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Letter to the Editor

Comments on "Homotopy Perturbation Method for Solving Reaction-Diffusion Equations"

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The paper entitled *"Homotopy perturbation method for solving reaction diffussion equation"* contains some mistakes and misinterpretations along with a false conclusion. Applying the homotopy perturbation method (HPM) in an incorrect manner, the authors have drawn the false conclusion that this approach is efficient for reaction-diffusion type of equation. We show that HPM in the proposed form is not efficient in most cases, and hence, we will introduce the correct form of HPM.

1. Introduction

In [1] Wang et al. proposed a new approach to apply the HPM for solving the reaction-diffusion problem. We can easily identify in [1] some mistakes, which can mislead the readers, as we show in Section 3 of this paper. The main difficulties in the application of HPM in proposed form are that the corresponding approximate solution is a divergent series on $[0, 1]$ and the method of weighted residuals applied in an incorrect form.

We consider a reaction-diffusion process governed by the equation

$$u'' + u^n = 0, \quad 0 < x < L, \tag{1.1}$$

with boundary conditions

$$u(0) = u(L) = 0, \tag{1.2}$$

where $u(x)$ represents the steady-state temperature for the corresponding reaction-diffusion equation with the reaction term u^n, n is the power of the reaction term (heat source), and L is the length of the sample (heat conductor). The physical interpretation of (1.1) was given in [2]. For $n = 3$, the problem (1.1)-(1.2) has a unique positive solution (e.g., as can be seen by phase-plane analysis) which is given in terms of elliptic functions. For an arbitrary real number $n \neq 3$, the general reaction-diffusion process is more complicated. The problem (1.1)-(1.2) was studied by Lesnic using Adomian decomposition method [2] and by Mo [3] using variational method. In this paper, we analyze the HPM used in [1] and demonstrate the correct form of HPM.

2. Homotopy Perturbation Method

In [1] the authors used the HPM in standard form:

$$(1-p)\left(u'' - u_0''\right) + p\left(u'' + u^n\right) = 0, \tag{2.1}$$

with initial approximation

$$u_0 = ax(1-x), \tag{2.2}$$

where a is an unknown constant. The HPM supposes that the solution of (1.1)-(1.2) has the form

$$u = u_0 + pu_1 + p^2 u_2 + \cdots . \tag{2.3}$$

And then equating the terms with identical powers of p, the components $u_1, u_2, \ldots$ of u can be calculated easily. This standard process for $n = 2$ with $L = 1$ gives an approximate solution for u [1]:

$$u = ax(1-x) + ax^2 - a^2\left(\frac{1}{30}x^6 - \frac{1}{10}x^5 + \frac{1}{12}x^4\right) - \left(a - \frac{1}{60}a^2\right)x. \tag{2.4}$$

To find a the authors in [1] then used the method of weighted residuals. Substituting (2.4) into (1.1) results in the following residual:

$$R(x, a) = u''(x) + u^n(x). \tag{2.5}$$

Then they set $R(1/3, a) = 0$ and found $a = 45.4205$ (why 1/3?).

It is not clear why the number 1/3 (but not 1/2) plays some special role for determining a. Actually, the problem is that the solution corresponding to 1/2 in fact has no relationship with the exact solution; that is error is large enough comparatively with the case 1/3. But it is not difficult to show that the error (in case of $n = 2$) for 1/3 is also large enough. This solution is not even so good around 1/3. Tables 1 and 2 demonstrate the errors in $u'' + u^n = 0$ for some values of x. If we use one more term (component) in the HPM in [1] applying the method of weighted residuals (for $x = 1/3$ and $n = 2$), we obtain $a = 23.949$ and an approximate solution which in fact has no relationship with normal solution: for example, the "error" for $x = 0.9$ is 43.780. That is the result received in [1] cannot be improved by taking more terms in approximation. The failure of the method related also with the convergence

Table 1: Errors in approximate solution for Example 3.1.

x	$u_{\text{appr}} = \sum_{k=0}^{9} u_k$ ($u_0 = b$)	$u_{\text{appr}} = \sum_{k=0}^{3} u_k$ ($u_0 = a\sin \pi x$)	Error in $u'' + u^2 = 0$ ($u_0 = b$)	Error in $u'' + u^2 = 0$ ($u_0 = a\sin x$)	Error in $u'' + u^2 = 0$ in [1]
0.0	0	0.0	0.0	$2.566\,5 \times 10^{-3}$	0.0
0.1	3. 3009	3. 297 6	0.1145	2.5942×10^{-2}	5. 004 5
0.2	6. 4712	6. 469 1	6.061×10^{-3}	9.8435×10^{-2}	8. 412 7
0.3	9. 212 3	9. 210 9	5.5852×10^{-3}	0.136 82	3. 109 7
0.4	11. 108	11. 107	4.5057×10^{-3}	3.5474×10^{-2}	6. 237 2
0.5	11. 79	11. 788	0.00	5.4003×10^{-2}	10.76
0.6	11. 108	11. 107	4.5057×10^{-3}	3.5474×10^{-2}	6. 232 1
0.7	9. 212 3	9. 210 9	5.5852×10^{-3}	0.136 82	3. 110 9
0.9	3. 300 9	3. 297 6	0.114 5	2.5942×10^{-2}	5. 004 5

Table 2: Errors in approximate solution for Example 3.2.

x	Error in $u'' + u^3 = 0$	Error in $u'' + u^2 = 0$ in [1]
0.0	0.0	0.0
0.1	2.7632×10^{-3}	1.0055
0.2	1.7240×10^{-2}	3.3603
0.3	0.136 73	1.7968
0.4	0.12186	4.3907
0.5	0.0	8.076
0.6	0.12186	4.3845
0.7	0.13673	1.8083
0.8	1.7240×10^{-2}	3.3749
0.9	2.7632×10^{-3}	1.0180
1.0	0.0	0.0

rate of the series solution. It is not difficult to show that the coefficients in the highest degree terms of the second derivative are large enough and therefore go to infinity for x close to 1.

To overcome these difficulties we offer the next two versions of the HPM.

(1) Since the solution must be symmetric with respect to the line $x = L/2$, try to represent the solution as a series of $\sum (x - L/2)^k$. Since the solution reaches the maximum at $L/2$ take $u'(L/2) = 0$. We construct a homotopy as in (2.1) but with initial approximation:

$$u_0 = b, \tag{2.6}$$

where b is an unknown constant to be further determined. Instead of initial boundary conditions we consider initial conditions:

$$u_k\left(\frac{L}{2}\right) = 0, \qquad u_k'\left(\frac{L}{2}\right) = 0, \tag{2.7}$$

for each component u_k, $k = 1, 2, \ldots$.

(2) To get rapidly convergent series solution we offer to take an initial approximation as $u_0 = a\sin(\pi x/L)$ where a is an unknown constant to be further determined (by taking $x = L/2$ in $u'' + u^n = 0$, where $u = u_{\text{approx}}$).

3. Applications

Example 3.1. Consider the problem

$$u'' + u^2 = 0,$$
$$u(0) = u(1) = 0 \tag{3.1}$$

and apply version 1 of HPM. Substituting $u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots$ into (2.1) and equating the terms with the identical powers of p, we obtain

$$u_0'' + pu_1'' + p^2u_2'' + p^3u_3'' + \cdots - v_0'' + p\left(v_0'' + \left(u_0 + pu_1 + p^2u_2 + p^3u_3 + p^4u_4 + p^5u_5\right)^2\right) = 0,$$

$$u_0 = b,$$

$$u_1'' = -u_0'' - u_0^2 = -(b)^2, \qquad u_1(0.5) = u_1'(0.5) = 0,$$

$$u_1 = -0.5b^2(x-0.5)^2,$$

$$u_2'' = -2u_0u_1 = -2(b)\left(-0.5b^2(x-0.5)^2\right), \qquad u_2(0.5) = u_2'(0.5) = 0,$$

$$u_2 = \frac{1}{12}b^3(x-0.5)^4,$$

$$u_3'' = -2u_0u_2 - u_1^2 = -2(b)\left(8.\,3333\times10^{-2}b^3(x-0.5)^4\right) - \left(-0.5b^2(x-0.5)^2\right)^2,$$

$$u_3 = -1.\,3889\times10^{-2}b^4(x-0.5)^6,$$

$$u_4'' = -2u_0u_3 - 2u_1u_2,$$

$$u_4 = \frac{0.111\,11}{56}b^5(x-0.5)^8,$$

$$u_5'' = -2u_0u_4 - 2u_1u_3 - u_2^2,$$

$$u_5 = -2.\,7558\times10^{-4}b^6(x-0.5)^{10},$$

$$u_6'' = -2u_0u_5 - 2u_1u_4 - 2u_2u_3,$$

$$u_6 = 3.\,6743\times10^{-5}b^7(x-0.5)^{12},$$

$$u_7'' = -2u_0u_6 - 2u_1u_5 - 2u_2u_4 - u_3^2$$

$$u_7 = -4.\,7948\times10^{-6}b^8(x-0.5)^{14},$$

$$u_8'' = -2u_0u_7 - 2u_1u_6 - 2u_2u_5 - 2u_3u_4^2,$$
$$u_8 = 6.\ 1407 \times 10^{-7} b^9 (x-0.5)^{16},$$
$$u_9'' = -2u_0u_8 - 2u_1u_7 - 2u_2u_6 - 2u_3u_5 - (u_4)^2,$$
$$u_9 = -7.\ 7577 \times 10^{-8} b^{10} (x-0.5)^{18}. \tag{3.2}$$

Let us take $u_{\text{approx}} = u_0 + u_1 + \cdots + u_9$ and find b from $u_{\text{approx}}(0) = 0$—some numeric method can be used for finding b (we used standard LaTex computation): $b = 11.\ 79$. Thus we get an approximate solution:

$$\begin{aligned} u = {} & (11.79) - 0.5(11.79)^2(x-0.5)^2 + \frac{1}{12}(11.79)^3(x-0.5)^4 - 1.\ 3889 \times 10^{-2}(11.79)^4(x-0.5)^6 \\ & + 1.\ 9841 \times 10^{-3}(11.79)^5(x-0.5)^8 - 2.\ 7558 \times 10^{-4}(11.79)^6(x-0.5)^{10} \\ & + 3.\ 6743 \times 10^{-5}(11.79)^7(x-0.5)^{12} - 4.\ 7948 \times 10^{-6}(11.79)^8(x-0.5)^{14} \\ & + 6.\ 1407 \times 10^{-7}(11.79)^9(x-0.5)^{16} - 7.\ 7577 \times 10^{-8}(11.79)^{10}(x-0.5)^{18}. \end{aligned} \tag{3.3}$$

Now let us apply version 2 of HPM. We take $u_0 = a\ \sin \pi x$, and equating the terms with the identical powers of p, we obtain

$$u_0 = a\sin(\pi x),$$
$$u_1'' = -u_0'' - u_0^2 = \pi^2 a \sin(\pi x) - a^2\sin^2(\pi x), \qquad u_1(0) = u_1'(1) = 0,$$
$$u_1 = -\frac{1}{4}a^2x^2 + \frac{1}{4}a^2x - \frac{1}{8\pi^2}a^2\cos^2\pi x + \frac{1}{8\pi^2}a^2\sin^2\pi x + \frac{1}{8\pi^2}a^2 - a\sin\pi x,$$
$$u_2'' = -2u_0u_1 = -2(a\sin(\pi x))\left(-\frac{1}{4}a^2x^2 + \frac{1}{4}a^2x - \frac{a^2\cos^2\pi x}{8\pi^2} + \frac{a^2\sin^2\pi x}{8\pi^2} + \frac{a^2}{8\pi^2} - a\sin\pi x\right),$$
$$\begin{aligned} u_2 = {} & -\frac{1}{2\pi^2}a^3x^2\sin\pi x - \frac{2}{\pi^3}a^3x\cos\pi x + \frac{1}{2\pi^2}a^3x\sin\pi x - \frac{1}{24\pi^4}a^3\cos^2\pi x\sin\pi x + \frac{a^3\cos\pi x}{\pi^3} \\ & + \frac{a^3\sin^3\pi x}{72\pi^4} + \frac{27}{8\pi^4}a^3\sin\pi x + \frac{a^2x^2}{2} + \frac{a^2\cos^2\pi x}{4\pi^2} - \frac{a^2\sin^2\pi x}{4\pi^2} - \frac{a^2x}{2} - \frac{a^2(\pi+4a)}{4\pi^3}, \end{aligned}$$
$$u_3'' = -2u_0u_2 - u_1^2,$$
$$\begin{aligned} u_3 = {} & \frac{1}{160}a^4x^5 - \frac{1}{192}a^4x^4 - \frac{1}{4}a^2x^2 - \frac{1}{480}a^4x^6 - \frac{3}{\pi^3}a^3\cos\pi x - \frac{81}{8\pi^4}a^3\sin\pi x - \frac{2}{\pi^5}a^4\sin\pi x \\ & - \frac{3}{32\pi^2}a^4x^3 + \frac{3}{64\pi^2}a^4x^4 - \frac{435}{256\pi^4}a^4x^2 - \frac{1}{8\pi^2}a^2\cos^2\pi x - \frac{451}{288\pi^6}a^4\cos^2\pi x \\ & + \frac{25}{18\ 432\pi^6}a^4\cos^4\pi x + \frac{1}{8\pi^2}a^2\sin^2\pi x - \frac{1}{24\pi^4}a^3\sin^3\pi x + \frac{451}{288\pi^6}a^4\sin^2\pi x \end{aligned}$$

$$+\frac{25}{18\ 432\pi^6}a^4\sin^4\pi x+\frac{1}{8\pi^4}a^3\cos^2\pi x\sin\pi x+\frac{9}{64\pi^4}a^4x^2\cos^2\pi x-\frac{9}{64\pi^4}a^4x^2\sin^2\pi x$$
$$+\frac{6}{\pi^3}a^3x\cos\pi x-\frac{3}{2\pi^2}a^3x\sin\pi x-\frac{25}{3072\pi^6}a^4\cos^2\pi x\sin^2\pi x+\frac{25}{32\pi^5}a^4\cos\pi x\sin\pi x$$
$$-\frac{9}{64\pi^4}a^4x\cos^2\pi x+\frac{3}{2\pi^2}a^3x^2\sin\pi x+\frac{9}{64\pi^4}a^4x\sin^2\pi x$$
$$-\frac{25}{16\pi^5}a^4x\cos\pi x\sin\pi x+\frac{1}{3840\pi^4}a^2\left(180\pi^2a^2+4\pi^4a^2+960\pi^4+6525a^2\right)x$$
$$+\frac{9613}{6144\pi^6}a^4+\frac{3}{\pi^3}a^3+\frac{1}{8\pi^2}a^2. \tag{3.4}$$

(We used very easy LaTex tools for finding integrals.) Now we calculate the number a from the relationship

$$u''+u^2=0, \tag{3.5}$$

where $u\approx u_0+u_1+u_2+u_3$ and $x=0.5$. We obtain $a=11.\ 569$. Table 1 shows the errors in the solutions and comparison with the errors in [1].

Example 3.2. Consider the problem

$$u''+u^3=0,$$
$$u(0)=u(1)=0. \tag{3.6}$$

Substituting $u=u_0+pu_1+p^2u_2+p^3u_3+\cdots$ into (2.1), where $u_0=a\sin(\pi x)$, and equating the terms with the identical powers of p, we obtain from

$$u_0''+pu_1''+p^2u_2''+p^3u_3''+\cdots-v_0''+p\left(v_0''+\left(u_0+pu_1+p^2u_2+p^3u_3+p^4u_4+p^5u_5\right)^3\right)=0 \tag{3.7}$$

that

$$u_0=a\sin(\pi x),$$
$$u_1''=-u_0''-u_0^3=\pi^2a\sin(\pi x)-a^3\sin^3(\pi x),\qquad u_1(0)=u_1'(1)=0,$$
$$u_1=-\frac{1}{12\pi}\left(\frac{1}{3\pi}a^3\sin 3\pi x-\frac{9}{\pi}a^3\sin\pi x+12\pi a\sin\pi x\right),$$
$$u_2''=-3u_0^2u_1=-3(a\sin(\pi x))^2\left(-\frac{a^3\cos^2\pi x\sin\pi x}{12\pi^2}+\frac{a^3\sin^3\pi x}{36\pi^2}+\frac{3a^3\sin\pi x}{4\pi^2}-a\sin\pi x\right),$$

$$u_2 = \frac{1}{1440\pi^3}\left(\frac{2460}{\pi}a^5\sin\pi x - \frac{290}{3\pi}a^5\sin 3\pi x + \frac{6a^5\sin 5\pi x}{5\pi} - 3240\pi a^3\sin\pi x + 120\pi a^3\sin 3\pi x\right),$$

$$u_3'' = -3u_0^2u_2 - 3u_0u_1^2,$$

$$u_3 = -\frac{1}{756000\pi^5}\left(63000\pi^3a^3\sin 3\pi x - \frac{3925250}{\pi}a^7\sin\pi x - 1701000\pi^3a^3\sin\pi x + \frac{470470}{3\pi}a^7\sin 3\pi x - \frac{12614}{5\pi}a^7\sin 5\pi x + \frac{130}{7\pi}a^7\sin 7\pi x + 6457500\pi a^5\sin\pi x - 253750\pi a^5\sin 3\pi x + 3150\pi a^5\sin 5\pi x\right). \tag{3.8}$$

To find a we take $u = u_0 + u_1 + u_2 + u_3$ in $u'' + u^3 = 0$ and replace $x = 0.5$. Thus we obtain $a = 3.6033$. Table 2 demonstrates the errors in the solution and the comparison with the errors in [1].

4. Conclusions

We proved that Wang et al. who intended to develop in [1] a new approach to apply the HPM for solving reaction-diffusion equations, included in their paper some mistakes and misinterpretations. They used the method of weighted residuals in an incorrect form, and initial approximation is not so good for the convergence of the series solution. We introduced two versions of the HPM, which are very efficient for initial-boundary type of problems.

The main strength of HPM is in fact its fast convergence, but the form of initial approximation and initial-boundary conditions is very important for the efficiency of the HPM.

References

[1] Y.-X. Wang, H.-Y. Si, and L.-F. Mo, "Homotopy perturbation method for solving reaction-diffusion equations," *Mathematical Problems in Engineering*, vol. 2008, Article ID 795838, 5 pages, 2008.

[2] D. Lesnic, "A nonlinear reaction-diffusion process using the Adomian decomposition method," *International Communications in Heat and Mass Transfer*, vol. 34, no. 2, pp. 129–135, 2007.

[3] L.-F. Mo, "Variational approach to reaction-diffusion process," *Physics Letters A*, vol. 368, no. 3-4, pp. 263–265, 2007.