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Research Article

Representing Smoothed Spectrum Estimate with the Cauchy Integral

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Estimating power spectrum density (PSD) is essential in signal processing. This short paper gives a theorem to represent a smoothed PSD estimate with the Cauchy integral. It may be used for the approximation of the smoothed PSD estimate.

1. Introduction

Estimating power spectrum density (PSD) of signals plays a role in signal processing. It has applications to many issues in engineering [1–21]. Examples include those in biomedical signal processing, see, for example, [1–3, 6, 12, 13]. Smoothing an estimate of PSD is commonly utilized for the purpose of reducing the estimate variance, see, for example, [22–29]. By smoothing a PSD estimate, one means that a smoothed estimate of PSD of a signal is the PSD estimate convoluted by a smoother function [30, 31]. This short paper aims at providing a representation of a smoothed PSD estimate based on the Cauchy's integral.

2. Cauchy Representation of Smoothed PSD Estimate

Let $x(t)$ be a signal for $-\infty < t < \infty$. Let $S_{xx}(\omega)$ be its PSD, where $\omega = 2\pi f$ is radian frequency and f is frequency. Then, by using the Fourier transform, $S_{xx}(\omega)$ is computed by

$$S_{xx}(\omega) = \left| \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \right|^2, \quad j = \sqrt{-1}. \tag{2.1}$$

In practical terms, if $x(t)$ is a random signal, $S_{xx}(\omega)$ may never be achieved exactly because a PSD is digitally computed only in a finite interval, say, (T_1, T_2) for $T_1 \neq T_2$. Therefore, one can only attain an estimate of $S_{xx}(\omega)$.

Denote by $\widehat{S}_{xx}(\omega)$ an estimate of $S_{xx}(\omega)$. Then,

$$\widehat{S}_{xx}(\omega) = \left| \int_{T_1}^{T_2} x(t) e^{-j\omega t}\, dt \right|^2. \tag{2.2}$$

Without generality losing, we assume $T_1 = 0$ and $T_2 = T$. Thus, the above becomes

$$\widehat{S}_{xx}(\omega) = \left| \int_{0}^{T} x(t) e^{-j\omega t}\, dt \right|^2. \tag{2.3}$$

In the discrete case, one has the following for a discrete signal $x(n)$ [21–23]:

$$\widehat{S}_{xx}(\omega) = \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2. \tag{2.4}$$

Because

$$\left| \sum_{n=L}^{N+L-1} x(n) e^{-j\omega n} \right|^2 \neq \left| \sum_{n=M}^{N+M-1} x(n) e^{-j\omega n} \right|^2 \quad \text{for } L \neq M, \tag{2.5}$$

$\widehat{S}_{xx}(\omega)$ is usually a random variable. One way of reducing the variance of $\widehat{S}_{xx}(\omega)$ is to smooth $\widehat{S}_{xx}(\omega)$ by a smoother function denoted by $G(\omega)$. Denote by $\widetilde{S}_{xx}(\omega)$ the smoothed PSD estimate. Let $*$ imply the operation of convolution. Then, $\widetilde{S}_{xx}(\omega)$ is given by

$$\widetilde{S}_{xx}(\omega) = \widehat{S}_{xx}(\omega) * G(\omega). \tag{2.6}$$

Assume that $\widetilde{S}_{xx}(\omega)$ is differentiable any time for $-\infty < \omega < \infty$. Then, by using the Taylor series at $\omega = \omega_0$, $\widehat{S}_{xx}(\omega)$ is expressed by

$$\widehat{S}_{xx}(\omega) = \sum_{l=0}^{\infty} \frac{\widehat{S}_{xx}^{(l)}(\omega_0)}{l!} (\omega - \omega_0)^n. \tag{2.7}$$

Therefore,

$$\widetilde{S}_{xx}(\omega) = \sum_{l=0}^{\infty} \frac{\widehat{S}_{xx}^{(l)}(\omega_0)}{l!} (\omega - \omega_0)^n * G(\omega). \tag{2.8}$$

Let $\omega - \omega_0 = \omega_1$. Then,

$$(\omega - \omega_0)^n * G(\omega) = \omega_1^n * G(\omega_1 + \omega_0). \tag{2.9}$$

Thus, we have a theorem to represent $\widetilde{S}_{xx}(\omega)$ based on the Cauchy integral.

Theorem 2.1. *Suppose $\widehat{S}_{xx}(\omega)$ is differentiable any time at ω_0. Then, the smoothed PSD, that is, $\widetilde{S}_{xx}(\omega)$, may be expressed by*

$$\widetilde{S}_{xx}(\omega) = \sum_{l=0}^{\infty} \frac{\widehat{S}_{xx}^{(l)}(\omega_0)}{l!} \omega_1^l * G(\omega_1 + \omega_0) = \sum_{l=0}^{\infty} \widehat{S}_{xx}^{(l)}(\omega_0) \int_0^{\omega_1} \frac{(\omega_1 - \omega_\tau)^l}{l!} G(\omega_\tau + \omega_0)\, d\omega_\tau. \tag{2.10}$$

Proof. The Cauchy integral in terms of $G(\omega_\tau + \omega_0)$ is in the form

$$\int_0^{\omega_1} \frac{(\omega_1 - \omega_\tau)^l}{l!} G(\omega_\tau + \omega_0)\, d\omega_\tau = \underbrace{\int_0^{\omega_\tau} d\omega_\tau \cdots \int_0^{\omega_\tau} G(\omega_\tau + \omega_0)\, d\omega_\tau.}_{l+1} \tag{2.11}$$

That may be taken as the convolution between $\omega_1^l / l!$ and $G(\omega_1 + \omega_0)$. Thus,

$$\frac{\omega_1^l}{l!} * G(\omega_1 + \omega_0) = \int_0^{\omega_1} \frac{(\omega_1 - \omega_\tau)^l}{l!} G(\omega_\tau + \omega_0)\, d\omega_\tau. \tag{2.12}$$

Therefore, (2.10) holds. This completes the proof. □

The present theorem is a theoretic representation of a smoothed PSD estimate. It may yet be a method to be used in the approximation of a smoothed PSD estimate. As a matter of fact, we may approximate $\widetilde{S}_{xx}(\omega)$ by a finite series given by

$$\widetilde{S}_{xx}(\omega) \approx \sum_{l=0}^{L} \widehat{S}_{xx}^{(l)}(\omega_0) \int_0^{\omega_1} \frac{(\omega_1 - \omega_\tau)^l}{l!} G(\omega_\tau + \omega_0)\, d\omega_\tau. \tag{2.13}$$

From the above theorem, we have the following corollary.

Corollary 2.2. *Suppose $\widehat{S}_{xx}(\omega)$ is differentiable any time at $\omega = 0$. Then, $\widetilde{S}_{xx}(\omega)$ may be expressed by*

$$\widetilde{S}_{xx}(\omega) = \sum_{l=0}^{\infty} \frac{\widehat{S}_{xx}^{(l)}(0)}{l!} \omega^l * G(\omega) = \sum_{l=0}^{\infty} \widehat{S}_{xx}^{(l)}(0) \int_0^{\omega} \frac{(\omega - \omega_\tau)^l}{l!} G(\omega_\tau)\, d\omega_\tau. \tag{2.14}$$

The proof is omitted since it is straightforward when one takes into account the proof of theorem.

3. Conclusions

We have presented a theorem with respect to a representation of a smoothed PSD estimate of signals based on the Cauchy integral. The theorem constructively implies that the design of a smoother function $G(\omega)$ may consider the approximation described by the Cauchy integral with the finite Taylor series (2.13). In addition, the smoother function $G(\omega)$ can also be taken as a solution to the integral equation (2.14), which is worth being investigated in the future.

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