D_t^{\alpha_{22}-1}\, y_2(t)\right]_{t=0}, \tag{3.13}$$

$$z_2' = b_{23}\frac{\lambda_{23}^{-1}t^{-1}}{\Gamma(-\alpha_{23})} + p\int_0^t (t-\tau)^{\alpha_{23}-1}E_{\alpha_{23},\alpha_{23}}\left(\lambda_{23}(t-\tau)^{\alpha 23}\right)d\tau, \tag{3.14}$$

$$b_{23} = \left[{}_0D_t^{\alpha_{23}-1}\, z_2(t)\right]_{t=0}, \tag{3.15}$$

By using Lemma 2.2, we obtain

$$z_2' = b_{23}\frac{\lambda_{23}^{-1}t^{-1}}{\Gamma(-\alpha_{23})} + pt^{\alpha_{23}}E_{\alpha_{23},\alpha_{23}+1}(\lambda_{23}t^{\alpha_{23}}). \tag{3.16}$$

Taking (2.10) into account, we obtain

$$z_2' = b_{23}\frac{\lambda_{23}^{-1}t^{-1}}{\Gamma(-\alpha_{23})} + \frac{p}{\lambda_{23}}\left(\frac{1}{\Gamma(1-\alpha_{23})} - 1\right). \tag{3.17}$$

Using (3.8), we can obtain

$$x_2^2 + y_2^2 = \frac{1}{2}\left(-x_2'^2 + y_2'^2\right), \qquad z_2 = z_2', \tag{3.18}$$

taking (3.10) and (3.12) into account, we can obtain

$$x_2^2 + y_2^2 = \frac{1}{2}t^{-2}\left(-\frac{1}{\Gamma^2(-\alpha_{21})}\frac{b_{21}^2}{(a_2+ib_2)^2} + \frac{1}{\Gamma^2(-\alpha_{22})}\frac{b_{22}^2}{(a_2-ib_2)^2}\right). \tag{3.19}$$

when

$$t \longrightarrow +\infty, \quad \sqrt{x_2^2+y_2^2} \longrightarrow 0, \quad z_2 \longrightarrow \frac{p}{\lambda_{23}}\left(\frac{1}{\Gamma(1-\alpha_{23})} - 1\right). \tag{3.20}$$

So we see that the trajectory of S_2 is a spiral line. And $(0, 0, (p/\lambda_{23})(1/\Gamma(1-\alpha_{23})-1))$ is the fixed point of S_2. And we can slightly change the restricted condition of the parameters $|p/c_2| < 1$ to

$$\left|\frac{p}{-c_2}\left(\frac{1}{\Gamma(1-\alpha_{23})} - 1\right)\right| < 1. \tag{3.21}$$

For S_1, taking $c_1 > 0$, we can obtain

$$\frac{\alpha_{13}\pi}{2} < \left|\arg(\lambda_{13})\right|. \tag{3.22}$$

Then we can quickly write

$$z_1 = b_{13}\frac{\lambda_{13}^{-1}t^{-1}}{\Gamma(-\alpha_{13})}, \tag{3.23}$$

when

$$t \longrightarrow +\infty, \quad z_1 \longrightarrow 0 \tag{3.24}$$

So we see that the trajectory of S_1 will fall into the plane $z = 0$. For x_1, y_1, because of their complexity, we do not give their analytic solutions.

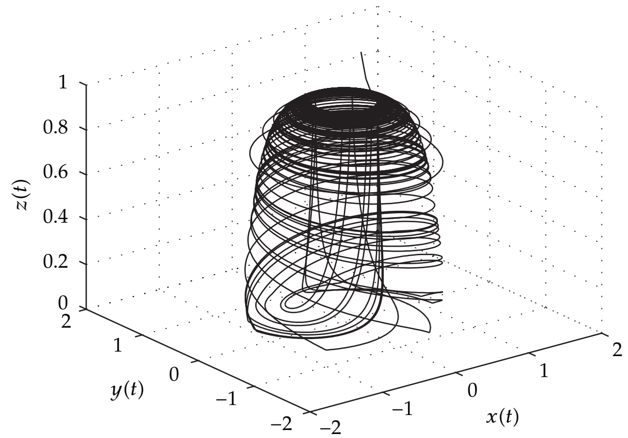


Figure 1: The chaotic attractor generated by the switching fractional system (3.1).

3.2. Numerical Simulations

Now we can do some numerical simulations and research in quantities of relationship between the parameters and the dynamic behaviours of the chaos. We take all the parameters under the restricted conditions we proposed above, so that the switching fractional system (3.1) will show chaos behaviours. Below in this part, we take $\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{22} = \alpha_{23} = 0.9$. Taking the parameters: $a_1 = 1$, $b_1 = 2$, $c_1 = 5$, $a_2 = 0.7$, $b_2 = 6$, $c_2 = 1$, $p = 1$, the switching fractional system (3.1) has a chaotic attractor, which is shown in Figure 1. The maximum Lyapunov exponent of this attractor is LE = 0.0155. And Figure 2 shows the directions of the trajectory of the switching fractional system (3.1) under the parameters: $a_1 = 1$, $b_1 = 2$, $c_1 = 5$, $a_2 = 0.6$, $b_2 = 5$, $c_2 = 1$, $p = 1$. The trajectory is denoted by the arrows.

We first focus on the parameters a_2 and b_2. In this segment, we fix the parameters: $a_1 = 1$, $b_1 = 2$, $c_1 = 5$, $c_2 = 1$, $p = 1$. We first take the parameter $a_2 = 0.7$ and increase b_2 from 6 to 7 with step 1. Figure 3 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 3, we can conclude that the increase of b_2 makes S_2 converge faster. This conclusion is established in all conditions, which satisfy $a_2 \in (-\infty, +\infty)$ and $|b_2| > a_2 \max\{\tan(\alpha_{21}\pi/2), \tan(\alpha_{22}\pi/2)\}$. Then we take the parameter $b_2 = 6$ and increase a_2 from 0.5 to 0.6 with step 0.1. Figure 4 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 4, we can conclude that the increase of a_2 makes S_2 converge slower. This conclusion is established in all conditions, which satisfy $a_2 \in (-\infty, +\infty)$ and $|b_2| > a_2 \max\{\tan(\alpha_{21}\pi/2), \tan(\alpha_{22}\pi/2)\}$.

Then we focus on the parameters c_2 and p. In this segment, we fix the parameters: $a_2 = 0.7$, $b_2 = 6$, $a_1 = 1$, $b_1 = 2$, $c_1 = 5$. We increase c_2 and p simultaneously from 1 to 2 with step 1. Figure 5 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 5, we can conclude that the increase of c_2 makes S_2 reach the plane

$$XYZ' = \left(0, 0, \frac{p}{-c_2}\left(\frac{1}{\Gamma(1-\alpha_{23})} - 1\right)\right) \tag{3.25}$$

faster. This conclusion is established in all conditions, which satisfy $c_2 > 0$.

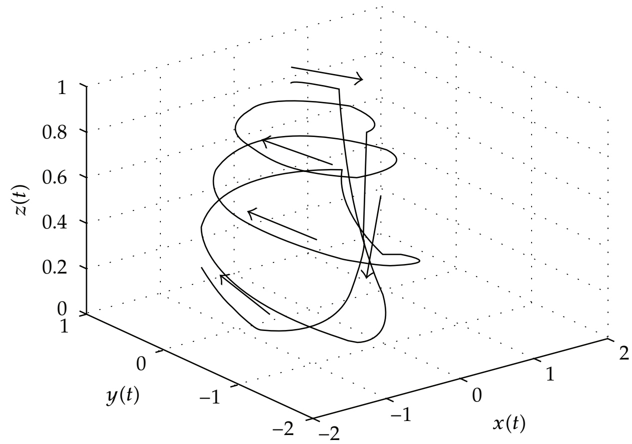
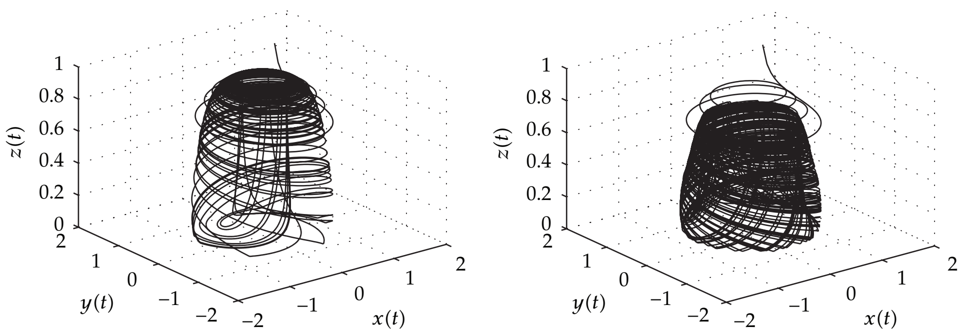


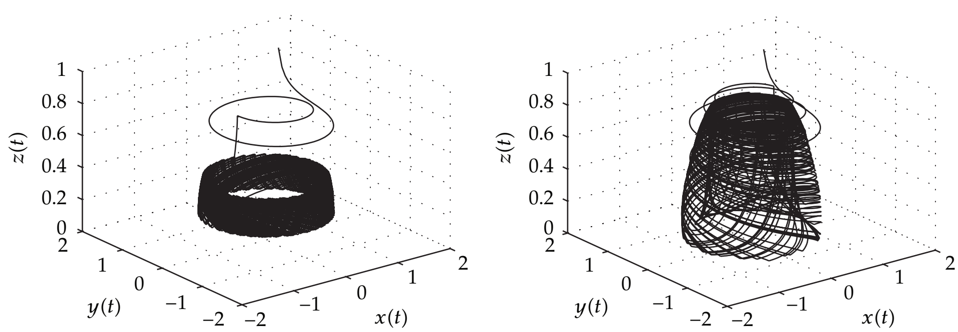
Figure 2: The trajectory of the switching fractional system (3.1).



(a) $b_2 = 6$

(b) $b_2 = 7$

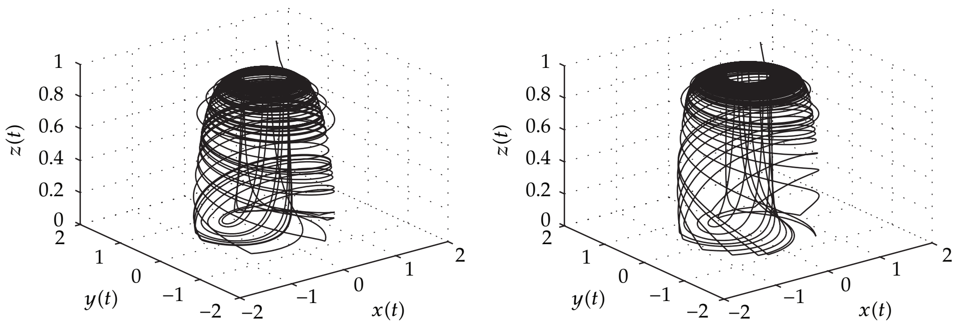
Figure 3: Phase portraits of the switching fractional system (3.1).



(a) $a_2 = 0.5$

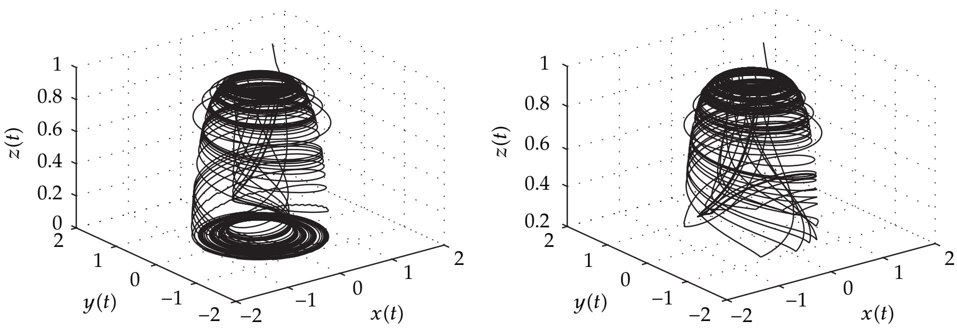
(b) $a_2 = 0.6$

Figure 4: Phase portraits of the switching fractional system (3.1).



(a) $c_2 = 1$ (b) $c_2 = 2$

Figure 5: Phase portraits of the switching fractional system (3.1).



(a) $a_1 = 1$ (b) $a_1 = 3$

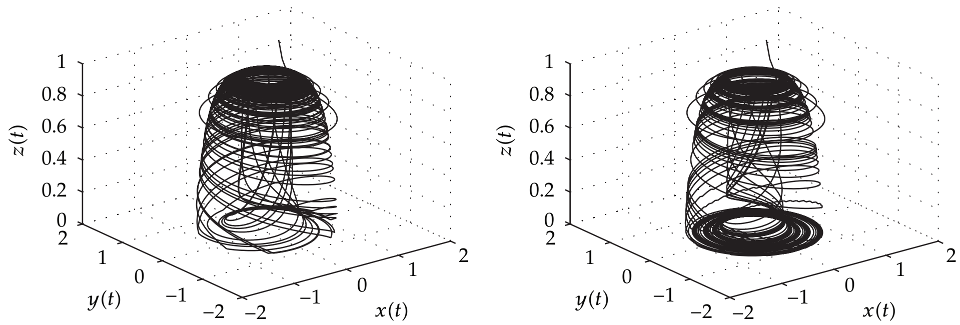
Figure 6: Phase portraits of the switching fractional system (3.1).

For S_1, we first focus on the parameters a_1 and b_1. In this segment, we fix the parameters: $c_1 = 4$, $a_2 = 0.7$, $b_2 = 6$, $c_2 = 1$, $p = 1$. We first take the parameter $b_1 = 6$ and increase a_1 from 1 to 3 with step 2. Figure 6 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 6, we can conclude that the increase of a_1 makes S_1 diverge faster. This conclusion is established in all conditions, which satisfy $a_1 \in (-\infty, +\infty)$ and $|b_1| < a_1 \min\{\tan(\alpha_{11}\pi/2), \tan(\alpha_{12}\pi/2)\}$. Then we take the parameter $a_1 = 1$ and increase b_1 from 3 to 6 with step 3. Figure 7 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 7, we can conclude that the increase of b_1 makes S_1 diverge slower. This conclusion is established in all conditions, which satisfy $a_1 \in (-\infty, +\infty)$ and $|b_1| < a_1 \min\{\tan(\alpha_{11}\pi/2), \tan(\alpha_{12}\pi/2)\}$.

Then we focus on the parameters c_1. In this segment, we fix the parameters: $a_1 = 1$, $b_1 = 2$, $a_2 = 0.7$, $b_2 = 6$, $c_2 = 1$, $p = 1$. We increase c_1 from 1 to 2 with step 1. Figure 8 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 8, we can conclude that the increase of c_1 makes S_1 reach the plane

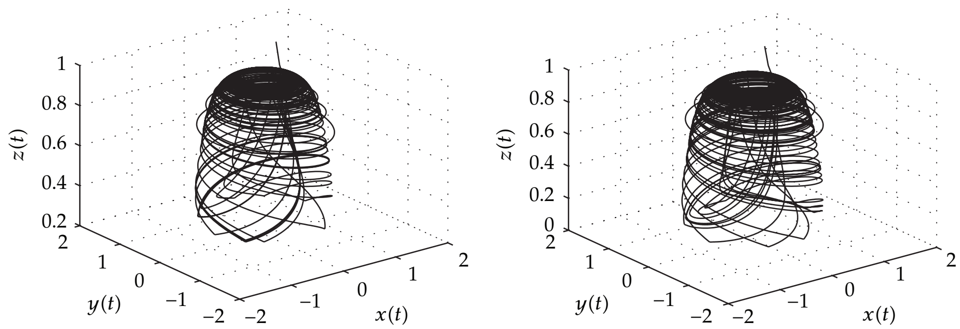
$$XYZ = (0, 0, 0) \tag{3.26}$$

faster. This conclusion is established in all conditions, which satisfy $c_1 > 0$.



(a) $b_1 = 3$ (b) $b_1 = 6$

Figure 7: Phase portraits of the switching fractional system (3.1).



(a) $c_1 = 2$ (b) $c_1 = 1$

Figure 8: Phase portraits of the switching fractional system (3.1).

3.3. More Detailed Rules to Design Chaotic or Chaos-Like Fractional Switching Systems

Here we epitomize some more detailed rules of relationship of the dynamic behaviours of the switching fractional systems and the parameters of the switching fractional systems. Take the switching fractional system (3.1) with the parameters under the restricted conditions of the parameters, which we proposed previously, for example. There are two fixed points of the switching fractional system (3.1). The parameters c_1, c_2, and p control the speed of the switching fractional system to reach the fixed points, respectively. The parameters a_1 and b_1 control the speed of S_1 to diverge. The parameters a_2 and b_2 control the speed of S_2 to converge. Taking the experiments, we did above, into account, we conclude that if we want the switching fractional system to arrive at the fixed points faster, we can increase the parameters c_1, c_2, and p. However, c_1, c_2, and p should also not be so large that the orbits of the switching fractional system will cross through the plane $x^2 + y^2 + z^2 = 1$, because the sum $x_1^2+y_1^2+z_1^2-1$ will be bigger or smaller than 0, respectively; if we want S_1 to diverge faster, we can increase the parameter a_1 or decrease the parameter b_1; if we want S_2 to converge faster, we can increase the parameter b_2 or decrease the parameter a_2.

So the rules are that we should balance all the parameters when we apply the rules obtained in the literature [36]. If we find that S_2 converges too fast, we may decrease the value of either b_2 or c_2 or increase a_2; it is also very similar to S_1. But remember if either S_1 or S_2 converges or diverges so fast that it cannot reach its own fixed point plane, the pattern we get may not show two attractors. Even if we obey these rules, we can design a chaotic or chaos-like switching fractional system with a wide range of the parameter values.

3.4. A New Chaotic or Chaos-Like Switching Fractional System

Under the rules above, we here propose a new chaotic or chaos-like switching fractional system S_1, S_2, and S_3 with the switching function (3.2). Actually, under the rules above, lots of chaotic or chaos-like switching fractional systems can also be proposed:

$$
\begin{gathered}
S_1 : \begin{cases} D^{\alpha_{11}} x_1 = a_1 x_1 + b_1 y_1, \\ D^{\alpha_{12}} y_1 = -b_1 x_1 + a_1 y_1, \\ D^{\alpha_{13}} z_1 = -c_1 z_1 + p_1, \end{cases} \\
S_2 : \begin{cases} D^{\alpha_{21}} x_2 = a_2 x_2 + b_2 y_2, \\ D^{\alpha_{22}} y_2 = -b_2 x_2 + a_2 y_2, \\ D^{\alpha_{23}} z_2 = -c_2 z_2, \end{cases} \\
S_3 : \begin{cases} D^{\alpha_{31}} x_3 = a_3 x_3 + b_3 y_3, \\ D^{\alpha_{32}} y_3 = -b_3 x_3 + a_3 y_3, \\ D^{\alpha_{33}} z_3 = -c_3 z_3 + p_2, \end{cases}
\end{gathered}
\tag{3.27}
$$

where $0 < \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{31}, \alpha_{32}, \alpha_{33} < 1$, $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, p_1, p_2 \neq 0$,

$$
g_1(x, y, z) = x^2 + y^2 + z^2 - 1, \qquad g_2(x, y) = xy. \tag{3.28}
$$

We take the switching rule as follows: when S_2 is active, the system will switch to S_1 at the time $g_1(x_2(t), y_2(t), z_2(t)) \geq 0$ and $g_2(x_2(t), y_2(t)) \geq 0$ with the initial condition of S_1 being $(x_2(t), y_2(t), z_2(t))$ and the system will switch to S_3 at the time $g_1(x_2(t), y_2(t), z_2(t)) \geq 0$ and $g_2(x_2(t), y_2(t)) < 0$ with the initial condition of S_1 being $(x_2(t), y_2(t), z_2(t))$. When S_1 is active, the system will switch to S_2 at the time $g_1(x_1(t), y_1(t), z_1(t)) < 0$ with the initial condition of S_2 being $(x_1(t), y_1(t), z_1(t))$. When S_3 is active, the system will switch to S_2 at the time $g_1(x_3(t), y_3(t), z_3(t)) < 0$ with the initial condition of S_2 being $(x_3(t), y_3(t), z_3(t))$.

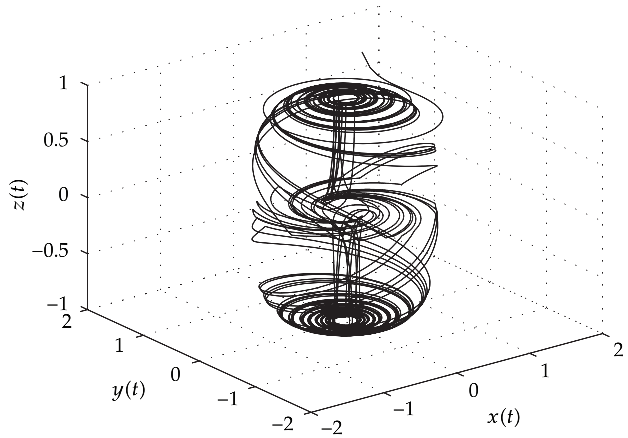


Figure 9: The chaotic attractor generated by the switching fractional system (3.27).

We take S_1, S_3 to be asymptotically stable, while we take S_2 to be unstable. Then we determine the parameters:

$$|b_1| > a_1 \max\left\{\tan\left(\frac{\alpha_{11}\pi}{2}\right), \tan\left(\frac{\alpha_{12}\pi}{2}\right)\right\}, \qquad c_1 > 0, \qquad \left|\frac{p_1}{-c_1}\left(\frac{1}{\Gamma(1-\alpha_{13})} - 1\right)\right| < 1,$$
$$|b_2| < a_2 \min\left\{\tan\left(\frac{\alpha_{21}\pi}{2}\right), \tan\left(\frac{\alpha_{22}\pi}{2}\right)\right\}, \qquad c_2 > 0,$$
$$|b_3| > a_3 \max\left\{\tan\left(\frac{\alpha_{31}\pi}{2}\right), \tan\left(\frac{\alpha_{32}\pi}{2}\right)\right\}, \qquad c_3 > 0, \qquad \left|\frac{p_3}{-c_3}\left(\frac{1}{\Gamma(1-\alpha_{33})} - 1\right)\right| < 1. \tag{3.29}$$

We take the parameters: $a_1 = 0.6$, $b_1 = 9$, $c_1 = 5$, $p_1 = 5$, $a_2 = 1$, $b_2 = 2$, $c_2 = 5$, $a_3 = 0.6$, $b_3 = 9$, $c_3 = 5$, $p_2 = -5$, $\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{22} = \alpha_{23} = \alpha_{31} = \alpha_{32} = \alpha_{33} = 0.9$. The switching fractional system (3.27), has a chaotic attractor, which is shown in Figure 9. The maximum Lyapunov exponent of this attractor is LE = 0.0434.

We take the parameters: $a_1 = 0.6$, $b_1 = 9$, $c_1 = 5$, $p_1 = 5$, $a_2 = 1$, $b_2 = 2$, $c_2 = 5$, $a_3 = 0.6$, $b_3 = 9$, $c_3 = 5$, $p_2 = -5$, $\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = \alpha_{31} = \alpha_{32} = 0.9$, $\alpha_{13} = \alpha_{23} = \alpha_{33} = 0.5$. The switching fractional system (3.27), has a chaotic attractor, which is shown in Figure 10. The maximum Lyapunov exponent of this attractor is LE = 0.0255. From Figure 10, we can obtain that the switching fractional system (3.27), under variant parameters α_{ij} $(i = 1, 2, 3,\ \ j = 1, 2, 3)$ can also perform chaotic or chaos-like dynamic behaviours.

4. Conclusions

In this paper, we for the first time study the basic behaviours of a chaotic switching fractional system both theoretically and numerically. We also analyse the parameterization of the switching fractional system and the dynamics of the system's trajectory. Then we try to write down some more detailed rules for designing chaotic or chaos-like systems by switching fractional systems based on the basic rules given in the literature [36]. At last, for the first

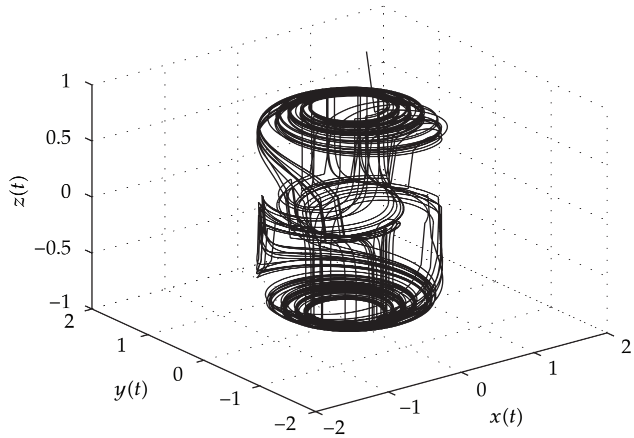


Figure 10: The chaotic attractor generated by the switching fractional system (3.27).

time, we proposed a new switching fractional system, which can generate three attractors with the positive largest Lyapunov exponent.

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