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Research Article

Numerical Solution of Nonlinear Volterra Integral Equations System Using Simpson's 3/8 Rule

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The Simpson's 3/8 rule is used to solve the nonlinear Volterra integral equations system. Using this rule the system is converted to a nonlinear block system and then by solving this nonlinear system we find approximate solution of nonlinear Volterra integral equations system. One of the advantages of the proposed method is its simplicity in application. Further, we investigate the convergence of the proposed method and it is shown that its convergence is of order $O(h^4)$. Numerical examples are given to show abilities of the proposed method for solving linear as well as nonlinear systems. Our results show that the proposed method is simple and effective.

1. Introduction

We consider the system of second kind Volterra integral equations (VIE) given by

$$f(x) = g(x) + \int_0^x K(x,s,f(s))ds, \quad 0 \le s \le x \le X, \tag{1.1}$$

where

$$f(x) = \left[f_1(x), f_2(x), \ldots, f_n(x)\right]^T, \qquad g(x) = \left[g_1(x), g_2(x), \ldots, g_n(x)\right]^T,$$
$$K(x,s,f(s)) = \begin{bmatrix} k_{1,1}(x,s,f_1,\ldots,f_n) & \cdots & k_{1,n}(x,s,f_1,\ldots,f_n) \\ \vdots & \ddots & \vdots \\ k_{n,1}(x,s,f_1,\ldots,f_n) & \cdots & k_{n,n}(x,s,f_1,\ldots,f_n) \end{bmatrix}. \tag{1.2}$$

Numerical solution of Volterra integral equations system has been considered by many authors. See for example [1–5].

In recent years, application of HPM (Homotopy Perturbation Method) and ADM (Adomian Decomposition Method) in nonlinear problems has been undertaken by several scientists and engineers [6–8]. HPM [8] was proposed by He in 1999 for the first time and recently Yusufoğlu has proposed this method [9] for solving a system of Fredholm-Volterra type integral equations. Block by block method was suggested by Young [10] for the first time in connection with product integration techniques. On the other side, the engineers are facing certain challenges to deal with complexity and efficient mathematical modeling. Thus, there are several research have being carried out related to these problems. For example, more details on the modeling of complexity we refer to [11], direct operational method to solve a system of linear in-homogenous couple fractional differential equations, see [12] and for a class of fractional oscillatory system, see [13].

In this paper, we consider block by block method by using Simpson's 3/8 rule for solving linear and nonlinear systems of Volterra integral equations. And we make a block by block method comparison between our method and HPM.

This paper is organized as follow: in Section 2, we present some background material on the use of this method. In Section 3, we prove the convergence result. Finally, numerical results are given in Section 4.

2. Starting Method

Consider a system of nonlinear Volterra integral equations in (1.1) and further, we suppose that the system equation (1.1) has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of (1.1) can be found in [14]. The idea behind the block by block methods is quite general but is most easily understood by considering a specific. Let us assume that $m = 2$ in (1.1) and use the Simpson's 3/8 rule as a numerical integration formulae. Let $F_{i,j} \simeq f_i(x_j)$ then

$$\begin{aligned} F_{1,3} \simeq f_1(x_3) &= g_1(x_3) + \int_0^{x_3} k_{1,1}\big(x_3, s, f_1(s)\big)ds + \int_0^{x_3} k_{1,2}\big(x_3, s, f_2(s)\big)ds, \\ F_{2,3} \simeq f_2(x_3) &= g_2(x_3) + \int_0^{x_3} k_{2,1}\big(x_3, s, f_1(s)\big)ds + \int_0^{x_3} k_{2,2}\big(x_3, s, f_2(s)\big)ds, \end{aligned} \tag{2.1}$$

approximating the integrals by Simpson's 3/8 rule, we have

$$\begin{aligned} F_{1,3} = g_1(x_3) &+ \frac{3h}{8} \\ &\times \{k_{1,1}(x_3, x_0, F_{1,0}) + 3k_{1,1}(x_3, x_1, F_{1,1}) + 3k_{1,1}(x_3, x_2, F_{1,2}) + k_{1,1}(x_3, x_3, F_{1,3})\} + \frac{3h}{8} \\ &\times \{k_{1,2}(x_3, x_0, F_{2,0}) + 3k_{1,2}(x_3, x_1, F_{2,1}) + 3k_{1,2}(x_3, x_2, F_{2,2}) + k_{1,2}(x_3, x_3, F_{2,3})\}, \end{aligned}$$

$$
\begin{aligned}
F_{2,3} = {} & g_2(x_3) + \frac{3h}{8} \\
& \times \{k_{2,1}(x_3, x_0, F_{1,0}) + 3k_{2,1}(x_3, x_1, F_{1,1}) + 3k_{2,1}(x_3, x_2, F_{1,2}) + k_{2,1}(x_3, x_3, F_{1,3})\} + \frac{3h}{8} \\
& \times \{k_{2,2}(x_3, x_0, F_{2,0}) + 3k_{2,2}(x_3, x_1, F_{2,1}) + 3k_{2,2}(x_3, x_2, F_{2,2}) + k_{2,2}(x_3, x_3, F_{2,3})\},
\end{aligned}
\tag{2.2}
$$

where

$$
F_{1,0} = g_1(x_0), \qquad F_{2,0} = g_2(x_0). \tag{2.3}
$$

Further we also get

$$
\begin{aligned}
F_{1,2} &\simeq f_1(x_2) = g_1(x_2) + \int_0^{x_2} k_{1,1}\big(x_2, s, f_1(s)\big)ds + \int_0^{x_2} k_{1,2}\big(x_2, s, f_2(s)\big)ds, \\
F_{2,2} &\simeq f_2(x_2) = g_2(x_2) + \int_0^{x_2} k_{2,1}\big(x_2, s, f_1(s)\big)ds + \int_0^{x_2} k_{2,2}\big(x_2, s, f_2(s)\big)ds.
\end{aligned}
\tag{2.4}
$$

In order to evaluate the integrals on the right hand sides, we introduce the points $x_{2/3} = 2h/3$, $x_{4/3} = 4h/3$ and the corresponding values $F_{2/3}$, $F_{4/3}$ and use the Simpson's $3/8$ rule with step size $2h/3$, then we gain

$$
\begin{aligned}
F_{1,2} = {} & g_1(x_2) + \frac{h}{4} \\
& \times \{k_{1,1}(x_2, x_0, F_{1,0}) + 3k_{1,1}(x_2, x_{2/3}, F_{1,2/3}) + 3k_{1,1}(x_2, x_{4/3}, F_{1,4/3}) + k_{1,1}(x_2, x_2, F_{1,2})\} \\
& + \frac{h}{4}\{k_{1,2}(x_2, x_0, F_{2,0}) + 3k_{1,2}(x_2, x_{2/3}, F_{2,2/3}) + 3k_{1,2}(x_2, x_{4/3}, F_{2,4/3}) + k_{1,2}(x_2, x_2, F_{2,2})\}, \\
F_{2,2} = {} & g_2(x_2) + \frac{h}{4} \\
& \times \{k_{2,1}(x_2, x_0, F_{1,0}) + 3k_{2,1}(x_2, x_{2/3}, F_{1,2/3}) + 3k_{2,1}(x_2, x_{4/3}, F_{1,4/3}) + k_{2,1}(x_2, x_2, F_{1,2})\} \\
& + \frac{h}{4}\{k_{2,2}(x_2, x_0, F_{2,0}) + 3k_{2,2}(x_2, x_{2/3}, F_{2,2/3}) + 3k_{2,2}(x_2, x_{4/3}, F_{2,4/3}) + k_{2,2}(x_2, x_2, F_{2,2})\},
\end{aligned}
\tag{2.5}
$$

where $F_{1,2/3}$, $F_{1,4/3}$, $F_{2,2/3}$, $F_{2,4/3}$ have unknown values, which can be estimated by Lagrange interpolation points x_0, x_1, x_2, x_3. Therefore we obtain

$$
\begin{aligned}
l_0(x_{2/3}) &= \frac{14}{81}, \qquad l_1(x_{2/3}) = \frac{28}{27}, \qquad l_2(x_{2/3}) = -\frac{7}{27}, \qquad l_3(x_{2/3}) = \frac{4}{81}, \\
l_0(x_{4/3}) &= -\frac{5}{81}, \qquad l_1(x_{4/3}) = \frac{20}{27}, \qquad l_2(x_{4/3}) = \frac{10}{27}, \qquad l_3(x_{4/3}) = -\frac{4}{81}.
\end{aligned}
\tag{2.6}
$$

Thus,

$$\begin{aligned}
F_{1,2/3} &= \frac{14}{81}F_{1,0} + \frac{28}{27}F_{1,1} - \frac{7}{27}F_{1,2} + \frac{4}{81}F_{1,3},\\
F_{1,4/3} &= -\frac{5}{81}F_{1,0} + \frac{20}{27}F_{1,1} + \frac{10}{27}F_{1,2} - \frac{4}{81}F_{1,3},\\
F_{2,2/3} &= \frac{14}{81}F_{2,0} + \frac{28}{27}F_{2,1} - \frac{7}{27}F_{2,2} + \frac{4}{81}F_{2,3},\\
F_{2,4/3} &= -\frac{5}{81}F_{2,0} + \frac{20}{27}F_{2,1} + \frac{10}{27}F_{2,2} - \frac{4}{81}F_{2,3}.
\end{aligned} \tag{2.7}$$

Substituting from (2.7) into (2.5) we obtain the following values for $F_{1,2}$, $F_{2,2}$

$$\begin{aligned}
F_{1,2} &= g_1(x_2) + \frac{h}{4}\\
&\quad\times\left\{k_{1,1}(x_2,x_0,F_{1,0}) + 3k_{1,1}\times\left(x_2,x_{2/3},\frac{14}{81}F_{1,0}+\frac{28}{27}F_{1,1}-\frac{7}{27}F_{1,2}+\frac{4}{81}F_{1,3}\right)\right.\\
&\qquad\left.+3k_{1,1}\left(x_2,x_{4/3},-\frac{5}{81}F_{1,0}+\frac{20}{27}F_{1,1}+\frac{10}{27}F_{1,2}-\frac{4}{81}F_{1,3}\right)+k_{1,1}(x_2,x_2,F_{1,2})\right\}+\frac{h}{4}\\
&\quad\times\left\{k_{1,2}(x_2,x_0,F_{2,0}) + 3k_{1,2}\left(x_2,x_{2/3},\frac{14}{81}F_{2,0}+\frac{28}{27}F_{2,1}-\frac{7}{27}F_{2,2}+\frac{4}{81}F_{2,3}\right)\right.\\
&\qquad\left.+3k_{1,2}\left(x_2,x_{4/3},-\frac{5}{81}F_{2,0}+\frac{20}{27}F_{2,1}+\frac{10}{27}F_{2,2}-\frac{4}{81}F_{2,3}\right)+k_{1,2}(x_2,x_2,F_{2,2})\right\},\\
F_{2,2} &= g_2(x_2) + \frac{h}{4}\\
&\quad\times\left\{k_{2,1}(x_2,x_0,F_{1,0}) + 3k_{2,1}\times\left(x_2,x_{2/3},\frac{14}{81}F_{1,0}+\frac{28}{27}F_{1,1}-\frac{7}{27}F_{1,2}+\frac{4}{81}F_{1,3}\right)\right.\\
&\qquad\left.+3k_{2,1}\left(x_2,x_{4/3},-\frac{5}{81}F_{1,0}+\frac{20}{27}F_{1,1}+\frac{10}{27}F_{1,2}-\frac{4}{81}F_{1,3}\right)+k_{2,1}(x_2,x_2,F_{1,2})\right\}\\
&\quad+\frac{h}{4}\left\{k_{2,2}(x_2,x_0,F_{2,0}) + 3k_{2,2}\left(x_2,x_{2/3},\frac{14}{81}F_{2,0}+\frac{28}{27}F_{2,1}-\frac{7}{27}F_{2,2}+\frac{4}{81}F_{2,3}\right)\right.\\
&\qquad\left.+3k_{2,2}\left(x_2,x_{4/3},-\frac{5}{81}F_{2,0}+\frac{20}{27}F_{2,1}+\frac{10}{27}F_{2,2}-\frac{4}{81}F_{2,3}\right)+k_{2,2}(x_2,x_2,F_{2,2})\right\}.
\end{aligned} \tag{2.8}$$

Also we get

$$\begin{aligned}
F_{1,1} &\simeq f_1(x_1) = g_1(x_1) + \int_0^{x_1} k_{1,1}\big(x_1,s,f_1(s)\big)ds + \int_0^{x_1} k_{1,2}\big(x_1,s,f_2(s)\big)ds,\\
F_{2,1} &\simeq f_2(x_1) = g_2(x_1) + \int_0^{x_1} k_{2,1}\big(x_1,s,f_1(s)\big)ds + \int_0^{x_1} k_{2,2}\big(x_1,s,f_2(s)\big)ds,
\end{aligned} \tag{2.9}$$

to evaluate the integrals on the right-hand sides, we introduce points $x_{1/3} = h/3$, $x_{2/3} = 2h/3$ and the corresponding values $F_{1/3}$, $F_{2/3}$ and use the Simpson's 3/8 rule with step size $h/3$. Therefore we have

$$\begin{aligned}
F_{1,1} &= g_1(x_1) + \frac{h}{8} \\
&\quad \times \{k_{1,1}(x_1, x_0, F_{1,0}) + 3k_{1,1}(x_1, x_{1/3}, F_{1,1/3}) + 3k_{1,1}(x_1, x_{2/3}, F_{1,2/3}) + k_{1,1}(x_1, x_1, F_{1,1})\} \\
&\quad + \frac{h}{8}\{k_{1,2}(x_1, x_0, F_{2,0}) + 3k_{1,2}(x_1, x_{1/3}, F_{2,1/3}) + 3k_{1,2}(x_1, x_{2/3}, F_{2,2/3}) + k_{1,2}(x_1, x_1, F_{2,1})\}, \\
F_{2,1} &= g_2(x_1) + \frac{h}{8} \\
&\quad \times \{k_{2,1}(x_1, x_0, F_{1,0}) + 3k_{2,1}(x_1, x_{1/3}, F_{1,1/3}) + 3k_{2,1}(x_1, x_{2/3}, F_{1,2/3}) + k_{2,1}(x_1, x_1, F_{1,1})\} \\
&\quad + \frac{h}{8}\{k_{2,2}(x_1, x_0, F_{2,0}) + 3k_{2,2}(x_1, x_{1/3}, F_{2,1/3}) + 3k_{2,2}(x_1, x_{2/3}, F_{2,2/3}) + k_{2,2}(x_1, x_1, F_{2,1})\},
\end{aligned} \tag{2.10}$$

where $F_{1,1/3}$, $F_{1,2/3}$, $F_{2,1/3}$, $F_{2,2/3}$ have unknown values, similarly, that can be estimated by Lagrange interpolation points x_0, x_1, x_2, x_3. As a result we obtain

$$\begin{aligned}
l_0(x_{1/3}) &= \frac{40}{81}, \quad l_1(x_{1/3}) = \frac{20}{27}, \quad l_2(x_{1/3}) = -\frac{8}{27}, \quad l_3(x_{1/3}) = \frac{5}{81}, \\
l_0(x_{2/3}) &= \frac{14}{81}, \quad l_1(x_{2/3}) = \frac{28}{27}, \quad l_2(x_{2/3}) = -\frac{7}{27}, \quad l_3(x_{2/3}) = \frac{4}{81},
\end{aligned} \tag{2.11}$$

and so

$$\begin{aligned}
F_{1,1/3} &= \frac{40}{81}F_{1,0} + \frac{20}{27}F_{1,1} - \frac{8}{27}F_{1,2} + \frac{5}{81}F_{1,3}, \\
F_{1,2/3} &= \frac{14}{81}F_{1,0} + \frac{28}{27}F_{1,1} - \frac{7}{27}F_{1,2} + \frac{4}{81}F_{1,3}, \\
F_{2,1/3} &= \frac{40}{81}F_{2,0} + \frac{20}{27}F_{2,1} - \frac{8}{27}F_{2,2} + \frac{5}{81}F_{2,3}, \\
F_{2,2/3} &= \frac{14}{81}F_{2,0} + \frac{28}{27}F_{2,1} - \frac{7}{27}F_{2,2} + \frac{4}{81}F_{2,3}.
\end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.10), we obtain the following values for $F_{1,1}$ and $F_{2,1}$:

$$\begin{aligned}
F_{1,1} &= g_1(x_1) + \frac{h}{8} \\
&\quad \times \left\{ k_{1,1}(x_1, x_0, F_{1,0}) + 3k_{1,1} \times \left(x_1, x_{1/3}, \frac{40}{81}F_{1,0} + \frac{20}{27}F_{1,1} - \frac{8}{27}F_{1,2} + \frac{5}{81}F_{1,3} \right) \right. \\
&\qquad \left. + 3k_{1,1} \times \left(x_1, x_{2/3}, \frac{14}{81}F_{1,0} + \frac{28}{27}F_{1,1} - \frac{7}{27}F_{1,2} + \frac{4}{81}F_{1,3} \right) + k_{1,1}(x_1, x_1, F_{1,1}) \right\} \\
&\quad + \frac{h}{8} \left\{ k_{1,2}(x_1, x_0, F_{2,0}) + 3k_{1,2} \times \left(x_1, x_{1/3}, \frac{40}{81}F_{2,0} + \frac{20}{27}F_{2,1} - \frac{8}{27}F_{2,2} + \frac{5}{81}F_{2,3} \right) \right. \\
&\qquad \left. + 3k_{1,2} \left(x_1, x_{2/3}, \frac{14}{81}F_{2,0} + \frac{28}{27}F_{2,1} - \frac{7}{27}F_{2,2} + \frac{4}{81}F_{2,3} \right) + k_{1,2}(x_1, x_1, F_{2,1}) \right\}, \\
F_{2,1} &= g_2(x_1) + \frac{h}{8} \\
&\quad \times \left\{ k_{2,1}(x_1, x_0, F_{1,0}) + 3k_{2,1} \left(x_1, x_{1/3}, \frac{40}{81}F_{1,0} + \frac{20}{27}F_{1,1} - \frac{8}{27}F_{1,2} + \frac{5}{81}F_{1,3} \right) \right. \\
&\qquad \left. + 3k_{2,1} \left(x_1, x_{2/3}, \frac{14}{81}F_{1,0} + \frac{28}{27}F_{1,1} - \frac{7}{27}F_{1,2} + \frac{4}{81}F_{1,3} \right) + k_{2,1}(x_1, x_1, F_{1,1}) \right\} \\
&\quad + \frac{h}{8} \left\{ k_{2,2}(x_1, x_0, F_{2,0}) + 3k_{2,2} \times \left(x_1, x_{1/3}, \frac{40}{81}F_{2,0} + \frac{20}{27}F_{2,1} - \frac{8}{27}F_{2,2} + \frac{5}{81}F_{2,3} \right) \right. \\
&\qquad \left. + 3k_{2,2} \left(x_1, x_{2/3}, \frac{14}{81}F_{2,0} + \frac{28}{27}F_{2,1} - \frac{7}{27}F_{2,2} + \frac{4}{81}F_{2,3} \right) + k_{2,2}(x_1, x_1, F_{2,1}) \right\}.
\end{aligned} \tag{2.13}$$

Equations (2.2), (2.8), and (2.13) are three pairs of simultaneous equations in terms of unknowns $F_{1,1}$, $F_{2,1}$, $F_{1,2}$, $F_{2,2}$, $F_{1,3}$, and $F_{2,3}$ for the nonlinear system of VIE. The solutions of these equations may be found by the method of successive approximation or by a suitable software package such as Maple. For the linear system of VIE a direct method can be used for solving system of linear algebraic equations.

3. The General Scheme

Consider the system of VIE

$$f(x) = g(x) + \int_0^x K(x, s, f(s))ds, \quad 0 \le x \le a. \tag{3.1}$$

Let $0 = x_0 < x_1 < \cdots < x_N = a$ be a partition of $[0, a]$ with the step size h, such that $x_i = ih$ for $i = 0, 1, \ldots, N$. Then we can construct a block by block approach so that a system

of p simultaneous equations is obtained and thus a block of p values of F is also obtained simultaneously. We put $p = 6$ for simplicity. Setting $x = x_{3m+1}$ in (3.1) we get

$$\begin{aligned} F_{1,3m+1} &\simeq f_1(x_{3m+1}) = g_1(x_{3m+1}) + \int_0^{x_{3m+1}} k_{1,1}(x_{3m+1}, s, f_1(s))ds + \int_0^{x_{3m+1}} k_{1,2}(x_{3m+1}, s, f_2(s))ds, \\ F_{2,3m+1} &\simeq f_2(x_{3m+1}) = g_2(x_{3m+1}) + \int_0^{x_{3m+1}} k_{2,1}(x_{3m+1}, s, f_1(s))ds + \int_0^{x_{3m+1}} k_{2,2}(x_{3m+1}, s, f_2(s))ds \end{aligned} \tag{3.2}$$

or equivalently

$$\begin{aligned} F_{1,3m+1} &= g_1(x_{3m+1}) + \int_0^{x_{3m}} k_{1,1}(x_{3m+1}, s, f_1(s))ds + \int_0^{x_{3m}} k_{1,2}(x_{3m+1}, s, f_2(s))ds \\ &\quad + \int_{x_{3m}}^{x_{3m+1}} k_{1,1}(x_{3m+1}, s, f_1(s))ds + \int_{x_{3m}}^{x_{3m+1}} k_{1,2}(x_{3m+1}, s, f_2(s))ds, \\ F_{2,3m+1} &= g_2(x_{3m+1}) + \int_0^{x_{3m}} k_{2,1}(x_{3m+1}, s, f_1(s))ds + \int_0^{x_{3m}} k_{2,2}(x_{3m+1}, s, f_2(s))ds \\ &\quad + \int_{x_{3m}}^{x_{3m+1}} k_{2,1}(x_{3m+1}, s, f_1(s))ds + \int_{x_{3m}}^{x_{3m+1}} k_{2,2}(x_{3m+1}, s, f_2(s))ds. \end{aligned} \tag{3.3}$$

Now, integration over $[0, x_{3m}]$ can be accomplished by Simpson's $3/8$ rule and the integral over $[x_{3m}, x_{3m+1}]$ is computed by using a cubic interpolation. Hence

$$\begin{aligned} F_{1,3m+1} &= g_1(x_{3m+1}) + \frac{3h}{8} \\ &\quad \times \{k_{1,1}(x_{3m+1}, x_0, F_{1,0}) + 3k_{1,1}(x_{3m+1}, x_1, F_{1,1}) \\ &\quad +3k_{1,1}(x_{3m+1}, x_2, F_{1,2}) + 2k_{1,1}(x_{3m+1}, x_3, F_{1,3}) + \cdots + k_{1,1}(x_{3m+1}, x_{3m}, F_{1,3m})\} \\ &\quad + \frac{3h}{8}\{k_{1,2}(x_{3m+1}, x_0, F_{2,0}) + 3k_{1,2}(x_{3m+1}, x_1, F_{2,1}) \\ &\quad +3k_{1,2}(x_{3m+1}, x_2, F_{2,2}) + 2k_{1,2}(x_{3m+1}, x_3, F_{2,3}) + \cdots + k_{1,2}(x_{3m+1}, x_{3m}, F_{2,3m})\} \\ &\quad + \frac{h}{8}\Bigg\{k_{1,1}(x_{3m+1}, x_{3m}, F_{1,3m}) + 3k_{1,1} \\ &\quad \times \left(x_{3m+1}, x_{3m+1/3}, \frac{40}{81}F_{1,3m} + \frac{20}{27}F_{1,3m+1} - \frac{8}{27}F_{1,3m+2} + \frac{5}{81}F_{1,3m+3}\right) \\ &\quad + 3k_{1,1}\left(x_{3m+1}, x_{3m+2/3}, \frac{14}{81}F_{1,3m} + \frac{28}{27}F_{1,3m+1} - \frac{7}{27}F_{1,3m+2} + \frac{4}{81}F_{1,3m+3}\right) \\ &\quad +k_{1,1}(x_{3m+1}, x_{3m+1}, F_{1,3m+1})\Bigg\} + \frac{h}{8} \end{aligned}$$

$$\times \Bigg\{ k_{1,2}(x_{3m+1}, x_{3m}, F_{2,3m}) + \\ + 3k_{1,2}\left(x_{3m+1}, x_{3m+1/3}, \frac{40}{81}F_{2,3m} + \frac{20}{27}F_{2,3m+1} - \frac{8}{27}F_{2,3m+2} + \frac{5}{81}F_{2,3m+3} \right) \\ + 3k_{1,2}\left(x_{3m+1}, x_{3m+2/3}, \frac{14}{81}F_{2,3m} + \frac{28}{27}F_{2,3m+1} - \frac{7}{27}F_{2,3m+2} + \frac{4}{81}F_{2,3m+3} \right) \\ + k_{1,2}(x_{3m+1}, x_{3m+1}, F_{2,3m+1}) \Bigg\}, \tag{3.4}$$

and in a similar way the right-hand side is obtained for $F_{2,3m+1}$, where $F_{1,0} = g_1(x_0)$, $F_{2,0} = g_2(x_0)$.

By setting $x = x_{3m+2}$ in (3.1) we get

$$\begin{aligned} F_{1,3m+2} \simeq f_1(x_{3m+2}) &= g_1(x_{3m+2}) + \int_0^{x_{3m+2}} k_{1,1}(x_{3m+2}, s, f_1(s))ds \\ &\quad + \int_0^{x_{3m+2}} k_{1,2}(x_{3m+2}, s, f_2(s))ds, \\ F_{2,3m+2} \simeq f_2(x_{3m+2}) &= g_2(x_{3m+2}) + \int_0^{x_{3m+2}} k_{2,1}(x_{3m+2}, s, f_1(s))ds \\ &\quad + \int_0^{x_{3m+2}} k_{2,2}(x_{3m+2}, s, f_2(s))ds \end{aligned} \tag{3.5}$$

or equivalently

$$\begin{aligned} F_{1,3m+2} &= g_1(x_{3m+2}) + \int_0^{x_{3m}} k_{1,1}(x_{3m+2}, s, f_1(s))ds + \int_0^{x_{3m}} k_{1,2}(x_{3m+2}, s, f_2(s))ds \\ &\quad + \int_{x_{3m}}^{x_{3m+2}} k_{1,1}(x_{3m+2}, s, f_1(s))ds + \int_{x_{3m}}^{x_{3m+2}} k_{1,2}(x_{3m+2}, s, f_2(s))ds, \\ F_{2,3m+2} &= g_2(x_{3m+2}) + \int_0^{x_{3m}} k_{2,1}(x_{3m+2}, s, f_1(s))ds + \int_0^{x_{3m}} k_{2,2}(x_{3m+2}, s, f_2(s))ds \\ &\quad + \int_{x_{3m}}^{x_{3m+2}} k_{2,1}(x_{3m+2}, s, f_1(s))ds + \int_{x_{3m}}^{x_{3m+2}} k_{2,2}(x_{3m+2}, s, f_2(s))ds. \end{aligned} \tag{3.6}$$

Now, the integration over $[0, x_{3m}]$ can be accomplished by Simpson's 3/8 rule and the integral over $[x_{3m}, x_{3m+2}]$ is computed by using a cubic interpolation. Hence

$$F_{1,3m+2} = g_1(x_{3m+2}) + \frac{3h}{8} \\ \times \{k_{1,1}(x_{3m+2}, x_0, F_{1,0}) + 3k_{1,1}(x_{3m+2}, x_1, F_{1,1}) + 3k_{1,1}(x_{3m+2}, x_2, F_{1,2}) \\ + 2k_{1,1}(x_{3m+2}, x_3, F_{1,3}) + \cdots + k_{1,1}(x_{3m+2}, x_{3m}, F_{1,3m})\}$$

$$
\begin{aligned}
&+\frac{3h}{8}\{k_{1,2}(x_{3m+2},x_0,F_{2,0})+3k_{1,2}(x_{3m+2},x_1,F_{2,1})+3k_{1,2}(x_{3m+2},x_2,F_{2,2})\\
&\quad+2k_{1,2}(x_{3m+2},x_3,F_{2,3})+\cdots+k_{1,2}(x_{3m+2},x_{3m},F_{2,3m})\}\\
&+\frac{h}{4}\Bigg\{k_{1,1}(x_{3m+2},x_{3m},F_{1,3m})+3k_{1,1}\\
&\quad\times\left(x_{3m+2},x_{3m+2/3},\frac{14}{81}F_{1,3m}+\frac{28}{27}F_{1,3m+1}-\frac{7}{27}F_{1,3m+2}+\frac{4}{81}F_{1,3m+3}\right)+3k_{1,1}\\
&\quad\times\left(x_{3m+2},x_{3m+4/3},-\frac{5}{81}F_{1,3m}+\frac{20}{27}F_{1,3m+1}+\frac{10}{27}F_{1,3m+2}-\frac{4}{81}F_{1,3m+3}\right)\\
&\quad+k_{1,1}(x_{3m+2},x_{3m+2},F_{1,3m+2})\Bigg\}+\frac{h}{4}\\
&\times\Bigg\{k_{1,2}(x_{3m+2},x_{3m},F_{2,3m})+3k_{1,2}\\
&\quad\times\left(x_{3m+2},x_{3m+2/3},\frac{14}{81}F_{2,3m}+\frac{28}{27}F_{2,3m+1}-\frac{7}{27}F_{2,3m+2}+\frac{4}{81}F_{2,3m+3}\right)\\
&\quad+3k_{1,2}\left(x_{3m+2},x_{3m+4/3},-\frac{5}{81}F_{2,3m}+\frac{20}{27}F_{2,3m+1}+\frac{10}{27}F_{2,3m+2}-\frac{4}{81}F_{2,3m+3}\right)\\
&\quad+k_{1,2}(x_{3m+2},x_{3m+2},F_{2,3m+2})\Bigg\},
\end{aligned}
\tag{3.7}
$$

and a similar right-hand side obtains for $F_{2,3m+2}$, where $F_{1,0}=g_1(x_0)$, $F_{2,0}=g_2(x_0)$. In a similar manner we obtain

$$
\begin{aligned}
F_{1,3m+3}\simeq f_1(x_{3m+3})&=g_1(x_{3m+3})+\int_0^{x_{3m+3}}k_{1,1}\big(x_{3m+3},s,f_1(s)\big)ds\\
&\quad+\int_0^{x_{3m+3}}k_{1,2}\big(x_{3m+3},s,f_2(s)\big)ds,\\
F_{2,3m+3}\simeq f_2(x_{3m+3})&=g_2(x_{3m+3})+\int_0^{x_{3m+3}}k_{2,1}\big(x_{3m+3},s,f_1(s)\big)ds\\
&\quad+\int_0^{x_{3m+3}}k_{2,2}\big(x_{3m+3},s,f_2(s)\big)ds,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
F_{1,3m+3}&=g_1(x_{3m+3})+\frac{3h}{8}\\
&\quad\times\{k_{1,1}(x_{3m+3},x_0,F_{1,0})+3k_{1,1}(x_{3m+3},x_1,F_{1,1})\\
&\quad+3k_{1,1}(x_{3m+3},x_2,F_{1,2})+2k_{1,1}(x_{3m+3},x_3,F_{1,3})\\
&\quad+\cdots+k_{1,1}(x_{3m+3},x_{3m+3},F_{1,3m+3})\}+\frac{3h}{8}
\end{aligned}
$$

$$\times \{k_{1,2}(x_{3m+3}, x_0, F_{2,0}) + 3k_{1,2}(x_{3m+3}, x_1, F_{2,1}) + 3k_{1,2}(x_{3m+3}, x_2, F_{2,2}) + 2k_{1,2}(x_{3m+3}, x_3, F_{2,3}) + \cdots + k_{1,2}(x_{3m+3}, x_{3m+3}, F_{2,3m+3})\},$$

$$F_{2,3m+3} = g_2(x_{3m+3}) + \frac{3h}{8} \times \{k_{2,1}(x_{3m+3}, x_0, F_{1,0}) + 3k_{2,1}(x_{3m+3}, x_1, F_{1,1}) + 3k_{2,1}(x_{3m+3}, x_2, F_{1,2}) + 2k_{2,1}(x_{3m+3}, x_3, F_{1,3}) + \cdots + k_{2,1}(x_{3m+3}, x_{3m+3}, F_{1,3m+3})\} + \frac{3h}{8} \times \{k_{2,2}(x_{3m+3}, x_0, F_{2,0}) + 3k_{2,2}(x_{3m+3}, x_1, F_{2,1}) + 3k_{2,2}(x_{3m+3}, x_2, F_{2,2}) + 2k_{2,2}(x_{3m+3}, x_3, F_{2,3}) + \cdots + k_{2,2}(x_{3m+3}, x_{3m+3}, F_{2,3m+3})\}. \tag{3.9}$$

Equations (3.4)–(3.9) form a system with six unknowns for $m = 1, 2, \ldots$. In fact, we have six simultaneous equations at each step.

4. Convergence Analysis

In this section we investigate the convergence of the proposed method. The following theorem shows that the order of convergence is at least four.

Theorem 4.1. *The approximate method given by the systems Equations* (3.4), (3.7), *and* (3.9), *is convergent and its order of convergence is at least four.*

Proof. We have

$$\begin{aligned}
|\varepsilon_{1,3m+1}| &= \left|F_{1,3m+1} - f_1(x_{3m+1})\right| \\
&= \left| h\sum_{i=0}^{3m} w_i k_{1,1}(x_{3m+1}, x_i, F_{1,i}) + h\sum_{i=0}^{3m} w_i k_{1,2}(x_{3m+1}, x_i, F_{2,i}) \right. \\
&\quad + \frac{h}{8}k_{1,1}(x_{3m+1}, x_{3m}, F_{1,3m}) + \frac{3h}{8}k_{1,1} \\
&\quad \times \left(x_{3m+1}, x_{3m+1/3}, \frac{40}{81}F_{1,3m} + \frac{20}{27}F_{1,3m+1} - \frac{8}{27}F_{1,3m+2} + \frac{5}{81}F_{1,3m+3}\right) \\
&\quad + \frac{3h}{8}k_{1,1}\left(x_{3m+1}, x_{3m+2/3}, \frac{14}{81}F_{1,3m} + \frac{28}{27}F_{1,3m+1} - \frac{7}{27}F_{1,3m+2} + \frac{4}{81}F_{1,3m+3}\right) \\
&\quad \left. + \frac{h}{8}k_{1,1}(x_{3m+1}, x_{3m+1}, F_{1,3m+1}) + \frac{h}{8}k_{1,2}(x_{3m+1}, x_{3m}, F_{2,3m}) \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3h}{8}k_{1,2}\left(x_{3m+1},x_{3m+1/3},\frac{40}{81}F_{2,3m}+\frac{20}{27}F_{2,3m+1}-\frac{8}{27}F_{2,3m+2}+\frac{5}{81}F_{2,3m+3}\right)\\
&+\frac{3h}{8}k_{1,2}\left(x_{3m+1},x_{3m+2/3},\frac{14}{81}F_{2,3m}+\frac{28}{27}F_{2,3m+1}-\frac{7}{27}F_{2,3m+2}+\frac{4}{81}F_{2,3m+3}\right)\\
&+\frac{h}{8}k_{1,2}(x_{3m+1},x_{3m+1},F_{2,3m+1})\\
&-\int_0^{x_{3m+1}}k_{1,1}(x_{3m+1},s,f_1(s))ds-\int_0^{x_{3m+1}}k_{1,2}(x_{3m+1},s,f_2(s))ds\Bigg|,
\end{aligned}
\tag{4.1}
$$

using the Lipschitz condition it can be written as

$$
\begin{aligned}
|\varepsilon_{1,3m+1}|\le{}& hc_1\sum_{i=0}^{3m}|\varepsilon_{1,i}|+hc_2\sum_{i=0}^{3m}|\varepsilon_{2,i}|+hc_3|\varepsilon_{1,3m+1}|+hc_4|\varepsilon_{2,3m+1}|\\
&+hc_5|\varepsilon_{1,3m+2}|+hc_6|\varepsilon_{2,3m+2}|+hc_7|\varepsilon_{1,3m+3}|+hc_8|\varepsilon_{2,3m+3}|\\
&+|R_{1,3m+1}|+|R_{2,3m+1}|+|R_{1,3m+2}|+|R_{2,3m+2}|,
\end{aligned}
\tag{4.2}
$$

where $R_{i,3m+1}$, $R_{i,3m+2}$ $(i=1,2)$ are the errors of integration rule. Without loss of generality, we assume that

$$
\left\|\varepsilon_{l,j}\right\|_\infty=\max_{l=1,2}\max_{j=3m+1,3m+2,3m+3}\left|\varepsilon_{l,j}\right|=|\varepsilon_{1,3m+1}|,
\tag{4.3}
$$

then let $R=\max_i[|R_{1,i}|,|R_{2,i}|]$, hence

$$
\begin{aligned}
\left\|\varepsilon_{l,j}\right\|_\infty&\le hc\sum_{i=0}^{3m}(|\varepsilon_{1,i}|+|\varepsilon_{2,i}|)+6hc'\left\|\varepsilon_{l,j}\right\|_\infty+4R,\\
\left\|\varepsilon_{l,j}\right\|_\infty&\le\frac{hc}{1-6hc'}\sum_{i=0}^{3m}(|\varepsilon_{1,i}|+|\varepsilon_{2,i}|)+\frac{4R}{1-6hc'},
\end{aligned}
\tag{4.4}
$$

then from Gronwall inequality, we have

$$
\left\|\varepsilon_{l,j}\right\|_\infty\le\frac{4R}{1-6hc'}e^{(c/(1-6hc'))x_n}.
\tag{4.5}
$$

For functions k and f with at least fourth-order derivatives, we have $R=o(h^4)$ and so $\|\varepsilon_m\|_\infty=o(h^4)$ and the proof is completed. □

5. Numerical Results

In this section, some examples are given to certify the convergence and error bounds of the presented method. All results are computed using the well-known symbolic software

Table 1: Numerical results of Example 5.1.

x	e_{HPM} in [9]		$h = 0.1$		$h = 0.05$	
	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$
0.1	$9.678e^{-07}$	$4.711e^{-05}$	0	0	0	0
0.2	$3.002e^{-05}$	$7.111e^{-04}$	0	0	0	0
0.3	$2.211e^{-04}$	$3.399e^{-03}$	0	0	0	0
0.4	$9.053e^{-04}$	$1.015e^{-02}$	0	0	0	0
0.5	$2.687e^{-03}$	$2.346e^{-02}$	0	$1.0e^{-10}$	0	$1.0e^{-10}$
0.6	$6.511e^{-03}$	$4.605e^{-02}$	0	0	0	0
0.7	$1.372e^{-02}$	$8.084e^{-02}$	0	$1.0e^{-10}$	0	$1.0e^{-10}$
0.8	$2.609e^{-02}$	0.131	$1.0e^{-10}$	$1.0e^{-10}$	0	$1.0e^{-10}$
0.9	$2.929e^{-02}$	0.198	0	0	0	0
1.0	$7.601e^{-02}$	0.287	0	0	0	0

Table 2: Numerical results of Example 5.2.

x	$h = 0.1$		$h = 0.05$		$h = 0.025$	
	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$
0.1	$8.76e^{-08}$	$5.4267e^{-07}$	$2.5e^{-09}$	$4.23e^{-09}$	$3.e^{-10}$	$5.3e^{-10}$
0.2	$1.582e^{-07}$	$1.721e^{-07}$	$1.93e^{-08}$	$1.96e^{-08}$	$1.1e^{-09}$	$3.0e^{-10}$
0.3	$9.821e^{-07}$	$1.480e^{-07}$	$5.86e^{-08}$	$9.1e^{-09}$	$3.6e^{-09}$	$6.0e^{-10}$
0.4	$1.279e^{-06}$	$9.462e^{-07}$	$7.26e^{-08}$	$2.73e^{-08}$	$6.0e^{-09}$	$2.1e^{-09}$
0.5	$1.5475e^{-06}$	$8.662e^{-07}$	$1.432e^{-07}$	$6.44e^{-08}$	$8.7e^{-09}$	$3.1e^{-09}$
0.6	$3.7774e^{-06}$	$1.2556e^{-06}$	$2.320e^{-07}$	$8.35e^{-08}$	$1.44e^{-08}$	$5.3e^{-09}$
0.7	$4.4688e^{-06}$	$2.7058e^{-06}$	$2.707e^{-07}$	$1.421e^{-07}$	$1.92e^{-08}$	$9.4e^{-09}$
0.8	$5.1935e^{-06}$	$3.4142e^{-06}$	$3.935e^{-07}$	$2.295e^{-07}$	$2.43e^{-08}$	$1.38e^{-08}$
0.9	$8.4753e^{-06}$	$4.7428e^{-06}$	$5.286e^{-07}$	$3.101e^{-07}$	$3.30e^{-08}$	$2.0e^{-08}$
1.0	$9.8341e^{-06}$	$7.3373e^{-06}$	$6.107e^{-07}$	$4.427e^{-07}$	$4.04e^{-08}$	$2.86e^{-08}$

Maple 12. Tables 1–3 show the obtained results for nonlinear examples and Tables 4–6 contain the results for linear examples. The results in Tables 1–6 show the absolute errors $|f(x_i) - Fi|$, $i = 1, 2, \ldots, N$, where $f(x_i)$ is the exact solution evaluated at $x = x_i$ and F_i is the corresponding approximate solution.

Example 5.1. Consider the following system [9]:

$$\begin{aligned} x - x^2 + \int_0^x \left(f_1(s) + f_2(s)\right) ds &= f_1(x), \quad 0 \le x \le 1 \\ x - \frac{x^2}{2} - \frac{x^3}{3} + \int_0^x \left(f_1^2(s) + f_2(s)\right) ds &= f_2(x), \quad 0 \le x \le 1 \end{aligned} \tag{5.1}$$

with the exact solutions $f_1(x) = x$ and $f_2(x) = x$. In Table 1 we compare the errors $e(f_1)$ and $e(f_2)$ obtained using the present method and method in [9].

Table 3: Numerical results of Example 5.3.

x	$h = 0.1$		$h = 0.05$		$h = 0.025$	
	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$
0.1	$3.398e^{-06}$	$3.4931e^{-06}$	$3.4e^{-08}$	$8.21e^{-08}$	$3.0e^{-09}$	$1.49e^{-08}$
0.2	$1.853e^{-06}$	$3.2536e^{-06}$	$1.36e^{-07}$	$5.302e^{-07}$	$2.0e^{-09}$	$2.76e^{-08}$
0.3	$1.934e^{-06}$	$1.55679e^{-05}$	$8.8e^{-08}$	$9.580e^{-07}$	$5.0e^{-09}$	$5.88e^{-08}$
0.4	$1.0510e^{-05}$	$2.52780e^{-05}$	$2.53e^{-07}$	$1.0938e^{-06}$	$1.7e^{-08}$	$9.17e^{-08}$
0.5	$1.1155e^{-05}$	$2.21338e^{-05}$	$6.96e^{-07}$	$2.3675e^{-06}$	$2.9e^{-08}$	$1.245e^{-07}$
0.6	$1.7852e^{-05}$	$6.29575e^{-05}$	$9.93e^{-07}$	$3.9457e^{-06}$	$6.0e^{-08}$	$2.441e^{-07}$
0.7	$6.2599e^{-05}$	$1.201563e^{-04}$	$2.259e^{-06}$	$4.5288e^{-06}$	$1.44e^{-07}$	$4.090e^{-07}$
0.8	$8.9572e^{-05}$	$9.5467e^{-05}$	$5.569e^{-06}$	$1.2368e^{-05}$	$2.90e^{-07}$	$6.01e^{-07}$
0.9	$1.70559e^{-04}$	$3.70854e^{-04}$	$1.0249e^{-05}$	$2.4445e^{-05}$	$6.51e^{-07}$	$1.533e^{-06}$
1.0	$7.91069e^{-04}$	$1.185001e^{-03}$	$2.6732e^{-05}$	$3.0637e^{-05}$	$1.730e^{-06}$	$3.391e^{-06}$

Table 4: Numerical results of Example 5.4.

x	$h = 0.1$		$h = 0.05$		$h = 0.025$	
	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$
0.1	$2.848e^{-07}$	$2.375e^{-07}$	$2.39e^{-08}$	$9.0e^{-10}$	$4.7e^{-09}$	$2.0e^{-10}$
0.2	$7.777e^{-07}$	$3.0e^{-10}$	$1.835e^{-07}$	$5.0e^{-09}$	$1.07e^{-08}$	$9.0e^{-10}$
0.3	$5.6610e^{-06}$	$5.69e^{-08}$	$3.433e^{-07}$	$2.75e^{-08}$	$2.13e^{-08}$	$2.3e^{-09}$
0.4	$5.5049e^{-06}$	$7.029e^{-07}$	$3.371e^{-07}$	$8.39e^{-08}$	$2.36e^{-08}$	$5.6e^{-09}$
0.5	$5.5930e^{-06}$	$2.1033e^{-06}$	$4.788e^{-07}$	$1.254e^{-07}$	$2.96e^{-08}$	$1.30e^{-08}$
0.6	$1.03890e^{-05}$	$2.115e^{-06}$	$6.279e^{-07}$	$1.97e^{-07}$	$3.86e^{-08}$	$1.3e^{-08}$
0.7	$1.00566e^{-05}$	$5.211e^{-06}$	$6.128e^{-07}$	$4.05e^{-07}$	$4.12e^{-08}$	$2.4e^{-08}$
0.8	$1.01662e^{-05}$	$9.770e^{-06}$	$7.676e^{-07}$	$6.02e^{-07}$	$4.56e^{-08}$	$3.9e^{-08}$
0.9	$1.56375e^{-05}$	$1.2629e^{-05}$	$09.480e^{-07}$	$9.28e^{-07}$	$5.81e^{-08}$	$6.2e^{-08}$
1.0	$1.60249e^{-05}$	$2.3160e^{-05}$	$9.813e^{-07}$	$1.592e^{-06}$	$6.54e^{-08}$	$1.03e^{-07}$

Example 5.2. Consider the following system [7]:

$$\begin{aligned} \cos(x) - \frac{1}{2}\sin^2(x) + \int_0^x \left(f_1(s) f_2(s)\right) ds &= f_1(x), \quad 0 \le x \le 1 \\ \sin(x) - x + \int_0^x f_1^2(s) ds + \int_0^x f_2^2(s) ds &= f_2(x), \quad 0 \le x \le 1 \end{aligned} \tag{5.2}$$

the exact solutions are $f_1(x) = \cos(x)$ and $f_2(x) = \sin(x)$. The errors are given in Table 2.

Example 5.3. Consider the following system [15]:

$$\begin{aligned} \sec(x) - x + \int_0^x \left(f_1^2(s) - f_2^2(s)\right) ds &= f_1(x), \quad 0 \le x \le 1 \\ 3\tan(x) - x - \int_0^x \left(f_1^2(s) + f_2^2(s)\right) ds &= f_2(x), \quad 0 \le x \le 1, \end{aligned} \tag{5.3}$$

where the exact solutions are $f_1(x) = \sec(x)$ and $f_2(x) = \tan(x)$.

Table 5: Numerical results of Example 5.5.

x	$h = 0.1$		$h = 0.05$		$h = 0.025$	
	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$
0.1	$3.671e^{-07}$	$2.9894e^{-07}$	$1.30e^{-08}$	$1.057e^{-08}$	$7.0e^{-10}$	$2.11e^{-09}$
0.2	$4.491e^{-07}$	$3.780e^{-07}$	$8.39e^{-08}$	$7.33e^{-08}$	$5.0e^{-09}$	$4.6e^{-09}$
0.3	$2.1364e^{-06}$	$2.0417e^{-06}$	$1.439e^{-07}$	$1.335e^{-07}$	$1.07e^{-08}$	$8.4e^{-09}$
0.4	$3.1803e^{-06}$	$2.5989e^{-06}$	$1.946e^{-07}$	$1.627e^{-07}$	$1.22e^{-08}$	$1.17e^{-08}$
0.5	$3.8590e^{-06}$	$3.0494e^{-06}$	$2.841e^{-07}$	$2.441e^{-07}$	$1.82e^{-08}$	$1.54e^{-08}$
0.6	$5.3109e^{-06}$	$5.0008e^{-06}$	$3.539e^{-07}$	$3.262e^{-07}$	$2.28e^{-08}$	$2.06e^{-08}$
0.7	$7.0863e^{-06}$	$5.9892e^{-06}$	$4.518e^{-07}$	$3.856e^{-07}$	$2.86e^{-08}$	$2.52e^{-08}$
0.8	$8.5579e^{-06}$	$7.0265e^{-06}$	$5.596e^{-07}$	$4.947e^{-07}$	$3.55e^{-08}$	$3.15e^{-08}$
0.9	$9.9840e^{-06}$	$9.3934e^{-06}$	$6.597e^{-07}$	$6.118e^{-07}$	$4.28e^{-08}$	$3.89e^{-08}$
1.0	$1.29156e^{-05}$	$1.10909e^{-05}$	$8.334e^{-07}$	$7.207e^{-07}$	$5.42e^{-08}$	$4.64e^{-08}$

Table 6: Numerical results of Example 5.6.

x	e_{HPM}		$h = 0.1$		$h = 0.05$	
	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$	$e(f_1)$	$e(f_2)$
0.2	$3.947e^{-04}$	$6.333e^{-05}$	0	$4.0e^{-09}$	$6.7e^{-08}$	$6.0e^{-09}$
0.4	$6.060e^{-03}$	$1.962e^{-03}$	$4.445e^{-06}$	$8.83e^{-07}$	$2.25e^{-07}$	$1.23e^{-07}$
0.6	$2.855e^{-02}$	$1.409e^{-02}$	$1.3e^{-08}$	$7.150e^{-06}$	$2.0e^{-09}$	$5.69e^{-07}$
0.8	$8.088e^{-02}$	$5.465e^{-02}$	$1.4583e^{-05}$	$1.5812e^{-05}$	$3.62e^{-07}$	$1.199e^{-06}$
1.0	$1.685e^{-01}$	$1.483e^{-01}$	$1.3989e^{-05}$	$3.2971e^{-05}$	$7.27e^{-07}$	$2.346e^{-06}$
1.2	$2.770e^{-01}$	$3.135e^{-01}$	$2.173e^{-06}$	$7.3764e^{-05}$	$1.51e^{-07}$	$4.865e^{-06}$
1.4	$3.567e^{-01}$	$5.369e^{-01}$	$3.4677e^{-05}$	$9.9081e^{-05}$	$1.006e^{-06}$	$7.222e^{-06}$
1.6	$3.034e^{-01}$	$7.325e^{-01}$	$3.5568e^{-05}$	$1.62912e^{-04}$	$2.002e^{-06}$	$1.0807e^{-05}$
1.8	$6.562e^{-02}$	$6.821e^{-01}$	$2.7768e^{-05}$	$2.77606e^{-04}$	$1.765e^{-06}$	$1.7732e^{-05}$
2.0	1.043	$4.961e^{-02}$	$9.6763e^{-05}$	$3.46228e^{-04}$	$4.258e^{-06}$	$2.4040e^{-05}$

Example 5.4. Consider the following system [7]:

$$\begin{aligned} \cosh(x) + x\sin(x) - \int_0^x e^{x-s} f_1(s)ds - \int_0^x \cos(x-s) f_2(s)ds = f_1(x), \quad 0 \le x \le 1 \\ 2\sin(x) + x\left(\sin^2(x) + e^x\right) - \int_0^x e^{x+s} f_1(s)ds - \int_0^x x\cos(s) f_2(s)ds = f_2(x) \quad 0 \le x \le 1, \end{aligned} \tag{5.4}$$

with the exact solutions $f_1(x) = e^{-x}$ and $f_2(x) = 2\sin(x)$. Table 4 shows the errors.

Example 5.5. Consider the following Volterra system of integral equations [5]:

$$\begin{aligned} g_1(x) - \int_0^x (\sin(s-x) - 1) f_1(s)ds + \int_0^x (1 - s\cos(x)) f_2(s)ds = f_1(x), \quad 0 \le x \le 1 \\ g_2(x) + \int_0^x f_1(s)ds + \int_0^x (x-s) f_2(s)ds = f_2(x), \quad 0 \le x \le 1 \end{aligned} \tag{5.5}$$

the functions $g_1(x)$ and $g_2(x)$ are chosen such that the exact solutions are to be $f_1(x) = \cos(x)$ and $f_2(x) = \sin(x)$. The errors are given in Table 5.

Example 5.6. Consider the following Volterra system of integral equations [5]

$$\begin{aligned} g_1(x) + \int_0^x (x-s)^3 f_1(s)ds + \int_0^x (x-s)^2 f_2(s)ds &= f_1(x), \quad 0 \le x \le 2 \\ g_2(x) + \int_0^x (x-s)^4 f_1(s)ds + \int_0^x (x-s)^3 f_2(s)ds &= f_2(x), \quad 0 \le x \le 2 \end{aligned} \tag{5.6}$$

the functions $g_1(x)$ and $g_2(x)$ are chosen such that the exact solutions are to be $f_1(x) = x^2 + 1$ and $f_2(x) = 1 - x^3 + x$. Table 6 compares errors.

6. Conclusion

Now let $E(h) = ch^q$ be error of the block by block using Simpson's 3/8 rule where c and q are, respectively, a constant and order of the error, then we have $q = \ln(E(h)/c)/\ln(h)$. By computing the order q from this formulae for the errors reported in Tables 1–6, we conclude that $q \ge 4$. This confirms the claim that was stated in introduction and was proved by the convergence analysis. For example in Table 2 for $x = 0.1$, $E(f_1(h)) = 0.876e^{-07}$ where $h = 0.1$, so we get $q = 8$ or in Table 4 for $x = 0.5$, $E(f_2(h)) = 0.130e^{-07}$ where $h = 0.025$, so we get $q = 4.14268$ or in Table 6 for $x = 0.6$, $E(f_1(h)) = 0.2e^{-08}$, $h = 0.05$ then $q = 6.148974$.

Note that in the Table 1 we compare the order of convergence between the results which were obtained by Yusufoğlu [9] (by using HPM method) and block by block method. Further, the results of the Table 6 show that block by block method is efficient for the large values of x, whereas HPM method isn't applicable. Also, the time of computation in block by block method is less than HPM method whenever programming of both method is done in MAPLE package.

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