

*Research Article*

# Minimum-Energy Multiwavelet Frames with Arbitrary Integer Dilation Factor

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Received 15 May 2012; Accepted 26 June 2012

Academic Editor: Carlo Cattani

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In order to organically combine the minimum-energy frame with the significant properties of multiwavelets, minimum-energy multiwavelet frames with arbitrary integer dilation factor are studied. Firstly, we define the concept of minimum-energy multiwavelet frame with arbitrary dilation factor and present its equivalent characterizations. Secondly, some necessary conditions and sufficient conditions for minimum-energy multiwavelet frame are given. Thirdly, the decomposition and reconstruction formulas of minimum-energy multiwavelet frame with arbitrary integer dilation factor are deduced. Finally, we give several numerical examples based on B-spline functions.

## 1. Introduction

Wavelets transform has been widely applied to information processing, image processing, computer science, mathematical physics, engineering, and so on. As you all know, it is not possible for any orthogonal scaling wavelet function with compact support to be symmetric, except for the Haar wavelets. In 1993, Goodman and Lee [1] established the multiwavelet theory by introducing the multiresolution analysis (MRA) with multiplicity  $r$ , and gave the spline multiwavelet examples. Using the fractal interpolation technology, Geronimo et al. [2] constructed the GHM multiwavelet which have short support, (anti)symmetry, orthogonality and vanishing moment with order 2 in 1994. From then on, multiwavelet has been a hot research area. In 1996, Chui and Lian [3] reconstructed the GHM multiwavelet without using the fractal interpolation technology, and they gave the general method on constructing the multiwavelet with short support, (anti)symmetry, and orthogonality. After that, Plonka and Strela [4] used two-scale similarity transforms (TSTs) to raise the approximation order of multiwavelet and gave the important conclusions of the two-scale matrix symbol's

factorizations and so on. And, by Lawton et al. [5], the construction of multiwavelet has been transformed into matrix extension problem in 1996. The construction theory of multiwavelet had a great development after Jiang [6, 7] putting forward a series of effective methods. Whether wavelets or multiwavelet, they require that the integer shifts of the scaling function form Riesz bases, orthogonal basis, or biorthogonal basis for its span space. And this will cause some defects: (1) the computational complexity can be increased during the course of decomposition and reconstruction; (2) the numerical instability can be caused during the procedure of reconstructing original signal (3) in the biorthogonal case, the analysis filter bank can not be replaced by the synthetic filter bank, and vice versa.

Fortunately, besides orthogonal wavelets and multiwavelet minimum-energy frames can effectively avoid the difficulty which is caused by different bases functions during the course of decomposition and reconstruction, still use the same wavelets both for analysis and synthesis. The theory of frames comes from signal processing firstly. It was introduced by Duffin and Schaffer to deal with problems in nonharmonic Fourier series. But in a long time after that, people did not pay enough attention to it. After Daubechies et al. [8] defined affine frames (wavelets frames) by combining the theory of continuous wavelets transforms and frames while wavelets theory was booming, people start to research frames and its application again. Benedetto and Li [9] gave the definition of frame multiresolution analysis (FMRA), and their work laid the foundation for other people's further investigation. Frames cannot only overcome the disadvantages of wavelets and multiwavelet, but also increase redundancy properly, then the numerical computation become much more stable using frames to reconstruct signal. With well time-frequency localization and shift invariance, frames can be designed more easily than wavelets or multiwavelet. Nowadays frames have been used widely in theoretical and applied domain [10–22], such as signal analysis, image processing, numerical calculation, Banach space theory, Besov space theory, and so on.

In 2000, Chui and He [11] proposed the concept of minimum-energy wavelets frames. The minimum-energy wavelets frames reduce the computational complexity, maintain the numerical stability, and do not need to search dual frames in the decomposition and reconstruction of functions (or signals). Therefore, many people pay a lot of attention to the study of minimum-energy wavelets frames. Huang and Cheng [15] studied the construction and characterizations of the minimum-energy with arbitrary integer dilation factor. Gao and Cao [18] researched the structure of the minimum-energy wavelets frames on the interval and its application on signal denoising systematically. Liang and Zhao [23] studied the minimum-energy multiwavelet frames with dilation factor 2 and multiplicity 2 and gave a characterization and a necessary condition of minimum-energy multiwavelet frames. Unfortunately, the authors did not give the sufficient conditions of minimum-energy multiwavelet frames. In fact, people need to pay close attention to the existence of sufficient conditions of minimum-energy wavelet frames in most cases. On the other hand B-spline functions which are the convolution of Shannon wavelets [24–26]. It can be seen that also Shannon wavelets are minimum-energy wavelets. In this paper, in order to organically combine the minimum-energy frame with the significant properties of multiwavelet, minimum-energy multiwavelet frames with arbitrary integer dilation factor are studied. Firstly, we define the concept of minimum-energy multiwavelet frame with arbitrary dilation factor and present its equivalent characterizations. Secondly, some necessary conditions and sufficient conditions for minimum-energy multiwavelet frame are given; Thirdly, the decomposition and reconstruction formulas of minimum-energy multiwavelet frame with arbitrary integer dilation factor and the multiplicity  $r$  are deduced. Finally, we give several numerical examples based on B-spline functions.

Let us now describe the organization of the material that as follows. Section 2 is preliminaries and basic definitions. Section 3 is main result. In Section 4, we give the decomposition and reconstruction formulas of minimum-energy multiwavelet frame. Section 5 is numerical examples.

## 2. Preliminaries and Basic Definitions

Throughout this paper, let  $\mathbb{Z}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  denote the set of integers, real numbers, and complex numbers respectively;  $a \in \mathbb{Z}$  with  $a \geq 2$ ,  $\omega_j = \cos(2j\pi/a) + i \sin(2j\pi/a)$ ,  $j = 0, 1, \dots, a-1$ .

A multiscaling function vector (refinable function vector) is a vector-valued function:

$$\Phi = (\phi_1(x), \dots, \phi_r(x))^T, \quad \phi_l(x) \in L^2(\mathbb{R}), \quad l = 1, \dots, r, \quad (2.1)$$

which satisfies a two-scale matrix refinement equation of the form:

$$\Phi(x) = \sum_{k \in \mathbb{Z}} P_k \Phi(ax - k), \quad x \in \mathbb{R}, \quad (2.2)$$

$r$  is called the multiplicity of  $\Phi$ , the integer  $a$  is said to be dilation factor. The recursion coefficients  $\{P_k\}_{k \in \mathbb{Z}}$  are  $r \times r$  matrices.

The Fourier transform of the formula (2.2) is

$$\hat{\Phi}(\omega) = P(z) \hat{\Phi}\left(\frac{\omega}{a}\right), \quad z = e^{-i\omega/a}, \quad (2.3)$$

where

$$P(z) = \frac{1}{a} \sum_{k \in \mathbb{Z}} P_k z^k. \quad (2.4)$$

$P(z)$  is the symbol of the matrix sequence  $\{P_k\}_{k \in \mathbb{Z}}$ .

The multiresolution analysis (MRA) with multiplicity  $r$  and dilation factor  $a$  generated by  $\Phi(x)$  is defined as

$$\{V_j\} = \overline{\text{span}}\{\phi_{\tau,j,k} : 1 \leq \tau \leq r, k \in \mathbb{Z}, j \in \mathbb{Z}\}, \quad (2.5)$$

where  $\phi_{\tau,j,k} = a^{j/2} \phi_{\tau}(a^j x - k)$ , and the sequence of closed subspace of  $L^2(\mathbb{R})$  has the following properties:

- (1)  $V_j \subset V_{j+1}$ ,  $j \in \mathbb{Z}$ .
- (2)  $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$ ,  $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ ;
- (3)  $f(x) \in V_j \Leftrightarrow f(ax) \in V_{j+1}$ , for all  $j \in \mathbb{Z}$ ;
- (4)  $f(x) \in V_j \Leftrightarrow f(x - a^{-j}k) \in V_j$ , for all  $k, j \in \mathbb{Z}$ ;
- (5)  $\{\phi_{\tau,0,k} : 1 \leq \tau \leq r, k \in \mathbb{Z}\}$  forms a Riesz basis of  $V_0$ ;

*Definition 2.1.* A finite family vector-valued function  $\Psi^i = (\psi_1^i, \dots, \psi_r^i)^T, i = 1, \dots, N$  generates a multiwavelet frames for  $L^2(\mathbb{R})$ , if there exist constants  $0 < A \leq B < \infty$  such that for any  $f(x) \in L^2(\mathbb{R})$

$$A\|f\|^2 \leq \sum_{i=1}^N \sum_{\tau=1}^r \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{\tau,j,k}^i \rangle|^2 \leq B\|f\|^2, \quad (2.6)$$

where  $\psi_{\tau,j,k}^i = a^{j/2} \psi_{\tau}^i(a^j x - k)$ .

*Definition 2.2.* A nested subspace generated by a multiscaling vector-valued function  $\Phi(x)$  satisfies formula (2.5) and its additional conditions, then finite family vector-valued function  $\{\Psi^1, \dots, \Psi^N\}$  generates a frame multiresolution analysis associated the vector-valued function  $\Phi(x)$ , if the finite family  $\Psi^i = (\psi_1^i, \dots, \psi_r^i)^T, i = 1, \dots, N$  satisfies the formulation (2.6) with  $\psi_{\tau}^i \in V_1, i = 1, \dots, N; \tau = 1, \dots, r$ .

*Definition 2.3.* Let  $\Phi(x) = (\phi_1(x), \dots, \phi_r(x))^T$ , with  $\hat{\phi}_{\tau} \in L^{\infty}(\mathbb{R}) \cap L^2(\mathbb{R}), \tau = 1, \dots, r, \hat{\Phi}$  continuous at 0 and  $\hat{\Phi}(0) \neq 0$ , be a multiscaling vector-valued function that generates the nested subspace  $\{V_j\}_{j \in \mathbb{Z}}$  in the sense of (2.5). Then a finite family vector-valued function  $\{\Psi^1, \dots, \Psi^N\} \subset V_1$  is called a minimum-energy multiwavelet frames associated with  $\Phi(x)$ , if for all  $f \in L^2(\mathbb{R})$

$$\sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |\langle f, \hat{\phi}_{\tau,1,k} \rangle|^2 = \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |\langle f, \hat{\phi}_{\tau,0,k} \rangle|^2 + \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |\langle f, \psi_{\tau,0,k}^i \rangle|^2. \quad (2.7)$$

*Remark 2.4.* By the Parseval identity, minimum-energy multiwavelet frames  $\{\Psi^1, \dots, \Psi^N\}$  must be tight frames for  $L^2(\mathbb{R})$  with frames bound equal to 1.

*Remark 2.5.* The formula (2.7) is equivalent to the following formulas:

$$\sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} \langle f, \hat{\phi}_{\tau,1,k} \rangle \hat{\phi}_{\tau,1,k} = \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} \langle f, \hat{\phi}_{\tau,0,k} \rangle \hat{\phi}_{\tau,0,k} + \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} \langle f, \psi_{\tau,0,k}^i \rangle \psi_{\tau,0,k}^i. \quad (2.8)$$

The interpretation of minimum energy will be clarified later.

### 3. Main Result

In this section, we will give a complete characterization of minimum-energy multiwavelet frames associated with some given multiscaling vector-valued function in term of their two-scale symbols. Let  $\Phi(x) = (\phi_1(x), \dots, \phi_r(x))^T$  with  $\hat{\phi}_{\tau} \in L^{\infty}(\mathbb{R}) \cap L^2(\mathbb{R}), \tau = 1, \dots, r, \hat{\Phi}$  continuous at 0, and  $\hat{\Phi}(0) \neq 0$  be a multiscaling vector-valued function which satisfies (2.2)–(2.5). Consider  $\{\Psi^1, \dots, \Psi^N\} \subset V_1$ , then

$$\Psi^i(x) = \sum_{k \in \mathbb{Z}} Q_k^i \Phi(ax - k), \quad (3.1)$$

where  $\{Q_k^l\}_{k \in \mathbb{Z}}$ ,  $l = 1, \dots, N$  are  $r \times r$  matrices. Using Fourier transform on (3.1), we can get their symbols as follows:

$$Q_l(z) = \frac{1}{a} \sum_{k \in \mathbb{Z}} Q_k^l z^k, \quad l = 1, \dots, N. \quad (3.2)$$

With  $P(z)$ ,  $Q_l(z)$ ,  $l = 1, \dots, N$ , we formulate the  $(N+1)r \times ar$  block matrix as follows:

$$R(z) = \begin{bmatrix} P(z) & P(\omega_1 z) & \cdots & P(\omega_{a-1} z) \\ Q_1(z) & Q_1(\omega_1 z) & \cdots & Q_1(\omega_{a-1} z) \\ \vdots & \vdots & & \vdots \\ Q_N(z) & Q_N(\omega_1 z) & \cdots & Q_N(\omega_{a-1} z) \end{bmatrix}, \quad (3.3)$$

and the  $R^*(z)$  denotes the complex conjugate of the transpose of  $R(z)$ .

The following theorem presents the equivalent characterizations of the minimum-energy multiwavelet frames with arbitrary integer dilation factor.

**Theorem 3.1.** *Suppose that every element of the symbols,  $P(z)$ ,  $Q_l(z)$ ,  $l = 1, \dots, N$ , in (2.4) and (3.2) is a Laurent polynomial, and the multiscaling vector-valued function  $\Phi(x)$  associated with  $P(z)$  generates a nested subspace  $\{V_j\}_{j \in \mathbb{Z}}$ . Then the following statements are equivalent:*

- (1)  $\{\Psi^1, \dots, \Psi^N\}$  is a minimum-energy multiwavelet frames associated with  $\Phi(x)$ ;
- (2)

$$R^*(z)R(z) = I_{ar} \quad \text{for } \forall |z| = 1; \quad (3.4)$$

- (3)

$$\alpha_{ml,ij} = 0, \quad \forall m, l \in \mathbb{Z}; \quad i, j = 1, \dots, r, \quad (3.5)$$

where

$$\alpha_{ml,ij} = \sum_{k \in \mathbb{Z}} \sum_{\tau=1}^r \left( P_{l-ak}^{\tau i*} P_{m-ak}^{\tau j} + \sum_{t=1}^N Q_{l-ak}^{t, \tau i*} Q_{m-ak}^{t, \tau j} \right) - a \delta_{ml,ij}, \quad (3.6)$$

$$\delta_{ml,ij} = \begin{cases} 1, & m = l, \quad i = j, \\ 0, & \text{else.} \end{cases}$$

*Proof.* By using the two-scale relations (2.2) and (3.1) and notation  $\alpha_{ml,ij}$  for for all  $f \in L^2(\mathbb{R})$ , (2.8) can be written as

$$\sum_{l \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \sum_{i=1}^r \sum_{j=1}^r \alpha_{ml,ij} \langle f, \phi_i(ax - m) \rangle \phi_j(ax - l) = 0. \quad (3.7)$$

On the other hand, (3.4) can be reformulated as

$$\begin{aligned}
 P^*(z)P(z) + \sum_{t=1}^N Q_t^*(z)Q_t(z) &= I_r, \\
 P^*(z)P(\omega_j z) + \sum_{t=1}^N Q_t^*(z)Q_t(\omega_j z) &= 0_r, \\
 j &= 1, 2, \dots, a-1; \quad \forall |z| = 1,
 \end{aligned} \tag{3.8}$$

and it is equivalent to

$$\begin{aligned}
 \sum_{k=0}^{a-1} P^*(\omega_k z)P(z) + \sum_{t=1}^N \sum_{k=0}^{a-1} Q_t^*(\omega_k z)Q_t(z) &= I_r, \\
 \left( P^*(z) - \sum_{k=1}^{a-1} P^*(\omega_k z) \right) P(z) + \sum_{t=1}^N \left( Q_t^*(z) - \sum_{k=1}^{a-1} Q_t^*(\omega_k z) \right) Q_t(z) &= I_r, \\
 \left( \sum_{k=0}^{a-1} P^*(\omega_k z) - 2P^*(\omega_l z) \right) P(z) + \sum_{t=1}^N \left( \sum_{k=0}^{a-1} Q_t^*(\omega_k z) - 2Q_t^*(\omega_l z) \right) Q_t(z) &= I_r, \\
 l &= 1, 2, \dots, a-1; \quad \forall |z| = 1.
 \end{aligned} \tag{3.9}$$

With  $|z| = 1$ ,  $\bar{z}^k = z^{-k}$ ,  $\omega_l^k = \omega_k^l = \omega^{kl}$ , and

$$\sum_{l=0}^{a-1} \omega_l^k = \sum_{l=0}^{a-1} \omega_k^l = \begin{cases} 0 & \omega_k \neq 1 \\ a & \omega_k = 1, \end{cases} \tag{3.10}$$

the formulation (3.9) is equivalent to

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}} P_{-ak}^* z^{ak} P(z) + \sum_{t=1}^N \sum_{k \in \mathbb{Z}} Q_{-ak}^{t*} z^{ak} Q_t(z) &= I_r, \\
 \left( \sum_{l=1}^{a-1} \sum_{k \in \mathbb{Z}} P_{l-ak}^* z^{ak-l} \right) P(z) + \sum_{t=1}^N \left( \sum_{l=1}^{a-1} \sum_{k \in \mathbb{Z}} Q_{l-ak}^{t*} z^{ak-l} \right) Q_t(z) &= (a-1)I_r, \\
 \left( \sum_{l=1}^{a-1} e^{-(2sl\pi/a)i} \sum_{k \in \mathbb{Z}} P_{l-ak}^* z^{ak-l} \right) P(z) + \sum_{t=1}^N \left( \sum_{l=1}^{a-1} e^{-(2sl\pi/a)i} \sum_{k \in \mathbb{Z}} Q_{l-ak}^{t*} z^{ak-l} \right) Q_t(z) &= -I_r, \\
 s &= 1, 2, \dots, a-1; \quad \forall |z| = 1.
 \end{aligned} \tag{3.11}$$

Using the properties of roots of unity, the Vandermonde matrix and Cramer's rule, the above equation is equivalent to

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}} P_{-ak}^* z^{ak} P(z) + \sum_{t=1}^N \sum_{k \in \mathbb{Z}} Q_{-ak}^{t*} z^{ak} Q_t(z) &= I_r, \\
 \sum_{k \in \mathbb{Z}} P_{1-ak}^* z^{ak-1} P(z) + \sum_{t=1}^N \sum_{k \in \mathbb{Z}} Q_{1-ak}^{t*} z^{ak-1} Q_t(z) &= I_r, \\
 &\vdots \\
 \sum_{k \in \mathbb{Z}} P_{a-1-ak}^* z^{ak-a+1} P(z) + \sum_{t=1}^N \sum_{k \in \mathbb{Z}} Q_{a-1-ak}^{t*} z^{ak-a+1} Q_t(z) &= I_r.
 \end{aligned} \tag{3.12}$$

We multiply the identities in (3.12) by  $\widehat{\Phi}(\omega/a)z^l$ ,  $l = 0, 1, \dots, a-1$ , respectively, where  $z = e^{-i\omega/a}$ , to get

$$\sum_k \left\{ P_{l-ak}^* z^{ak} P(z) \widehat{\Phi}\left(\frac{\omega}{a}\right) + \sum_{t=1}^N Q_{l-ak}^{t*} z^{ak} Q_t(z) \widehat{\Phi}\left(\frac{\omega}{a}\right) \right\} = \widehat{\Phi}\left(\frac{\omega}{a}\right) z^l, \quad l = 0, \dots, a-1. \tag{3.13}$$

Hence, (3.12) is equivalent to

$$\sum_k \left\{ P_{l-ak}^* z^{ak} \widehat{\Phi}(\omega) + \sum_{t=1}^N Q_{l-ak}^{t*} z^{ak} \widehat{\Psi}^t(\omega) \right\} = \widehat{\Phi}\left(\frac{\omega}{a}\right) e^{-il\omega/a}, \quad l = 0, \dots, a-1 \tag{3.14}$$

or

$$\sum_k \left\{ P_{l-ak}^* z^{ak} \Phi(x-k) + \sum_{t=1}^N Q_{l-ak}^{t*} z^{ak} \Psi^t(x-k) \right\} = a\Phi(ax-l), \quad l = 0, \dots, a-1, \tag{3.15}$$

which can be reformulated as

$$\sum_k \left\{ P_{l-ak}^* z^{ak} \Phi(x-k) + \sum_{t=1}^N Q_{l-ak}^{t*} z^{ak} \Psi^t(x-k) \right\} = a\Phi(ax-l). \tag{3.16}$$

By using the two-scaling relations (2.2) and (3.1), we can rewrite (3.16) as

$$\sum_m \sum_{j=1}^a \alpha_{ml,ij} \phi_j(ax-m) = 0, \quad i = 1, \dots, r; \quad \forall l \in \mathbb{Z}. \tag{3.17}$$

In conclusion, the proof of Theorem 3.1 reduces to the proof of the equivalence of (3.5), (3.7), and (3.17).

It is obvious that (3.5) $\Rightarrow$ (3.17) $\Rightarrow$ (3.7). To show (3.7) $\Rightarrow$ (3.5), let  $f \in L^2(\mathbb{R})$  be any compactly supported function. By using the properties that for every fixed  $m$ ,  $\alpha_{ml,ij} = 0$  except for finitely many  $l, i, j$ , then the functional

$$\beta_{lj}(f) = \sum_m \sum_{i=1}^r \alpha_{ml,ij} \langle f, \phi_i(ax - m) \rangle \quad (3.18)$$

just has finite nonzero for  $l \in \mathbb{Z}, j = 1, \dots, r$ .

Using the property of Fourier transform, we obtain

$$\sum_l \sum_{j=1}^r \beta_{lj}(f) \widehat{\phi}_j(\omega) e^{-il\omega/a} = 0. \quad (3.19)$$

Since  $\widehat{\phi}_l(\omega)$  is nontrivial function, then  $\beta_{lj}(f) = 0, l \in \mathbb{Z}, j = 1, \dots, r$ , in other words, we have

$$\left\langle f, \sum_m \sum_{i=1}^r \alpha_{ml,ij} \phi_i(ax - m) \right\rangle = 0, \quad l \in \mathbb{Z}, j = 1, \dots, r. \quad (3.20)$$

Then the series in the above equation is a finite sum and hence represents a compactly supported function in  $L^2(\mathbb{R})$ . By choosing  $f$  to be this function, it follows that

$$\sum_m \sum_{i=1}^r \alpha_{ml,ij} \phi_i(ax - m) = 0, \quad (3.21)$$

which implies that the trigonometric polynomial  $\sum_m \sum_{i=1}^r \alpha_{ml,ij} \widehat{\phi}_i(\omega) e^{-im\omega}$  is identically equal to 0 so that  $\alpha_{ml,ij} = 0$ , for all  $m, l \in \mathbb{Z}; i, j = 1, \dots, r$ .

We complete the proof of Theorem 3.1 because the set of compactly supported functions is dense in  $L^2(\mathbb{R})$ .  $\square$

Theorem 3.1 characterizes the necessary and sufficient condition for the existence of the minimum-energy multiwavelet frames associated with  $\Phi$ . However it is not a good choice to use this theorem to construct the minimum-energy multiwavelet frames. For convenience, we need to present some sufficient conditions in terms of the symbols.

In this paper, we just discuss the minimum-energy frames with compact support, that is, every element of symbols is Laurent polynomial.

**Theorem 3.2.** *A compactly supported refinable vector-valued function  $\Phi(x) = (\phi_1(x), \dots, \phi_r(x))^T$ , with  $\widehat{\Phi}$  continuous at 0 and  $\widehat{\Phi}(0) \neq 0$ . Let  $\{\Psi^1, \dots, \Psi^N\}$  be the minimum-energy multiwavelet frames associated with it, then*

$$\sum_{i=1}^r |p_{ij}(\omega_l z)|^2 \leq 1 \quad \forall |z| = 1, 1 \leq j \leq r, 0 \leq l \leq a-1, \quad (3.22)$$

$$\sum_{l=0}^{a-1} \sum_{j=1}^r |p_{ij}(\omega_l z)|^2 \leq 1 \quad \forall |z| = 1, 1 \leq i \leq r. \quad (3.23)$$



*Proof.* Using Theorem 3.1, it is clear to show that the  $l^2$ -norm of every row vector of the symbol for  $\Phi$  is less than 1, in other words, (3.22) is valid. In order to prove (3.23), let  $i = 1$ . First, we set

$$f(z) = [p_{11}(z) \cdots p_{1r}(z) \cdots p_{11}(\omega_{a-1}z) \cdots p_{1r}(\omega_{a-1}z)], \quad (3.24)$$

and the rest of  $R(z)$  removed  $f(z)$  as  $F(z)$ . Then we can reformulate (3.4) as

$$f(z)^* f(z) + F(z)^* F(z) = I_{ar}, \quad (3.25)$$

or equivalently,  $F(z)^* F(z) = I_{ar} - f(z)^* f(z)$ , which is a nonnegative definite Hermitian matrix for  $|z| = 1$  so that

$$\det(I_{ar} - f(z)^* f(z)) \geq 0 \quad \forall |z| = 1, \quad (3.26)$$

and this gives

$$\sum_{l=0}^{a-1} \sum_{j=1}^r |p_{1j}(\omega_l z)|^2 \leq 1 \quad \forall |z| = 1. \quad (3.27)$$

In fact, we have

$$\begin{aligned} \begin{pmatrix} I_{ar} & f(z)^* \\ f(z) & 1 \end{pmatrix} \begin{pmatrix} I_{ar} & -f(z)^* \\ -f(z) & 1 \end{pmatrix} &= \begin{pmatrix} I_{ar} - f(z)^* f(z) & 0 \\ 0 & 1 - f(z) f(z)^* \end{pmatrix}, \\ \det \left( \begin{pmatrix} I_{ar} & f(z)^* \\ f(z) & 1 \end{pmatrix} \right) &= \det \left( \begin{pmatrix} I_{ar} & f(z)^* \\ 0 & 1 - f(z) f(z)^* \end{pmatrix} \right), \\ \det \left( \begin{pmatrix} I_{ar} & -f(z)^* \\ -f(z) & 1 \end{pmatrix} \right) &= \det \left( \begin{pmatrix} I_{ar} & -f(z)^* \\ 0 & 1 - f(z) f(z)^* \end{pmatrix} \right), \end{aligned} \quad (3.28)$$

then

$$\det(I_{ar} - f(z)^* f(z)) (1 - f(z) f(z)^*) = (1 - f(z) f(z)^*)^2, \quad (3.29)$$

and it gives  $1 - f(z) f(z)^* \geq 0$ , for all  $|z| = 1$ , that is,

$$\sum_{l=0}^{a-1} \sum_{j=1}^r |p_{ij}(\omega_l z)|^2 \leq 1 \quad \forall |z| = 1, \quad 1 \leq i \leq r. \quad (3.30)$$

The proof of Theorem 3.2 is completed.  $\square$

*Remark 3.3.* By the proof of Theorem 3.2, we know that the restriction in Theorem 3.2 on the two-scale symbol  $P(z)$  of a refinable vector-valued function  $\Phi(x)$  is a necessary condition for

the existence of a minimum-energy frames associated with  $\Phi(x)$  via the rectangular unitary matrix extension approach, even if  $\Phi(x)$  is not compactly supported.

*Remark 3.4.* For a certain compactly supported refinable vector-valued function, it cannot exist in minimum-energy frames.

We write  $P(z), Q_j(z), j = 1, \dots, N$  in their polyphase forms:

$$P(z) = \frac{\sqrt{a}}{a} \left( P_1(z^a) + zP_2(z^a) + \dots + z^{a-1}P_a(z^a) \right), \quad (3.31)$$

$$Q_j(z) = \frac{\sqrt{a}}{a} \left( Q_{j1}(z^a) + zQ_{j2}(z^a) + \dots + z^{a-1}Q_{ja}(z^a) \right), \quad j = 1, \dots, N, \quad (3.32)$$

where  $P_i(z), Q_{ij}(z), i = 1, \dots, a; j = 1, \dots, N$  are  $r \times r$  matrices and their every element is Laurent polynomial. Observe that

$$R(z) \frac{\sqrt{a}}{a} \begin{bmatrix} I_r & z^{-1}I_r & \dots & z^{1-a}I_r \\ I_r & (\omega_1 z)^{-1}I_r & \dots & (\omega_1 z)^{1-a}I_r \\ \vdots & \vdots & & \vdots \\ I_r & (\omega_{a-1} z)^{-1}I_r & \dots & (\omega_{a-1} z)^{1-a}I_r \end{bmatrix} = \begin{bmatrix} P_1(z^a) & P_2(z^a) & \dots & P_a(z^a) \\ Q_{11}(z^a) & Q_{12}(z^a) & \dots & Q_{1a}(z^a) \\ \vdots & \vdots & & \vdots \\ Q_{N1}(z^a) & Q_{N2}(z^a) & \dots & Q_{Na}(z^a) \end{bmatrix}. \quad (3.33)$$

Therefore, we have

$$\begin{aligned} & a \begin{bmatrix} P_1(z^a) & P_2(z^a) & \dots & P_a(z^a) \\ Q_{11}(z^a) & Q_{12}(z^a) & \dots & Q_{1a}(z^a) \\ \vdots & \vdots & & \vdots \\ Q_{N1}(z^a) & Q_{N2}(z^a) & \dots & Q_{Na}(z^a) \end{bmatrix}^* \begin{bmatrix} P_1(z^a) & P_2(z^a) & \dots & P_a(z^a) \\ Q_{11}(z^a) & Q_{12}(z^a) & \dots & Q_{1a}(z^a) \\ \vdots & \vdots & & \vdots \\ Q_{N1}(z^a) & Q_{N2}(z^a) & \dots & Q_{Na}(z^a) \end{bmatrix} \\ &= \begin{bmatrix} I_r & z^{-1}I_r & \dots & z^{1-a}I_r \\ I_r & (\omega_1 z)^{-1}I_r & \dots & (\omega_1 z)^{1-a}I_r \\ \vdots & \vdots & & \vdots \\ I_r & (\omega_{a-1} z)^{-1}I_r & \dots & (\omega_{a-1} z)^{1-a}I_r \end{bmatrix}^* R(z)^* R(z) \begin{bmatrix} I_r & z^{-1}I_r & \dots & z^{1-a}I_r \\ I_r & (\omega_1 z)^{-1}I_r & \dots & (\omega_1 z)^{1-a}I_r \\ \vdots & \vdots & & \vdots \\ I_r & (\omega_{a-1} z)^{-1}I_r & \dots & (\omega_{a-1} z)^{1-a}I_r \end{bmatrix}, \quad (3.34) \end{aligned}$$

and it follows from (3.4), that

$$\begin{bmatrix} P_1(z^a) & P_2(z^a) & \dots & P_a(z^a) \\ Q_{11}(z^a) & Q_{12}(z^a) & \dots & Q_{1a}(z^a) \\ \vdots & \vdots & & \vdots \\ Q_{N1}(z^a) & Q_{N2}(z^a) & \dots & Q_{Na}(z^a) \end{bmatrix}^* \begin{bmatrix} P_1(z^a) & P_2(z^a) & \dots & P_a(z^a) \\ Q_{11}(z^a) & Q_{12}(z^a) & \dots & Q_{1a}(z^a) \\ \vdots & \vdots & & \vdots \\ Q_{N1}(z^a) & Q_{N2}(z^a) & \dots & Q_{Na}(z^a) \end{bmatrix} = I_{ar}, \quad \forall |z| = 1. \quad (3.35)$$

And it is easy to obtain (3.35) from (3.4).

For convenience, we denote  $z^a = u$ . Next, we present some theorems to give several sufficient conditions for existence of minimum-energy multiwavelet frames.

**Theorem 3.5.** A compactly supported vector-valued function  $\Phi(x) = (\phi_1(x), \dots, \phi_r(x))^T$  with  $\widehat{\Phi}$  continuous at 0 and  $\widehat{\Phi}(0) \neq 0$ , its symbol  $P(z)$  satisfies

$$\sum_{i=1}^r \sum_{l=0}^{a-1} \sum_{j=1}^r |p_{ij}(\omega_l z)| < 1, \quad \forall |z| = 1. \quad (3.36)$$

Then there exist minimum-energy multiwavelet frames associated with  $\Phi$ .

*Proof.* Let  $P_j(z)$ ,  $j = 1, \dots, a$  be the polynomial components of  $P(z)$ , that is,

$$P(z) = \frac{\sqrt{a}}{a} \left( P_1(z^a) + zP_2(z^a) + \dots + z^{a-1}P_a(z^a) \right). \quad (3.37)$$

Using (3.34) and (3.35), we can get

$$\sum_{i=1}^r \sum_{l=1}^a \sum_{j=1}^r |p_l^{ij}(u)|^2 < 1. \quad (3.38)$$

Then we can find  $r$  real numbers  $x_1, x_2, \dots, x_r$ , with

$$\sum_{i=1}^r x_i = 1, \quad \sum_{l=1}^a \sum_{j=1}^r |p_l^{ij}(u)|^2 < x_i, \quad 1 \leq i \leq r. \quad (3.39)$$

By the Riesz lemma [27, Lemma 6.13], we can find Laurent polynomials  $P_{a+1}^i(z)$ ,  $i = 1, \dots, r$  satisfying

$$\sum_{l=1}^a \sum_{j=1}^r |p_l^{ij}(u)|^2 + |p_{a+1}^i(u)|^2 = x_i, \quad 1 \leq i \leq r. \quad (3.40)$$

For every  $i \in \{1, \dots, r\}$ , using the method in the reference [15, Theorem 3] on the unit vector

$$\frac{1}{\sqrt{x_i}} (P_1^{i1}(z) \cdots P_1^{ir}(z) \cdots P_a^{i1}(z) \cdots P_a^{ir}(z) P_{a+1}^i(z)), \quad (3.41)$$

we can get a matrix

$$\tilde{R}^i(z) = \frac{1}{\sqrt{x_i}} \begin{pmatrix} P_1^{i1}(z) & \cdots & P_1^{ir}(z) & \cdots & P_a^{i1}(z) & \cdots & P_a^{ir}(z) & P_{a+1}^i(z) \\ Q_{11}^{i1}(z) & \cdots & Q_{11}^{ir}(z) & \cdots & Q_{1a}^{i1}(z) & \cdots & Q_{1a}^{ir}(z) & Q_{1,a+1}^i(z) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ Q_{a1}^{i1}(z) & \cdots & Q_{a1}^{ir}(z) & \cdots & Q_{aa}^{i1}(z) & \cdots & Q_{aa}^{ir}(z) & Q_{a,a+1}^i(z) \end{pmatrix}, \quad (3.42)$$

which satisfies  $\tilde{R}^i(z)^* \tilde{R}^i(z) = I_{a+1}$ .

Therefore, the block matrix

$$\tilde{R}(z) = \begin{pmatrix} \sqrt{x_1} \tilde{R}^1(z) \\ \sqrt{x_2} \tilde{R}^2(z) \\ \vdots \\ \sqrt{x_a} \tilde{R}^a(z) \end{pmatrix} \quad (3.43)$$

satisfies  $\tilde{R}(z)^* \tilde{R}(z) = I_{ar+1}$ .

We can get matrix  $R(z)$  which satisfies  $R(z)^* R(z) = I_{ar}$ , after adjusting the rows of  $\tilde{R}(z)$  and removing the last column of it, and the  $r$  rows in the front of matrix  $R(z)$  are the polynomial components of the symbol  $P(z)$ .

Then we complete proof of Theorem 3.5 using the formulas (3.34), (3.32), and Theorem 3.1.  $\square$

Theorem 3.5 requests the sum of  $l^2$ -norm for every row in the matrix symbol  $P(z)$  associated with the vector-valued function  $\Phi$ . Then we can find a minimum-energy multiwavelet frames associated with the function using the theorem. The condition in Theorem 3.5 is too stringent compared with the sufficient conditions in Theorem 3.2. We can get the following theorem by strengthening the structure of the matrix symbol  $P(z)$ .

**Theorem 3.6.** *Let  $\Phi(x) = (\phi_1(x), \dots, \phi_r(x))^T$  with  $\hat{\Phi}$  continuous at 0 and  $\hat{\Phi}(0) \neq 0$  a compactly supported multiscaling vector-valued function. If the block matrix*

$$[P(z) \ P(\omega_1 z) \ \cdots \ P(\omega_{a-1} z)] \quad (3.44)$$

*satisfies standard orthogonal by row, then there exist a minimum-energy multiwavelet frames associated with the function  $\Phi$ .*

*Proof.* Let  $P_j(z)$ ,  $j = 1, \dots, a$  are the polynomial components of  $P(z)$ , that is,

$$P(z) = \frac{\sqrt{a}}{a} \left( P_1(z^a) + z P_2(z^a) + \cdots + z^{a-1} P_a(z^a) \right), \quad (3.45)$$

with (3.34) and (3.35), we can know that the block matrix

$$N(u) = [P_1(u) \ P_2(u) \ \cdots \ P_a(u)]_{r \times ar} \quad (3.46)$$

satisfies standard orthogonal by row.

Now, we use the method in the reference [15, Theorem 3] to deal with the first unit row vector  $N_1(u)$  in the matrix  $N(u)$ . And, we can find a paraunitary matrix  $H_1(u)$  which satisfies  $N_1(u)H_1(u) = e_1 = (1, 0, \dots, 0)_{ar}$  and

$$N(u)H_1(u) = \begin{pmatrix} 1 \\ \tilde{N}(u) \end{pmatrix}, \quad (3.47)$$

with  $\tilde{N}(u)$  also a matrix standard orthogonal by row.

By mathematical induction, there are  $r$  paraunitary matrices  $H_1(u), \dots, H_r(u)$  satisfying

$$N(u)H_1(u) \cdots H_a(u) = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}_{r \times ar}, \quad (3.48)$$

then the matrix  $N(u)$  is equivalent to the front  $r$  rows in the paraunitary matrix  $H_1(u)^* H_2(u)^* \cdots H_a(u)^*$ .

Using the formulation (3.34), (3.35), and Theorem 3.1, we completed the proof of this theorem.  $\square$

Theorem 3.6 requests that the multiscaling vector-valued function's symbol  $P(z)$  satisfies standard orthogonal by row. This means the  $l^2$ -norm of every row in  $P(z)$  is 1. If the  $l^2$ -norm of every row in  $P(z)$  is less than 1 strictly, and we can find a matrix  $P_{a+1}(u)$  to make the block matrix

$$[P_1(u) \ P_2(u) \ \cdots \ P_a(u) \ P_{a+1}(u)] \quad (3.49)$$

satisfy standard orthogonal by row, then there exist minimum-energy multiwavelet frames associated with the function  $\Phi$ .

**Corollary 3.7.** *Let  $\Phi(x) = (\phi_1(x), \dots, \phi_r(x))^T$  with  $\widehat{\Phi}$  continuous at 0 and  $\widehat{\Phi}(0) \neq 0$  a compactly supported multiscaling vector-valued function. If the  $l^2$ -norm of every row in  $P(z)$  is less than 1 strictly, that is,*

$$\sum_{l=0}^{a-1} \sum_{j=1}^r |p_{ij}(\omega_l z)|^2 < 1, \quad \forall |z| = 1, \quad 1 \leq i \leq r, \quad (3.50)$$

and there exists a matrix  $P_{a+1}(u)$  to make (3.49) satisfy standard orthogonal by row, then there exist minimum-energy multiwavelet frames associated with the function  $\Phi$ .

By Theorem 3.1, if we can find some row vectors  $\alpha_1(z), \dots, \alpha_n(z)$  with multiplicity  $ar$  and the matrix in (3.3) formed by the vectors and the symbol of  $\Phi$  satisfies standard orthogonal by column, there exist a minimum-energy multiwavelet frames associated with  $\Phi$ , and vice versa. However, the number of columns in the symbol of  $\Phi$  is so larger, that it is not easy to find the frames using Theorem 3.1. Corollary 3.7 requests some column vectors  $\beta_1(u), \dots, \beta_m(u)$  with multiplicity  $r$  and the matrix in (3.49) formed by the vectors and the polynomial components of  $P(z)$  satisfies standard orthogonal by row, then we can find a minimum-energy frames associated with  $\Phi$ . Obviously, the problem is vastly simplified.

For some multiscaling vector-valued function with small multiplicity which satisfies the conditions in Theorem 3.2, the matrix  $P_{a+1}(u)$  that makes the block matrix in (3.49) satisfied standard orthogonal by column can be found using the method of undetermined coefficients. We will give some examples later.

#### 4. Decomposition and Reconstruction Formulas of Minimum-Energy Multiwavelet Frames

Suppose the multiscaling vector-valued function  $\Phi$  has an associated minimum-energy multiwavelet frames  $\{\Psi^1, \dots, \Psi^N\}$ . Now, we consider the projection operators  $\mathbf{P}_j$  of  $L^2(\mathbb{R})$  onto the nested subspace  $V_j$  defined by

$$\mathbf{P}_j f := \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} \langle f, \phi_{\tau,j,k} \rangle \phi_{\tau,j,k}. \quad (4.1)$$

Then the formula (2.8) can be rewritten as

$$\mathbf{P}_{j+1} f - \mathbf{P}_j f := \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} \langle f, \psi_{\tau,j,k}^i \rangle \psi_{\tau,j,k}^i. \quad (4.2)$$

In other words, the error term  $g_j = \mathbf{P}_{j+1} f - \mathbf{P}_j f$  between consecutive projections is given by the frame expansion:

$$g_j = \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} \langle f, \psi_{\tau,j,k}^i \rangle \psi_{\tau,j,k}^i. \quad (4.3)$$

Suppose that the error term  $g_j$  has other expansion in terms of the frames  $\{\Psi^1, \dots, \Psi^N\}$ , that is,

$$g_j = \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} c_{\tau,j,k} \psi_{\tau,j,k}^i. \quad (4.4)$$

Then by using both (4.3) and (4.4), we have

$$\langle g_j, f \rangle = \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |\langle f, \psi_{\tau,j,k}^i \rangle|^2 = \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} c_{\tau,j,k} \overline{\langle f, \psi_{\tau,j,k}^i \rangle}, \quad (4.5)$$

and this derives

$$\begin{aligned} 0 &\leq \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |c_{\tau,j,k} - \langle f, \psi_{\tau,j,k}^i \rangle|^2 \\ &= \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |c_{\tau,j,k}|^2 - 2 \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} c_{\tau,j,k} \overline{\langle f, \psi_{\tau,j,k}^i \rangle} + \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |\langle f, \psi_{\tau,j,k}^i \rangle|^2 \\ &= \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |c_{\tau,j,k}|^2 - \sum_{i=1}^N \sum_{\tau=1}^r \sum_{k \in \mathbb{Z}} |\langle f, \psi_{\tau,j,k}^i \rangle|^2. \end{aligned} \quad (4.6)$$

This inequality means that the coefficients of the error term  $g_j$  in (4.3) have minimal  $l^2$ -norm among all sequences  $\{c_{\tau,j,k}\}$  which satisfy (4.4).

We next discuss minimum-energy multiwavelet frames decomposition and reconstruction. For any  $f \in L^2(\mathbb{R})$ , define the vector coefficients as follows:

$$\mathbf{c}_{j,k} := \langle f, \Phi_{j,k} \rangle, \quad \mathbf{d}_{j,k} := \langle f, \Psi_{j,k}^i \rangle \quad i = 1, \dots, N. \quad (4.7)$$

The inner product of  $f$  with vector-valued  $\Phi_{j,k}$ ,  $\Psi_{j,k}^i$ ,  $i = 1, \dots, N$  is a vector, its every component is the inner product of  $f$  with the corresponding component of  $\Phi_{j,k}$ ,  $\Psi_{j,k}^i$ ,  $i = 1, \dots, N$ .

### (1) Decomposition Algorithm

suppose the vector coefficients  $\{\mathbf{c}_{j+1,l} : l \in \mathbf{Z}\}$  are known. By the two-scale relations (2.2) and (3.1), we have

$$\Phi_{j,l}(x) = \frac{1}{\sqrt{a}} \sum_{k \in \mathbf{Z}} P_{k-al} \Phi_{j+1,k}(x), \quad \Psi_{j,l}^i(x) = \frac{1}{\sqrt{a}} \sum_{k \in \mathbf{Z}} Q_{k-al}^i \Psi_{j+1,k}^i(x), \quad i = 1, \dots, N. \quad (4.8)$$

Then, the decomposition algorithm is given as

$$\mathbf{c}_{j,l} = \frac{1}{\sqrt{a}} \sum_{k \in \mathbf{Z}} P_{k-al} \mathbf{c}_{j+1,k}, \quad \mathbf{d}_{j,l}^i = \frac{1}{\sqrt{a}} \sum_{k \in \mathbf{Z}} Q_{k-al}^i \mathbf{d}_{j+1,k}^i, \quad i = 1, \dots, N. \quad (4.9)$$

### (2) Reconstruction Algorithm

from (3.16), it follow that

$$\Phi_{j+1,l}(x) = \frac{1}{\sqrt{a}} \sum_k \left\{ P_{l-ak}^* \Phi_{j,k}(x) + \sum_{i=1}^N Q_{l-ak}^{i*} \Psi_{j,k}^i(x) \right\}. \quad (4.10)$$

Taking the inner products on both sides of this equality, we get

$$\mathbf{c}_{j+1,l} = \frac{1}{\sqrt{a}} \sum_k \left\{ P_{l-ak}^* \mathbf{c}_{j,k} + \sum_{i=1}^N Q_{l-ak}^{i*} \mathbf{d}_{j,k}^i \right\}. \quad (4.11)$$

## 5. Numerical Examples

By Theorem 3.6, the orthogonal multiwavelet always have minimum-energy multiwavelet frames associated with them, for example, DGHM multiwavelet and Chui-Lian multiwavelet. These examples are trivial. In this section, we will construct some minimum-energy multiwavelet frames in general sense.

It is well known that the  $m$ th-order cardinal B-spline  $N_m^a(x)$  with dilation factor  $a$  has the two-scale relation as follows:

$$\widehat{N}_m^a(\omega) = P_m^a(z) \widehat{N}_m^a\left(\frac{\omega}{a}\right), \quad P_m^a(z) = \left(\frac{1+z+\cdots+z^{a-1}}{a}\right)^m, \quad z = e^{-i\omega/a}. \quad (5.1)$$

In addition, if a scale wavelet  $\phi(x)$  satisfies the refinable function

$$\phi(x) = \sum_{k=k_0}^{k_1} p_k \phi(ax - k), \quad (5.2)$$

and let  $\Phi(x) = (\phi(x), \phi(x-1), \dots, \phi(x-r+1))^T$ , then the vector-valued function  $\Phi$  satisfies (2.2) with some matrixes  $\{P_k\}$ .

Below, upon these conclusions, using Theorem 3.5 and Corollary 3.7 in Section 3, the minimum-energy multiwavelet frames be presented with the dilation factors  $a = 2$ ,  $a = 3$ ,  $a = 4$ , respectively.

### 5.1. $a = 2$

*Example 5.1.* With  $a = 2$ , the symbol of the B-spline  $N_2^2(x)$  is

$$P_2^2(z) = \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^2. \quad (5.3)$$

Take  $\phi(x) = N_2^2(x)$ , and the support of this function is  $[0, 2]$ . The function satisfies

$$\phi(x) = \frac{1}{4}\phi(2x) + \frac{1}{2}\phi(2x-1) + \frac{1}{4}\phi(2x-2). \quad (5.4)$$

Let  $\Phi(x) = (\phi(x), \phi(x-1))^T$ , and

$$\Phi(x) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \Phi(2x) + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \Phi(2x-1) + \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Phi(2x-2) + \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Phi(2x-3). \quad (5.5)$$

The coefficient matrixes in (5.5) are not unique.

And the symbol of  $\Phi$  has polyphase components as follows:

$$P_1(u) = P_2(u) = \frac{\sqrt{2}}{2} \left( \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right). \quad (5.6)$$



Take

$$P_3(u) = \frac{\sqrt{2}}{2} \left( \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \right), \quad (5.7)$$

which satisfies

$$P_1(u)P_1(u)^* + P_2(u)P_2(u)^* + P_3(u)P_3(u)^* = I_2. \quad (5.8)$$

Using Theorem 3.5, we can get matrix the following:

$$C(u) = \frac{\sqrt{2}}{2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{u}{2} & \frac{u}{2} & \frac{u}{2} & \frac{u}{2} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{u}{2} & \frac{u}{2} & -\frac{u}{2} & \frac{u}{2} \end{pmatrix}, \quad (5.9)$$

which satisfy the formula (3.35). Then we take symbols as

$$Q_1(z) = \frac{1}{2} \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

$$Q_2(z) = \frac{1}{2} \left( \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + z^3 \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \right). \quad (5.10)$$

The graphs of  $\Phi$  and its minimum-energy frames are shown in Figure 1.

We may discover from Figure 1 that every component of minimum-energy frames is (anti)symmetrical.

*Example 5.2.* With  $a = 2$ , the symbol of the B-spline  $N_3^2(x)$  is

$$P_3^2(z) = \frac{1}{8} + \frac{3}{8}z + \frac{3}{8}z^2 + \frac{1}{8}z^3. \quad (5.11)$$

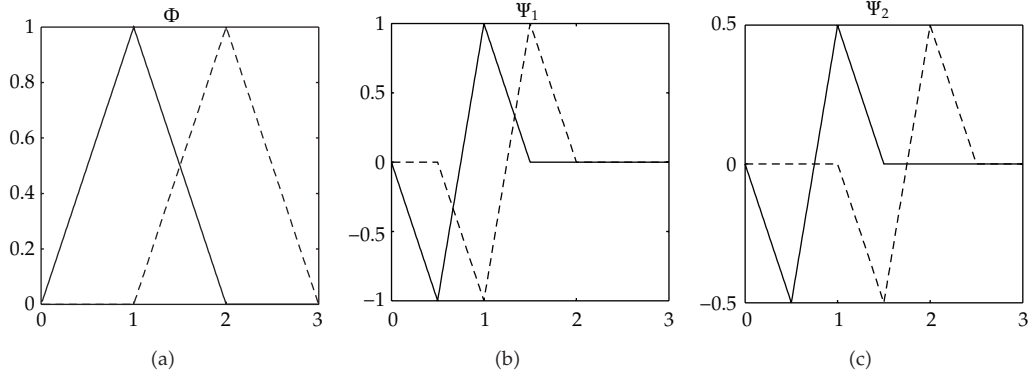


Figure 1

Take  $\phi(x) = N_3^2(x)$ , and the support of this function is  $[0,3]$ .

(1) Let  $\Phi(x) = (\phi(x), \phi(x-1))^T$ , and

$$\begin{aligned} \Phi(x) = & \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{pmatrix} \Phi(2x) + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \Phi(2x-1) + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \Phi(2x-2) \\ & + \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Phi(2x-3) + \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \Phi(2x-4). \end{aligned} \quad (5.12)$$

The symbol of  $\Phi$  has polyphase components as follows:

$$\begin{aligned} P_1(u) &= \frac{\sqrt{2}}{2} \left( \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} + u^2 \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \right), \\ P_2(u) &= \frac{\sqrt{2}}{2} \left( \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right). \end{aligned} \quad (5.13)$$

Take

$$P_3(u) = \begin{pmatrix} -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}u & \frac{\sqrt{2}}{2} & -\frac{1}{2} + \frac{u}{2} & 0 \\ \frac{\sqrt{2}}{4}u + \frac{\sqrt{2}}{4}u^2 & \frac{\sqrt{2}}{2}u & 0 & -\frac{u}{2} + \frac{u^2}{2} \end{pmatrix}, \quad (5.14)$$

which satisfies  $P_1(u)P_1(u)^* + P_2(u)P_2(u)^* + P_3(u)P_3(u)^* = I_2$ . Using Theorem 3.5, we can get the following symbols:

$$\begin{aligned}
 Q_1(z) &= \frac{\sqrt{2}}{2} \left( \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right), \\
 Q_2(z) &= \frac{\sqrt{2}}{2} \left( \begin{pmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix} + z^4 \begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \end{pmatrix} \right), \\
 Q_3(z) &= \frac{\sqrt{2}}{2} \left( \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} + z^3 \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + z^4 \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \right).
 \end{aligned} \tag{5.15}$$

Then, we get the minimum-wavelet frames associated with  $\Phi$ . The graphs of them are shown in Figure 2.

We can discover from Figure 2 that every component of the minimum-energy frames is (anti)symmetrical and smooth.

(2) Take  $\Phi(x) = (\phi(x), \phi(x-1), \phi(x-2))^T$ , which satisfies

$$\begin{aligned}
 \Phi(x) &= \frac{1}{4} \left( \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi(2x+2) + \begin{pmatrix} 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi(2x+1) + \begin{pmatrix} \frac{1}{3} & 1 & 1 \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} \Phi(2x) \right. \\
 &\quad + \begin{pmatrix} 1 & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 \end{pmatrix} \Phi(2x-1) + \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & 1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \Phi(2x-2) \\
 &\quad + \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 1 & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \Phi(2x-3) + \begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix} \Phi(2x-4) \\
 &\quad \left. + \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 1 & 1 & \frac{1}{3} \end{pmatrix} \Phi(2x-5) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 \end{pmatrix} \Phi(2x-6) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{pmatrix} \Phi(2x-7) \right)
 \end{aligned} \tag{5.16}$$

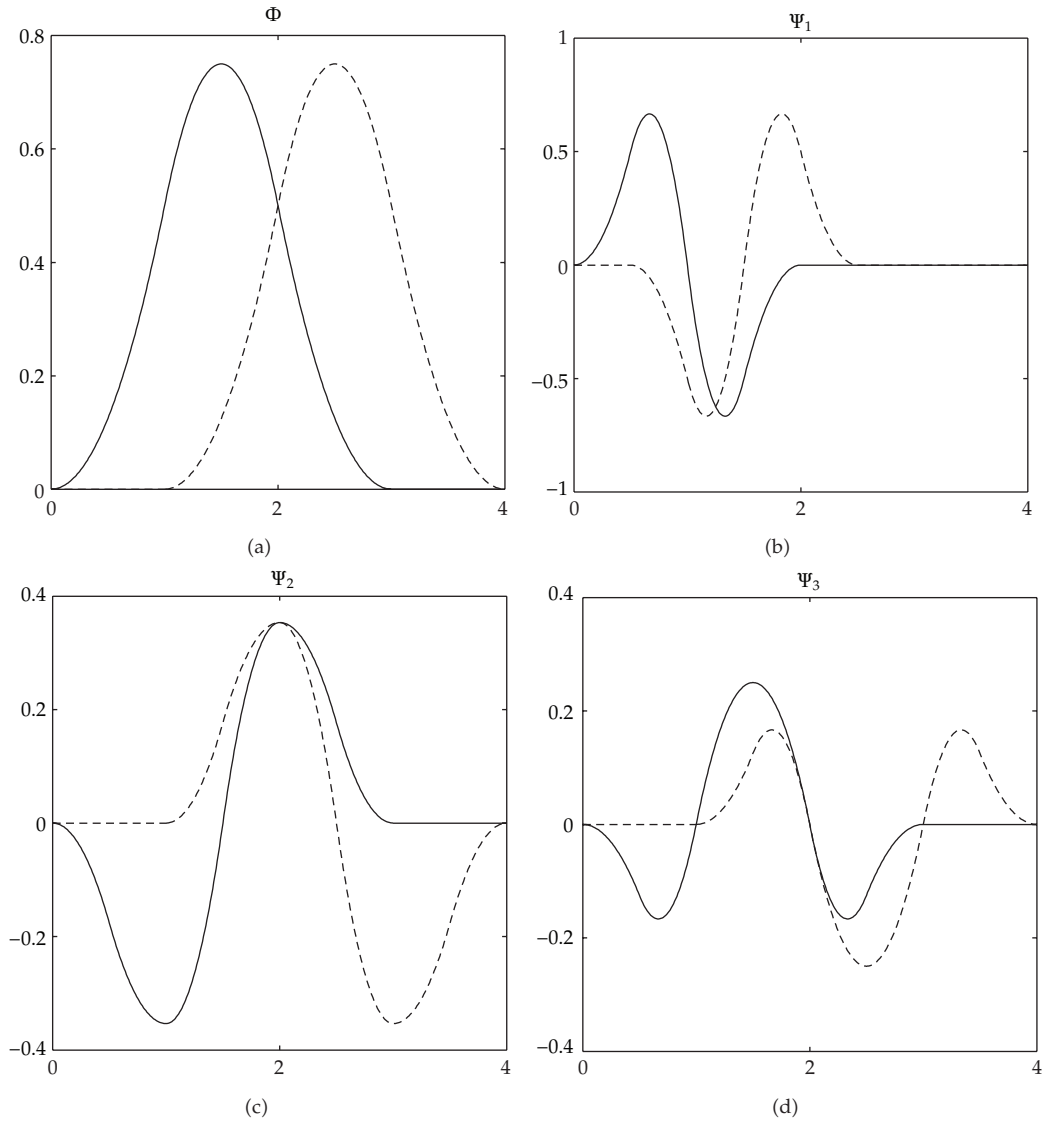


Figure 2

and the symbol of this multiscaling vector-valued function has the following polyphase components:

$$\begin{aligned}
 P_1(u) = \frac{\sqrt{2}}{8} & \left( \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 1 & 1 \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} u + \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & 1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} u^2 \right. \\
 & \left. + \begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix} u^3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 \end{pmatrix} u^4 \right),
 \end{aligned}$$

$$\begin{aligned}
P_2(u) = \frac{\sqrt{2}}{8} & \left( \begin{pmatrix} 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 \end{pmatrix} u + \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 1 & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 \end{pmatrix} u^2 \right. \\
& \left. + \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 1 & 1 & \frac{1}{3} \end{pmatrix} u^3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{pmatrix} u^4 \right).
\end{aligned} \tag{5.17}$$

Let

$$\begin{aligned}
P_3(u) = \frac{\sqrt{2}}{8} \\
\times \begin{pmatrix} \sqrt{6} + \frac{\sqrt{6}}{3}u & 0 & -\left(\sqrt{6} + \frac{\sqrt{6}}{3}u\right) & \sqrt{6}u - \sqrt{6} & 0 & 0 \\ -(\mathcal{A}) & \mathcal{A} & 0 & 0 & \sqrt{6}u^2 - \sqrt{6}u & 0 \\ 0 & -(\mathcal{B}) & \mathcal{B} & 0 & 0 & \sqrt{6}u^3 - \sqrt{6}u^2 \end{pmatrix},
\end{aligned} \tag{5.18}$$

where  $\mathcal{A}$  denotes  $\sqrt{6}u + (\sqrt{6}/3)u^2$ , and  $\mathcal{B}$  denotes  $\sqrt{6}u^2 + (\sqrt{6}/3)u^3$ , which satisfies  $P_1(u)P_1(u)^* + P_2(u)P_2(u)^* + P_3(u)P_3(u)^* = I_3$ . Using Theorem 3.5, we can get

$$\begin{aligned}
Q_1(z) \\
= \frac{\sqrt{2}}{2} & \left( \begin{pmatrix} -0.23547806816473105 & -0.1969247665800399 & -0.14891401559609693 \\ 0.08212785057744523 & -0.18555493781122898 & -0.19106026688126101 \\ 0.027375950192481735 & -0.061851645937076295 & -0.07928888488639749 \end{pmatrix} \right. \\
& + z \begin{pmatrix} 0.9033589710742332 & -0.07112938093948917 & -0.016506887733618497 \\ -0.0608286139211068 & 0.905683150972805 & -0.03808257781757404 \\ -0.020276204640368934 & -0.04704107893504214 & 0.9404994196162103 \end{pmatrix} \\
& + z^2 \begin{pmatrix} -0.20402395049705468 & -0.05175658881673616 & 0.01828281901756732 \\ -0.2489115696557434 & -0.0979600599918386 & -0.016863228869914655 \\ -0.0700638169224561 & -0.1222299444169772 & -0.11122693740098887 \end{pmatrix} \\
& + z^3 \begin{pmatrix} -0.05175658881673616 & 0.01828281901756732 & 0.006094273005855774 \\ -0.0979600599918386 & -0.016863228869914655 & -0.005621076289971551 \\ -0.1222299444169776 & -0.11122693740098887 & -0.03707564580032962 \end{pmatrix} \\
& + z^4 \begin{pmatrix} 0.018282819017567314 & 0.0060942730058557715 & 0 \\ -0.016863228869914648 & -0.005621076289971549 & 0 \\ -0.11122693740098877 & -0.03707564580032959 & 0 \end{pmatrix} \\
& \left. + z^5 \begin{pmatrix} 0.0060942730058557715 & 0 & 0 \\ -0.005621076289971549 & 0 & 0 \\ -0.03707564580032959 & 0 & 0 \end{pmatrix} \right),
\end{aligned}$$

$$\begin{aligned}
& Q_2(z) \\
&= \frac{\sqrt{2}}{2} \left( \begin{pmatrix} -0.09986325103103813 & 0.31589881995517033 & 0.0052538925646165024 \\ 0.0854124922234307 & -0.26079578287935323 & 0.2998893006031239 \\ 0.014450758807607454 & -0.05510303707581709 & -0.30514319316774036 \end{pmatrix} \right. \\
&\quad + z \begin{pmatrix} -0.016309066862055847 & 0.005908379170891813 & 0.019381713326762175 \\ 0.0045583062090611155 & -0.010950091637853966 & -0.0031215427310982043 \\ 0.011750760652994713 & 0.005041712466962157 & -0.016260170595664 \end{pmatrix} \\
&\quad + z^2 \begin{pmatrix} -0.20916114471771918 & -0.051515547612800944 & 0.02048043820474356 \\ -0.023639974479628924 & -0.01950964138878848 & -0.013083356132526147 \\ 0.23280111919734814 & 0.07102518900158937 & -0.007397082072217509 \end{pmatrix} \\
&\quad + z^3 \begin{pmatrix} -0.051515547612800944 & 0.02048043820474356 & 0.0068268127349145336 \\ -0.019509641388788437 & -0.013083356132526147 & -0.00436111871084204 \\ 0.07102518900158936 & -0.007397082072217509 & -0.0024656940240724914 \end{pmatrix} \\
&\quad + z^4 \begin{pmatrix} 0.020480438204743462 & 0.006826812734914494 & 0 \\ -0.013083356132526154 & -0.004361118710842047 & 0 \\ -0.0073970820722173754 & -0.002465694024072461 & 0 \end{pmatrix} \\
&\quad \left. + z^5 \begin{pmatrix} 0.006826812734914494 & 0 & 0 \\ -0.004361118710842047 & 0 & 0 \\ -0.002465694024072461 & 0 & 0 \end{pmatrix} \right),
\end{aligned}$$

$$\begin{aligned}
& Q_3(z) \\
&= \frac{\sqrt{2}}{2} \left( \begin{pmatrix} -0.04725690064321836 & 0.21949688473386203 & 0.1639130739275549 \\ 0.2523328524498961 & -0.7281995751316489 & 0.15918487102463674 \\ -0.003904624220396051 & 0.054187773506410915 & -0.7337722255518694 \end{pmatrix} \right. \\
&\quad + z \begin{pmatrix} -0.0777261828042069 & -0.06662667519576504 & 0.0330957179061223 \\ -0.02879898221803944 & -0.07331833791750912 & 0.008051002298630018 \\ -0.042473900845222755 & -0.03375725777108183 & 0.037548046190520684 \end{pmatrix} \\
&\quad + z^2 \begin{pmatrix} -0.6930584034579804 & -0.08791155686974436 & 0.16099639981828018 \\ -0.0287070169546376 & 0.06098379400257172 & 0.07937189961088224 \\ 0.06463681665260418 & 0.11607547646429853 & 0.10634610477760928 \end{pmatrix} \\
&\quad + z^3 \begin{pmatrix} -0.08791155686974436 & 0.16099639981828018 & 0.053665466606093386 \\ 0.06098379400257172 & 0.07937189961088224 & 0.026457299870294076 \\ 0.11607547646429858 & 0.10634610477760928 & 0.03544870159253642 \end{pmatrix} \\
&\quad + z^4 \begin{pmatrix} 0.16099639981828012 & 0.05366546660609337 & 0 \\ 0.07937189961088217 & 0.026457299870294056 & 0 \\ 0.10634610477760917 & 0.035448701592536394 & 0 \end{pmatrix} \\
&\quad \left. + z^5 \begin{pmatrix} 0.05366546660609337 & 0 & 0 \\ 0.026457299870294056 & 0 & 0 \\ 0.035448701592536394 & 0 & 0 \end{pmatrix} \right).
\end{aligned}$$

(5.19)

They are the symbols of the minimum-energy multiwavelet frames associated with  $\Phi$ . The graphs of them are shown in Figure 3.

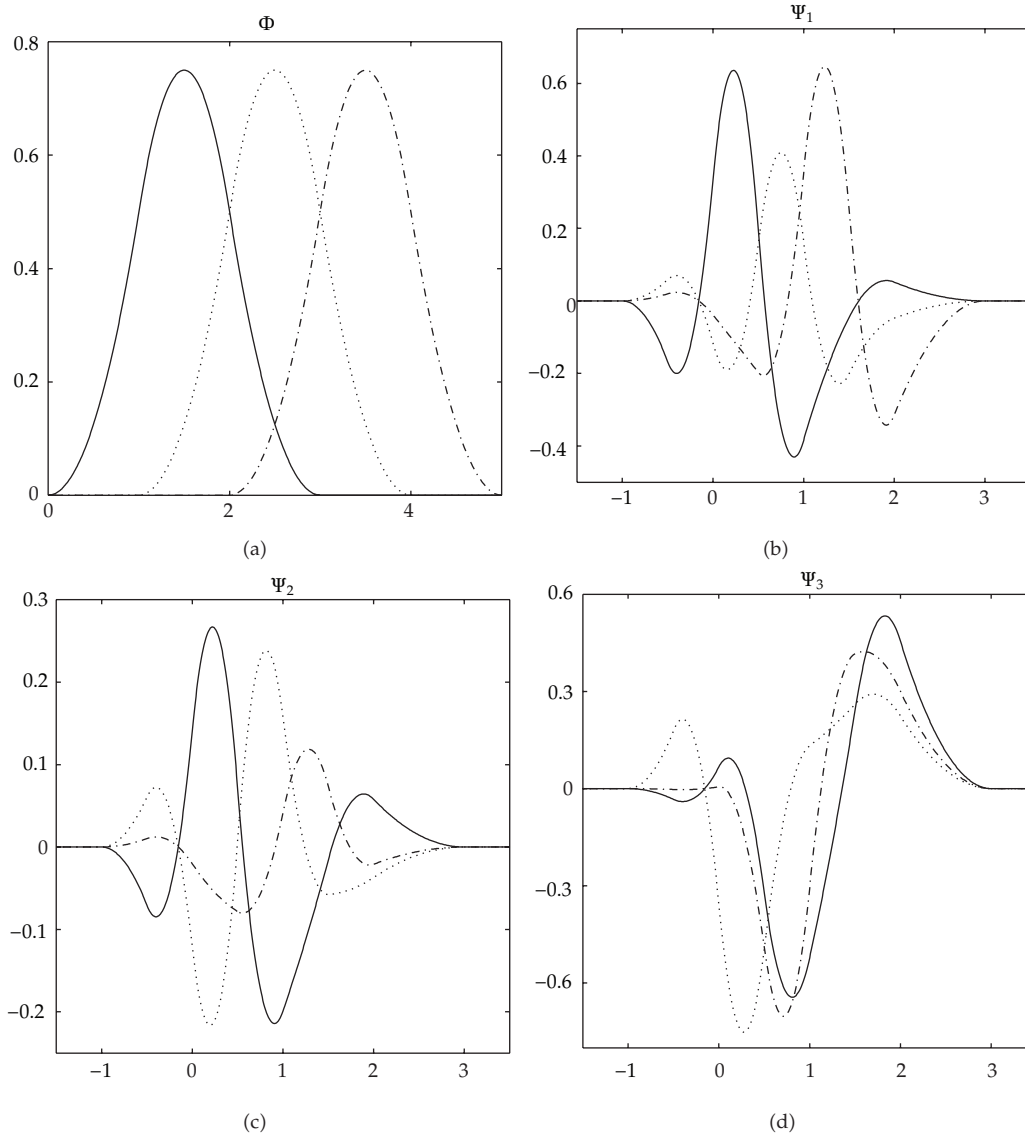


Figure 3

We can discover from Figure 3 that every component of the minimum-energy frames is smooth. When  $r = 3$ , it is very difficult to construct the minimum-energy multiwavelet frames with symmetry.

**5.2.  $a = 3$**

Example 5.3. With  $a = 3$ , the symbol of the B-spline  $\phi(x) = N_3^3(x)$  is

$$P_3^3(z) = \frac{1 + 3z + 6z^2 + 7z^3 + 6z^4 + 3z^5 + z^6}{27}. \tag{5.20}$$

Take  $\phi(x) = N_3^3(x)$ , and the support of this function is  $[0, 4]$ .

(1) Let  $\Phi(x) = (\phi(x), \phi(x-1))^T$ , and this vector-valued function satisfies

$$\begin{aligned} \Phi(x) = \frac{1}{9} & \left( \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \Phi(3x) + \begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix} \Phi(3x-1) + \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix} \Phi(3x-2) \right. \\ & + \begin{pmatrix} 7 & 3 \\ 2 & 3 \end{pmatrix} \Phi(3x-3) + \begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix} \Phi(3x-4) + \begin{pmatrix} 3 & 1 \\ 2 & 7 \end{pmatrix} \Phi(3x-5) \\ & \left. + \begin{pmatrix} 0 & 0 \\ 7 & 3 \end{pmatrix} \Phi(3x-6) + \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix} \Phi(3x-7) + \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \Phi(3x-8) \right). \end{aligned} \quad (5.21)$$

The symbol of  $\Phi(x)$  is

$$\begin{aligned} P(z) = \frac{1}{27} & \left( \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix} + z^2 \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix} + z^3 \begin{pmatrix} 7 & 3 \\ 2 & 3 \end{pmatrix} + z^4 \begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix} \right. \\ & \left. + z^5 \begin{pmatrix} 3 & 1 \\ 2 & 7 \end{pmatrix} + z^6 \begin{pmatrix} 0 & 0 \\ 7 & 3 \end{pmatrix} + z^7 \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix} + z^8 \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \right), \end{aligned} \quad (5.22)$$

and its polynomial components are

$$\begin{aligned} P_1(u) &= \frac{\sqrt{3}}{27} \left( \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 7 & 3 \\ 2 & 3 \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ 7 & 3 \end{pmatrix} u^2 \right), \\ P_2(u) &= \frac{\sqrt{3}}{27} \left( \begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix} u^2 \right), \\ P_3(u) &= \frac{\sqrt{3}}{27} \left( \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 7 \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} u^2 \right), \end{aligned} \quad (5.23)$$

then

$$P_1(u)P_1(u)^* + P_2(u)P_2(u)^* + P_3(u)P_3(u)^* = \frac{1}{486} \begin{pmatrix} \frac{50}{u} + 143 + 50u & \frac{50}{u^2} + \frac{143}{u} + 50 \\ 50 + 143u + 50u^2 & \frac{50}{u} + 143 + 50u \end{pmatrix}. \quad (5.24)$$

(i) This example satisfies the conditions in Theorem 3.5. Let

$$P_4(u) = \frac{\sqrt{3}}{27} \begin{pmatrix} 5 - 5u \\ 5u - 5u^2 \end{pmatrix}. \quad (5.25)$$



The sum of  $l^2$ -norm for every row in the matrix in (3.34) formed by  $P_1(u)$ ,  $P_2(u)$ ,  $P_3(u)$ , and  $P_4(u)$  is equivalent to 1. Using Theorem 3.5, we can get the symbols of the minimum-energy frames associated with  $\Phi$  the following:

$$\begin{aligned}
Q_1(z) &= \frac{1}{27} \left( \begin{pmatrix} -1 & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} -\frac{3}{2} & -3 \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} -3 & -\frac{7}{2} \\ 0 & 0 \end{pmatrix} + z^3 \begin{pmatrix} \frac{7}{2} & 3 \\ -1 & -\frac{3}{2} \end{pmatrix} \right. \\
&\quad + z^4 \begin{pmatrix} 3 & \frac{3}{2} \\ -\frac{3}{2} & -3 \end{pmatrix} + z^5 \begin{pmatrix} \frac{3}{2} & 1 \\ -3 & -\frac{7}{2} \end{pmatrix} \\
&\quad \left. + z^6 \begin{pmatrix} 0 & 0 \\ 7 & 3 \end{pmatrix} + z^7 \begin{pmatrix} 0 & 0 \\ 3 & \frac{3}{2} \end{pmatrix} + z^8 \begin{pmatrix} 0 & 0 \\ \frac{3}{2} & 1 \end{pmatrix} \right), \\
Q_2(z) &= \frac{1}{27} \left( \begin{pmatrix} 0 & -\frac{9\sqrt{3}}{2} \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} \frac{9\sqrt{3}}{2} & 0 \\ 0 & 0 \end{pmatrix} + z^3 \begin{pmatrix} 0 & 0 \\ 0 & -\frac{9\sqrt{3}}{2} \end{pmatrix} + z^4 \begin{pmatrix} 0 & 0 \\ \frac{9\sqrt{3}}{2} & 0 \end{pmatrix} \right), \\
Q_3(z) &= \frac{1}{27} \left( \begin{pmatrix} \frac{9\sqrt{3}}{2} & -3\sqrt{3} \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} -3\sqrt{3} & \frac{3\sqrt{3}}{2} \\ 0 & 0 \end{pmatrix} + z^3 \begin{pmatrix} 0 & 0 \\ \frac{9\sqrt{3}}{2} & -3\sqrt{3} \end{pmatrix} \right. \\
&\quad \left. + z^4 \begin{pmatrix} 0 & 0 \\ -3\sqrt{3} & \frac{3\sqrt{3}}{2} \end{pmatrix} \right), \\
Q_4(z) &= \frac{1}{27} \left( \begin{pmatrix} \frac{9}{2}\sqrt{\frac{3}{35}} & -3\sqrt{\frac{3}{35}} \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} -3\sqrt{\frac{3}{35}} & -\frac{51}{2}\sqrt{\frac{3}{35}} \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} 27\sqrt{\frac{3}{35}} & 0 \\ 0 & 0 \end{pmatrix} \right. \\
&\quad \left. + z^3 \begin{pmatrix} 0 & 0 \\ \frac{9}{2}\sqrt{\frac{3}{35}} & -3\sqrt{\frac{3}{35}} \end{pmatrix} + z^4 \begin{pmatrix} 0 & 0 \\ -3\sqrt{\frac{3}{35}} & -\frac{51}{2}\sqrt{\frac{3}{35}} \end{pmatrix} + z^5 \begin{pmatrix} 0 & 0 \\ 27\sqrt{\frac{3}{35}} & 0 \end{pmatrix} \right), \\
Q_5(z) &= \frac{1}{27} \left( \begin{pmatrix} 45\sqrt{\frac{2}{133}} & \frac{45}{\sqrt{266}} \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} \frac{45}{\sqrt{266}} & -45\sqrt{\frac{2}{133}} \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} -45\sqrt{\frac{2}{133}} & 0 \\ 0 & 0 \end{pmatrix} \right. \\
&\quad \left. + z^3 \begin{pmatrix} 0 & 0 \\ 45\sqrt{\frac{2}{133}} & \frac{45}{\sqrt{266}} \end{pmatrix} + z^4 \begin{pmatrix} 0 & 0 \\ \frac{45}{\sqrt{266}} & -45\sqrt{\frac{2}{133}} \end{pmatrix} + z^5 \begin{pmatrix} 0 & 0 \\ -45\sqrt{\frac{2}{133}} & 0 \end{pmatrix} \right),
\end{aligned}$$

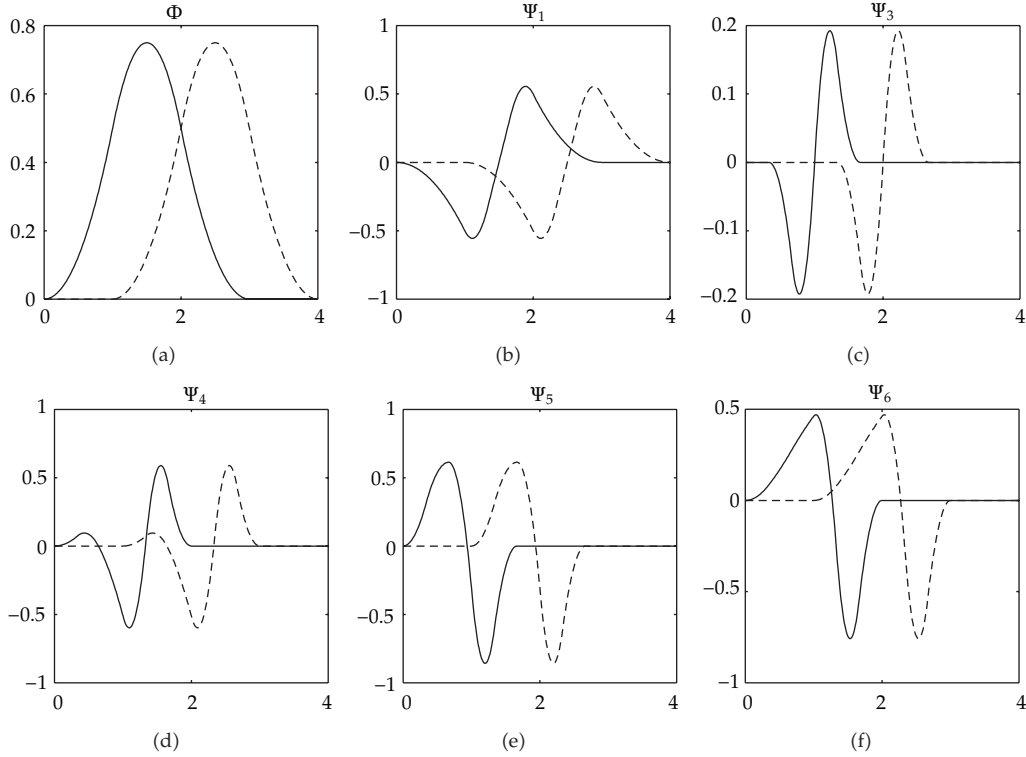


Figure 4

$$\begin{aligned}
 Q_6(z) = \frac{1}{27} & \left( \begin{pmatrix} \frac{14}{\sqrt{95}} & \frac{33}{2\sqrt{95}} \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} \frac{33}{2\sqrt{95}} & \frac{24}{\sqrt{95}} \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} \frac{24}{\sqrt{95}} & -\sqrt{95} \\ 0 & 0 \end{pmatrix} \right. \\
 & \left. + z^3 \begin{pmatrix} 0 & 0 \\ \frac{14}{\sqrt{95}} & \frac{33}{2\sqrt{95}} \end{pmatrix} + z^4 \begin{pmatrix} 0 & 0 \\ \frac{33}{2\sqrt{95}} & \frac{24}{\sqrt{95}} \end{pmatrix} + z^5 \begin{pmatrix} 0 & 0 \\ \frac{24}{\sqrt{95}} & -\sqrt{95} \end{pmatrix} \right).
 \end{aligned} \tag{5.26}$$

The graphs of  $\Phi$  and the minimum-energy frames associated with it are shown in Figure 4.

From this figure, we can discover that every component of the minimum-energy frames is smooth. The first vector-valued function of frames is antisymmetry and the second function vanishes.

(ii) In fact, this example also satisfies the conditions of Corollary 3.7. Take

$$P_4(u) = \frac{\sqrt{3}}{27} \begin{pmatrix} 5+5u & \sqrt{\frac{43}{2}} & 5\sqrt{2}-5\sqrt{2}u & 0 \\ -5u-5u^2 & -\sqrt{\frac{43}{2}}u & 0 & 5\sqrt{2}u-5\sqrt{2}u^2 \end{pmatrix} \tag{5.27}$$

and with  $P_1(u)$ ,  $P_2(u)$ , and  $P_3(u)$  to form matrix (3.49), which satisfies standard orthogonal by row. By Theorem 3.6, we can get minimum-energy multiwavelet frames associated with  $\Phi$ .

(2) Let  $\Phi(x) = (\phi(x), \phi(x-1), \phi(x-2))^T$ . This vector-valued function satisfies

$$\begin{aligned} \Phi(x) = \frac{1}{9} & \left( \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi(3x+2) + \begin{pmatrix} 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi(3x+1) + \begin{pmatrix} \frac{1}{3} & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi(3x) \right. \\ & + \begin{pmatrix} 1 & 2 & \frac{7}{3} \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} \Phi(3x-1) + \begin{pmatrix} 2 & \frac{7}{3} & 2 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 \end{pmatrix} \Phi(3x-2) + \begin{pmatrix} \frac{7}{3} & 2 & 1 \\ \frac{1}{3} & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Phi(3x-3) \\ & + \begin{pmatrix} 2 & 1 & \frac{1}{3} \\ 1 & 2 & \frac{7}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \Phi(3x-4) + \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 2 & \frac{7}{3} & 2 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \Phi(3x-5) + \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 2 & 1 \\ \frac{1}{3} & 1 & 2 \end{pmatrix} \Phi(3x-6) \\ & + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & \frac{1}{3} \\ 1 & 2 & \frac{7}{3} \end{pmatrix} \Phi(3x-7) + \begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 \\ 2 & \frac{7}{3} & 2 \end{pmatrix} \Phi(3x-8) + \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{7}{3} & 2 & 1 \end{pmatrix} \Phi(3x-9) \\ & \left. + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & \frac{1}{3} \end{pmatrix} \Phi(3x-10) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 \end{pmatrix} \Phi(3x-11) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{pmatrix} \Phi(3x-12) \right), \end{aligned}$$

$$\begin{aligned} P_1(u)P_1(u)^* + P_2(u)P_2(u)^* + P_3(u)P_3(u)^* &= \frac{1}{729} \\ & \times \begin{pmatrix} \frac{1}{u^2} + \frac{50}{u} + 141 + 50u + u^2 & \frac{1}{u^3} + \frac{50}{u^2} + \frac{141}{u} + 50 + u & \frac{1}{u^4} + \frac{50}{u^3} + \frac{141}{u^2} + \frac{50}{u} + 1 \\ \frac{1}{u} + 50 + 141u + 50u^2 + u^3 & \frac{1}{u^2} + \frac{50}{u} + 141 + 50u + u^2 & \frac{1}{u^3} + \frac{50}{u^2} + \frac{141}{u} + 50 + u \\ 1 + 50u + 141u^2 + 50u^3 + u^4 & \frac{1}{u} + 50 + 141u + 50u^2 + u^3 & \frac{1}{u^2} + \frac{50}{u} + 141 + 50u + u^2 \end{pmatrix}. \end{aligned} \quad (5.28)$$

Let

$$P_4(u) = \begin{pmatrix} x(u) & 0 & -x(u) & y(u) & 0 & 0 \\ -x(u)u & x(u) & 0 & 0 & y(u)u & 0 \\ 0 & -x(u)u & x(u)u^2 & 0 & 0 & y(u)u^2 \end{pmatrix}, \quad (5.29)$$

where

$$\begin{aligned} x(u) &= 0.4067366251768559 + 0.16724108784847952u + 0.003372556164290336u^2, \\ y(u) &= 0.46250686620877374 - 0.45360921162651374u - 0.008897654582258595u^2, \end{aligned} \quad (5.30)$$

then

$$P_1(u)P_1(u)^* + P_2(u)P_2(u)^* + P_3(u)P_3(u)^* + P_4(u)P_4(u)^* = I_3. \quad (5.31)$$

By Corollary 3.7 and Theorem 3.6, we know that the existence of the minimum-energy multiwavelet frames is associated with  $\Phi$ .

### 5.3. $a = 4$

With  $a = 4$ , the symbol of the B-spline  $\phi(x) = N_4^4(x)$  is

$$P_4^4(z) = \frac{1 + 4z + 10z^2 + 20z^3 + 31z^4 + 40z^5 + 44z^6 + 40z^7 + 31z^8 + 20z^9 + 10z^{10} + 4z^{11} + z^{12}}{256}. \quad (5.32)$$

Take  $\phi(x) = N_4^4(x)$ , and the support of this function is  $[0, 5]$ , the symbol is

$$\begin{aligned} P(z) &= \frac{1}{256} \left( \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} 5 & 10 \\ 0 & 0 \end{pmatrix} + z^3 \begin{pmatrix} 10 & 31 \\ 0 & 0 \end{pmatrix} \right. \\ &\quad + z^4 \begin{pmatrix} 31 & 20 \\ 1 & 2 \end{pmatrix} + z^5 \begin{pmatrix} 20 & 22 \\ 2 & 5 \end{pmatrix} + z^6 \begin{pmatrix} 22 & 20 \\ 5 & 10 \end{pmatrix} + z^7 \begin{pmatrix} 20 & 31 \\ 10 & 31 \\ 10 & 31 \\ 2 \end{pmatrix} \\ &\quad + z^8 \begin{pmatrix} 31 & 10 \\ 31 & 20 \\ 2 \end{pmatrix} z^9 \begin{pmatrix} 10 & 5 \\ 20 & 22 \end{pmatrix} + z^{10} \begin{pmatrix} 5 & 2 \\ 22 & 20 \end{pmatrix} + z^{11} \begin{pmatrix} 2 & 1 \\ 20 & 31 \\ 2 \end{pmatrix} \\ &\quad \left. + z^{12} \begin{pmatrix} 0 & 0 \\ 31 & 10 \\ 2 \end{pmatrix} + z^{13} \begin{pmatrix} 0 & 0 \\ 10 & 5 \end{pmatrix} + z^{14} \begin{pmatrix} 0 & 0 \\ 5 & 2 \end{pmatrix} + z^{15} \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \right). \end{aligned} \quad (5.33)$$

The polynomial components of  $P(z)A = \pi r^2$

$$\begin{aligned}
 P_1(u) &= \frac{1}{128} \left( \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} \frac{31}{2} & 20 \\ 1 & 2 \end{pmatrix} + u^2 \begin{pmatrix} \frac{31}{2} & 10 \\ \frac{31}{2} & 20 \end{pmatrix} + u^3 \begin{pmatrix} 0 & 0 \\ \frac{31}{2} & 10 \end{pmatrix} \right), \\
 P_2(u) &= \frac{1}{128} \left( \begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} 20 & 22 \\ 2 & 5 \end{pmatrix} + u^2 \begin{pmatrix} 10 & 5 \\ 20 & 22 \end{pmatrix} + u^3 \begin{pmatrix} 0 & 0 \\ 10 & 5 \end{pmatrix} \right), \\
 P_3(u) &= \frac{1}{128} \left( \begin{pmatrix} 5 & 10 \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} 22 & 20 \\ 5 & 10 \end{pmatrix} + u^2 \begin{pmatrix} 5 & 2 \\ 22 & 20 \end{pmatrix} + u^3 \begin{pmatrix} 0 & 0 \\ 5 & 2 \end{pmatrix} \right), \\
 P_4(u) &= \frac{1}{128} \left( \begin{pmatrix} 10 & \frac{31}{2} \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} 20 & \frac{31}{2} \\ 10 & \frac{31}{2} \end{pmatrix} + u^2 \begin{pmatrix} 2 & 1 \\ 20 & \frac{31}{2} \end{pmatrix} + u^3 \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \right),
 \end{aligned} \tag{5.34}$$

and they satisfy the conditions in Theorem 3.5. If we take

$$P_5(u) = \frac{1}{128} \begin{pmatrix} a + bu + cu^2 \\ au + bu^2 + cu^3 \end{pmatrix}, \tag{5.35}$$

where

$$\begin{aligned}
 a &= \frac{(-4467 + \sqrt{19539353})\sqrt{4467 + \sqrt{19539353}}}{1288}, \\
 b &= \frac{(-3823 + \sqrt{19539353})\sqrt{4467 + \sqrt{19539353}}}{1288}, \\
 c &= \frac{\sqrt{4467 + \sqrt{19539353}}}{2},
 \end{aligned} \tag{5.36}$$

then the sum of  $l^2$ -norm for every row of matrix in (3.49) formed by  $P_1(u)$ ,  $P_2(u)$ ,  $P_3(u)$ ,  $P_4(u)$ , and  $P_5(u)$  is equivalent to 1. Using the method in Theorem 3.5, we can get

$$\begin{aligned}
 Q_1(u) &= \begin{pmatrix} 0.02144655154326434 & 0.04289310308652868 \\ 0 & 0 \end{pmatrix} \\
 &+ z \begin{pmatrix} 0.04289310308652868 & 0.1072327577163217 \\ 0 & 0 \end{pmatrix} \\
 &+ z^2 \begin{pmatrix} 0.1072327577163217 & 0.2144655154326434 \\ 0 & 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
& + z^3 \begin{pmatrix} 0.2144655154326434 & 0.3324215489205973 \\ 0 & 0 \end{pmatrix} \\
& - z^4 \begin{pmatrix} 0.19702904865778179 & 0.15557470886671548 \\ -0.02144655154326434 & -0.04289310308652868 \end{pmatrix} \\
& - z^5 \begin{pmatrix} 0.15557470886671548 & 0.11193838675771434 \\ -0.04289310308652868 & -0.1072327577163217 \end{pmatrix} \\
& - z^6 \begin{pmatrix} 0.11193838675771434 & 0.07664965153915186 \\ -0.1072327577163217 & -0.2144655154326434 \end{pmatrix} \\
& - z^7 \begin{pmatrix} 0.07664965153915186 & 0.053977382251572635 \\ -0.2144655154326434 & -0.3324215489205973 \end{pmatrix} \\
& - z^8 \begin{pmatrix} 0.044111750085627274 & 0.0284591936036305 \\ 0.19702904865778179 & 0.15557470886671548 \end{pmatrix} \\
& - z^9 \begin{pmatrix} 0.0284591936036305 & 0.01422959680181525 \\ 0.15557470886671548 & 0.11193838675771434 \end{pmatrix} \\
& - z^{10} \begin{pmatrix} 0.01422959680181525 & 0.0056918387207261 \\ 0.11193838675771434 & 0.07664965153915186 \end{pmatrix} \\
& - z^{11} \begin{pmatrix} 0.0056918387207261 & 0.00284591936036305 \\ 0.07664965153915186 & 0.053977382251572635 \end{pmatrix} \\
& - z^{12} \begin{pmatrix} 0 & 0 \\ 0.044111750085627274 & 0.0284591936036305 \end{pmatrix} \\
& - z^{13} \begin{pmatrix} 0 & 0 \\ 0.0284591936036305 & 0.01422959680181525 \end{pmatrix} \\
& - z^{14} \begin{pmatrix} 0 & 0 \\ 0.01422959680181525 & 0.0056918387207261 \end{pmatrix} \\
& - z^{15} \begin{pmatrix} 0 & 0 \\ 0.0056918387207261 & 0.00284591936036305 \end{pmatrix}, \\
Q_2(u) = & - \begin{pmatrix} 0.1477957934872654 & 0.10094712082354547 \\ 0 & 0 \end{pmatrix} \\
& - z \begin{pmatrix} -0.6597141432730713 & 0.05159613960533249 \\ 0 & 0 \end{pmatrix} \\
& - z^2 \begin{pmatrix} 0.05159613960533249 & 0.06585963320155815 \\ 0 & 0 \end{pmatrix} \\
& - z^3 \begin{pmatrix} 0.06585963320155815 & 0.08467934092975693 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& -z^4 \begin{pmatrix} 0.028047431831489087 & 0.01809511731063812 \\ 0.1477957934872654 & 0.10094712082354547 \end{pmatrix} \\
& -z^5 \begin{pmatrix} 0.01809511731063812 & 0.00904755865531906 \\ -0.6597141432730713 & 0.05159613960533249 \end{pmatrix} \\
& -z^6 \begin{pmatrix} 0.00904755865531906 & 0.003619023462127624 \\ 0.05159613960533249 & 0.06585963320155815 \end{pmatrix} \\
& -z^7 \begin{pmatrix} 0.003619023462127624 & 0.001809511731063812 \\ 0.06585963320155815 & 0.08467934092975693 \end{pmatrix} \\
& -z^8 \begin{pmatrix} 0 & 0 \\ 0.028047431831489087 & 0.01809511731063812 \end{pmatrix} \\
& -z^9 \begin{pmatrix} 0 & 0 \\ 0.01809511731063812 & 0.00904755865531906 \end{pmatrix} \\
& -z^{10} \begin{pmatrix} 0 & 0 \\ 0.00904755865531906 & 0.003619023462127624 \end{pmatrix} \\
& -z^{11} \begin{pmatrix} 0 & 0 \\ 0.003619023462127624 & 0.001809511731063812 \end{pmatrix}, \\
Q_3(u) = & - \begin{pmatrix} 0.05761306818263398 & 0.17400836327560223 \\ 0 & 0 \end{pmatrix} \\
& -z \begin{pmatrix} 0.031575865006210874 & -0.6561977852588053 \\ 0 & 0 \end{pmatrix} \\
& -z^2 \begin{pmatrix} 0.05090899592774225 & 0.0849120583693081 \\ 0 & 0 \end{pmatrix} \\
& -z^3 \begin{pmatrix} 0.0849120583693081 & 0.12308355988310513 \\ 0 & 0 \end{pmatrix} \\
& -z^4 \begin{pmatrix} 0.015096022807838986 & 0.009739369553444507 \\ 0.05761306818263398 & 0.17400836327560223 \end{pmatrix} \\
& -z^5 \begin{pmatrix} 0.009739369553444507 & 0.004869684776722253 \\ 0.031575865006210874 & -0.6561977852588053 \end{pmatrix} \\
& -z^6 \begin{pmatrix} 0.004869684776722253 & 0.0019478739106889012 \\ 0.05090899592774225 & 0.0849120583693081 \end{pmatrix} \\
& -z^7 \begin{pmatrix} 0.0019478739106889012 & 0.0009739369553444506 \\ 0.0849120583693081 & 0.12308355988310513 \end{pmatrix} \\
& -z^8 \begin{pmatrix} 0 & 0 \\ 0.015096022807838986 & 0.009739369553444507 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& -z^9 \begin{pmatrix} 0 & 0 \\ 0.009739369553444507 & 0.004869684776722253 \end{pmatrix} \\
& -z^{10} \begin{pmatrix} 0 & 0 \\ 0.004869684776722253 & 0.0019478739106889012 \end{pmatrix} \\
& -z^{11} \begin{pmatrix} 0 & 0 \\ 0.0019478739106889012 & 0.0009739369553444506 \end{pmatrix}, \\
Q_4(u) = & - \begin{pmatrix} 0.05761306818263398 & 0.17400836327560223 \\ 0 & 0 \end{pmatrix} \\
& -z \begin{pmatrix} 0.031575865006210874 & 0.05090899592774225 \\ 0 & 0 \end{pmatrix} \\
& -z^2 \begin{pmatrix} -0.6561977852588053 & 0.0849120583693081 \\ 0 & 0 \end{pmatrix} \\
& -z^3 \begin{pmatrix} 0.0849120583693081 & 0.12308355988310513 \\ 0 & 0 \end{pmatrix} \\
& -z^4 \begin{pmatrix} 0.015096022807838986 & 0.009739369553444507 \\ 0.05761306818263398 & 0.17400836327560223 \end{pmatrix} \\
& -z^5 \begin{pmatrix} 0.009739369553444507 & 0.004869684776722253 \\ 0.031575865006210874 & 0.05090899592774225 \end{pmatrix} \\
& -z^6 \begin{pmatrix} 0.004869684776722253 & 0.0019478739106889012 \\ -0.6561977852588053 & 0.0849120583693081 \end{pmatrix} \\
& -z^7 \begin{pmatrix} 0.0019478739106889012 & 0.0009739369553444506 \\ 0.0849120583693081 & 0.12308355988310513 \end{pmatrix} \\
& -z^8 \begin{pmatrix} 0 & 0 \\ 0.015096022807838986 & 0.009739369553444507 \end{pmatrix} \\
& -z^9 \begin{pmatrix} 0 & 0 \\ 0.009739369553444507 & 0.004869684776722253 \end{pmatrix} \\
& -z^{10} \begin{pmatrix} 0 & 0 \\ 0.004869684776722253 & 0.0019478739106889012 \end{pmatrix} \\
& -z^{11} \begin{pmatrix} 0 & 0 \\ 0.0019478739106889012 & 0.0009739369553444506 \end{pmatrix}, \\
Q_5(u) = & \begin{pmatrix} 0.02609191544172444 & -0.2968153107875939 \\ 0 & 0 \end{pmatrix} \\
& -z \begin{pmatrix} 0.00853179785112384 & 0.04550078040666619 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$



$$\begin{aligned}
& -z^2 \begin{pmatrix} 0.04550078040666619 & -0.6008203844819463 \\ 0 & 0 \end{pmatrix} \\
& -z^3 \begin{pmatrix} 0.10628639670460124 & 0.17279291544901831 \\ 0 & 0 \end{pmatrix} \\
& +z^4 \begin{pmatrix} 0.014890951803385767 & 0.00960706567960372 \\ 0.02609191544172444 & -0.2968153107875939 \end{pmatrix} \\
& +z^5 \begin{pmatrix} 0.00960706567960372 & 0.00480353283980186 \\ -0.00853179785112384 & -0.04550078040666619 \end{pmatrix} \\
& +z^6 \begin{pmatrix} 0.00480353283980186 & 0.0019214131359207441 \\ -0.04550078040666619 & 0.6008203844819463 \end{pmatrix} \\
& +z^7 \begin{pmatrix} 0.0019214131359207441 & 0.0009607065679603721 \\ -0.10628639670460124 & -0.17279291544901831 \end{pmatrix} \\
& +z^8 \begin{pmatrix} 0 & 0 \\ 0.014890951803385767 & 0.00960706567960372 \end{pmatrix} \\
& +z^9 \begin{pmatrix} 0 & 0 \\ 0.00960706567960372 & 0.00480353283980186 \end{pmatrix} \\
& +z^{10} \begin{pmatrix} 0 & 0 \\ 0.00480353283980186 & 0.0019214131359207441 \end{pmatrix} \\
& +z^{11} \begin{pmatrix} 0 & 0 \\ 0.0019214131359207441 & 0.0009607065679603721 \end{pmatrix}, \\
Q_6(u) = & \begin{pmatrix} 0.02609191544172444 & -0.2968153107875939 \\ 0 & 0 \end{pmatrix} \\
& -z \begin{pmatrix} 0.00853179785112384 & 0.04550078040666619 \\ 0 & 0 \end{pmatrix} \\
& -z^2 \begin{pmatrix} 0.04550078040666619 & 0.10628639670460124 \\ 0 & 0 \end{pmatrix} \\
& -z^3 \begin{pmatrix} -0.6008203844819463 & 0.17279291544901831 \\ 0 & 0 \end{pmatrix} \\
& +z^4 \begin{pmatrix} 0.014890951803385767 & 0.00960706567960372 \\ 0.02609191544172444 & -0.2968153107875939 \end{pmatrix} \\
& +z^5 \begin{pmatrix} 0.00960706567960372 & 0.00480353283980186 \\ -0.00853179785112384 & -0.04550078040666619 \end{pmatrix} \\
& +z^6 \begin{pmatrix} 0.00480353283980186 & 0.0019214131359207441 \\ -0.04550078040666619 & -0.10628639670460124 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& + z^7 \begin{pmatrix} 0.0019214131359207441 & 0.0009607065679603721 \\ 0.6008203844819463 & -0.17279291544901831 \end{pmatrix} \\
& + z^8 \begin{pmatrix} 0 & 0 \\ 0.014890951803385767 & 0.00960706567960372 \end{pmatrix} \\
& + z^9 \begin{pmatrix} 0 & 0 \\ 0.00960706567960372 & 0.00480353283980186 \end{pmatrix} \\
& + z^{10} \begin{pmatrix} 0 & 0 \\ 0.00480353283980186 & 0.0019214131359207441 \end{pmatrix} \\
& + z^{11} \begin{pmatrix} 0 & 0 \\ 0.0019214131359207441 & 0.0009607065679603721 \end{pmatrix}, \\
Q_7(u) = & \begin{pmatrix} 0.08944633780381946 & -0.43235068115291514 \\ 0 & 0 \end{pmatrix} \\
& + z \begin{pmatrix} 0.01538612445300629 & -0.03771333490828642 \\ 0 & 0 \end{pmatrix} \\
& - z^2 \begin{pmatrix} 0.03771333490828642 & 0.12532192169119855 \\ 0 & 0 \end{pmatrix} \\
& + z^3 \begin{pmatrix} -0.12532192169119855 & 0.4857994473999439 \\ 0 & 0 \end{pmatrix} \\
& + z^4 \begin{pmatrix} 0.0514996814410752 & 0.033225600929725936 \\ 0.08944633780381946 & -0.43235068115291514 \end{pmatrix} \\
& + z^5 \begin{pmatrix} 0.033225600929725936 & 0.016612800464862968 \\ 0.01538612445300629 & -0.03771333490828642 \end{pmatrix} \\
& + z^6 \begin{pmatrix} 0.016612800464862968 & 0.006645120185945188 \\ -0.03771333490828642 & -0.12532192169119855 \end{pmatrix} \\
& + z^7 \begin{pmatrix} 0.006645120185945188 & 0.003322560092972594 \\ -0.12532192169119855 & 0.4857994473999439 \end{pmatrix} \\
& + z^8 \begin{pmatrix} 0 & 0 \\ 0.0514996814410752 & 0.033225600929725936 \end{pmatrix} \\
& + z^9 \begin{pmatrix} 0 & 0 \\ 0.033225600929725936 & 0.016612800464862968 \end{pmatrix} \\
& + z^{10} \begin{pmatrix} 0 & 0 \\ 0.016612800464862968 & 0.006645120185945188 \end{pmatrix} \\
& + z^{11} \begin{pmatrix} 0 & 0 \\ 0.006645120185945188 & 0.003322560092972594 \end{pmatrix}, \\
Q_8(u) = & \begin{pmatrix} -0.6202005681344304 & 0.0751156591403543 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& + z \begin{pmatrix} -0.043785112178583024 & 0.03234387705790018 \\ 0 & 0 \end{pmatrix} \\
& + z^2 \begin{pmatrix} 0.03234387705790018 & 0.1023493218758696 \\ 0 & 0 \end{pmatrix} \\
& + z^3 \begin{pmatrix} 0.1023493218758696 & 0.15482744823966293 \\ 0 & 0 \end{pmatrix} \\
& + z^4 \begin{pmatrix} 0.05053803393098198 & 0.032605183181278696 \\ -0.6202005681344304 & 0.0751156591403543 \end{pmatrix} \\
& + z^5 \begin{pmatrix} 0.032605183181278696 & 0.016302591590639348 \\ -0.043785112178583024 & 0.03234387705790018 \end{pmatrix} \\
& + z^6 \begin{pmatrix} 0.016612800464862968 & 0.006521036636255738 \\ 0.03234387705790018 & 0.1023493218758696 \end{pmatrix} \\
& + z^7 \begin{pmatrix} 0.006521036636255738 & 0.003260518318127869 \\ 0.1023493218758696 & 0.15482744823966293 \end{pmatrix} \\
& + z^8 \begin{pmatrix} 0 & 0 \\ 0.05053803393098198 & 0.032605183181278696 \end{pmatrix} \\
& + z^9 \begin{pmatrix} 0 & 0 \\ 0.032605183181278696 & 0.016302591590639348 \end{pmatrix} \\
& + z^{10} \begin{pmatrix} 0 & 0 \\ 0.016612800464862968 & 0.006521036636255738 \end{pmatrix} \\
& + z^{11} \begin{pmatrix} 0 & 0 \\ 0.006521036636255738 & 0.003260518318127869 \end{pmatrix}
\end{aligned} \tag{5.37}$$

they are the symbols of the minimum-energy multiwavelet frames associated with  $\Phi$ . The graphs of the vector-valued functions are shown in Figure 5.

From Figure 5, we can find that every component of the minimum-energy frames is smooth, but not (anti)symmetrical.

## 6. Conclusions

In this paper, minimum-energy multiwavelet frames with arbitrary integer dilation factor are studied. Firstly, we define the concept of minimum-energy multiwavelet frame with arbitrary dilation factor and present its equivalent characterizations. Secondly, some necessary conditions and sufficient conditions for minimum-energy multiwavelet frame are given, then the decomposition and reconstruction formulas of minimum-energy multiwavelet

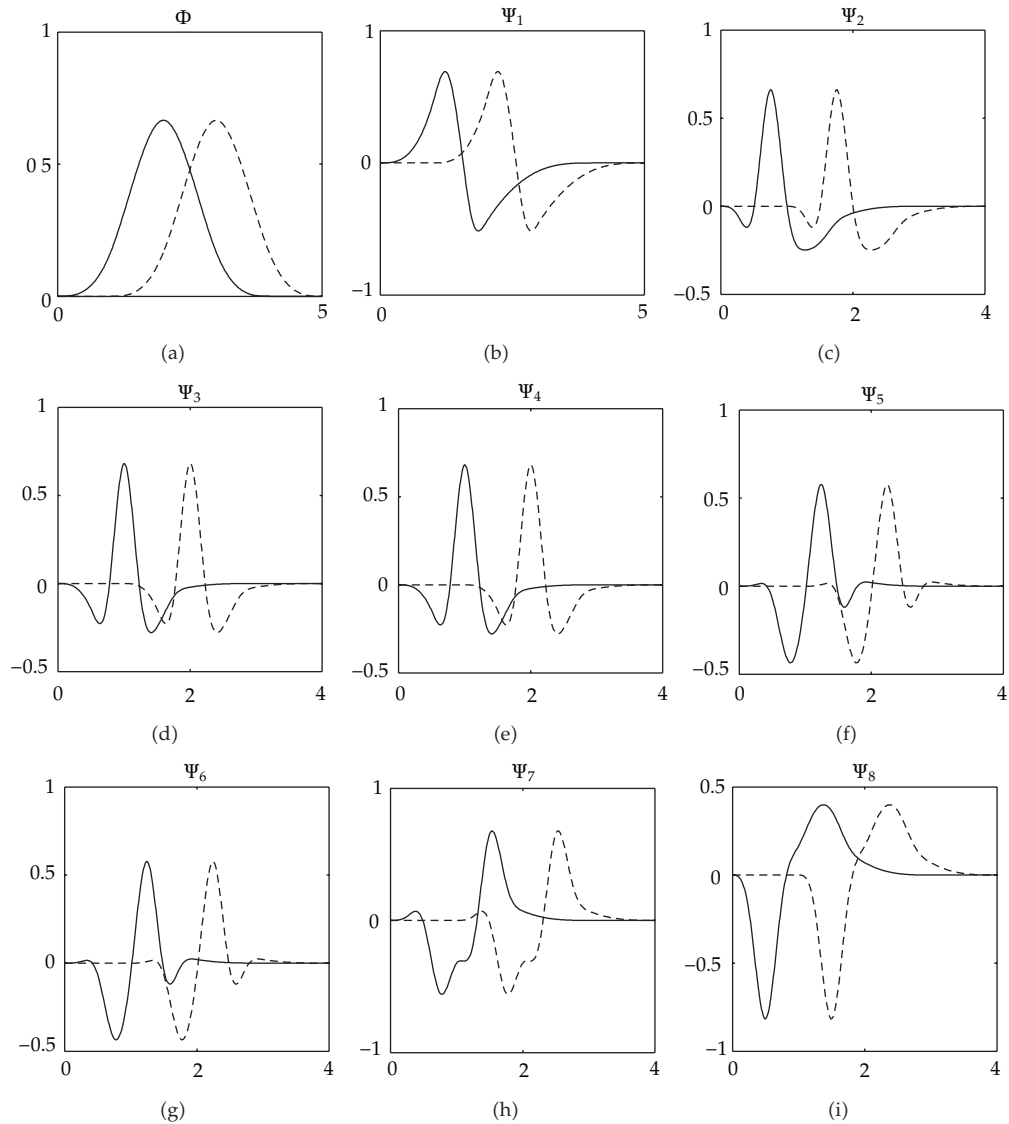


Figure 5

frame with arbitrary integer dilation factor are deduced. Finally, we give several numerical examples based on B-spline.

## Acknowledgments

The authors acknowledge the support from the National Natural Science Foundation of China under the Project Grant no. 10961001 and 61261043, the Research Project of Beifang University of Nationalities under the Project no. 2012Y036, and Graduate Students Innovative Project under the Project no. 2011XYC035. M. Li acknowledges the support from the 973 plan under the Project no. 2011CB302800 and the National Natural Science Foundation of China under the Project Grant no.s 61070214 and 60873264.

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