Research Article

Adaptive Output Feedback Control for a Class of Stochastic Nonlinear Systems with SiISS Inverse Dynamics

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The adaptive stabilization scheme based on tuning function for stochastic nonlinear systems with stochastic integral input-to-state stability (SiISS) inverse dynamics is investigated. By combining the stochastic LaSalle theorem and small-gain type conditions on SiISS, an adaptive output feedback controller is constructively designed. It is shown that all the closed-loop signals are bounded almost surely and the stochastic closed-loop system is globally stable in probability.

1. Introduction

Global stabilization control design of stochastic nonlinear systems is one of the most important topics in nonlinear control theory, which has received and is increasingly receiving a great deal of attention; see, for example, [1–37] and the references therein. For a class of stochastic nonlinear systems with stochastic inverse dynamics, much progress has been made in the design of the global stabilization controller [12, 13, 15, 16, 24, 25, 31]. However, all these controllers are only robust against stochastic inverse dynamics with stringent stability margin. To weaken the stringent condition on stochastic inverse dynamics, a natural idea is to benefit from input-to-state stability (ISS) in [38] and integral input-to-state stability (iISS) in [39] which are now recognized as the central unifying concepts in feedback design and stability analysis of deterministic nonlinear systems. Tsinias in [21], Tang and Basar in [19] first proposed the concept of stochastic input-to-state stability (SISS) independently. Further in-depth study on SISS and its applications are presented in [9–11, 18, 22]. Motivated by these aforementioned important results, [34] showed that SISS condition can be weakened to stochastic integral input-to-state stability (SiISS) and developed a unifying output feedback framework for global regulation.

Nonlinear small-gain theorem plays an important role in the controller design and stability analysis of deterministic nonlinear systems in [40, 41]. While for stochastic nonlinear systems, there are fewer results on the small-gain theorem. [25] firstly established a gain-function-based stochastic nonlinear small-gain theorem for ISpS in probability. In some succeeding research work, [9, 11, 35] presented some Lyapunov-based small-gain type conditions on SISS and SiISS, respectively. [4] further discusses the relationship of small-gain type conditions on SiISS and studies the problem of GAS in probability via output feedback.

In this paper, inspired by [4], a more general class of stochastic nonlinear systems with uncertain parameters and stochastic integral input-to-state stability (SiISS) inverse dynamics is investigated. By combining the stochastic LaSalle theorem and small-gain type conditions on SiISS, an adaptive output feedback controller is proposed to guarantee that all the closed-loop signals are bounded almost surely and the stochastic closed-loop system is globally stable in probability.

The paper is organized as follows. Section 2 begins with the mathematical preliminaries. Section 3 presents statement of the problem. The design of adaptive output feedback controller is given in Section 4. The corresponding analysis is given in Section 5. Section 6 concludes the paper.

2. Mathematical Preliminaries

The following notations are used throughout the paper. $\mathbb{R}_+$ stands for the set of all nonnegative real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{n \times m}$ is the space of real $n \times m$-matrices. For $x = (x_1, \ldots, x_n)$, one defines $\overline{x}_i = (x_1, \ldots, x_i)$, $i = 1, \ldots, n-1$. C^i denotes the family of all the functions with continuous ith partial derivatives. $L^1(\mathbb{R}_+; \mathbb{R}_+)$ is the family of all functions $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty l(t)dt < \infty$. For a given vector or matrix X, X^T denotes its transpose, $\text{Tr}\{X\}$ denotes its trace when X is square. $|X|$ denotes the Euclidean norm of a vector X, and $\|X\| = (\text{Tr}\{XX^T\})^{1/2}$ for a matrix X. $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing, and vanishing at zero; $\mathcal{K}_\infty$ is the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{KL}$ denotes the set of all functions $\beta(s, t)$: $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which are of $\mathcal{K}$ for each fixed t and decrease to zero as $t \to \infty$ for each fixed s.

Consider the stochastic nonlinear delay-free system

$$dx = f(x, t)dt + g(x, t)dw, \quad \forall x(0) = x_0 \in \mathbb{R}^n, \tag{2.1}$$

where $x \in \mathbb{R}^n$, w is an m-dimensional standard Wiener process defined in a complete probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P\}$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t \geq 0}$ being a filtration, and P being the probability measure. Borel measurable functions $f: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}$ are piecewise continuous in t and locally bounded and locally Lipschitz continuous in x uniformly in $t \in \mathbb{R}_+$. Let $\mathscr{L}V(x)$ denote infinitesimal generator of function $V \in C^2$ along stochastic system (2.1) with the definition of

$$\mathscr{L}V(x) = \frac{\partial V(x)}{\partial x} f(x, t) + \frac{1}{2} \text{Tr}\left\{ g^T(x, t) \frac{\partial^2 V(x)}{\partial x^2} g(x, t) \right\}. \tag{2.2}$$

Definition 2.1 (see [34]). Stochastic process $\{\xi(t)\}_{t \geq t_0}$ is said to be bounded almost surely if $\sup_{t \geq t_0} |\xi(t)| < \infty$ a.s.

Lemma 2.2 (Stochastic LaSalle Theorem [14]). *For system* (2.1), *if there exist functions* $V \in C^2$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $l \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, *and a continuous nonnegative function* $W : \mathbb{R}^n \rightarrow \mathbb{R}$ *such that for all* $x \in \mathbb{R}^n$, $t \geq 0$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \mathcal{L}V(x) \leq -W(x) + l(t), \tag{2.3}$$

then for each $x_0 \in \mathbb{R}^n$,

   (i) *system* (2.1) *has a unique strong solution on* $[0, \infty)$, *and solution* $x(t)$ *is bounded almost surely;*

   (ii) *when* $f(0, t) \equiv 0$, $g(0, t) \equiv 0$, $l(t) \equiv 0$, *the equilibrium* $x = 0$ *is globally stable in probability.*

   In the following, we cite two small-gain type conditions on SiISS in [35].
   Consider the following stochastic nonlinear system

$$dx = f(x, v, t)dt + g(x, v, t)dw, \quad \forall x(0) = x_0 \in \mathbb{R}^n, \tag{2.4}$$

where $x \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}^m$ is the input, and w is an r-dimensional standard Wiener process defined as in (2.1). Borel measurable functions $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times r}$ are locally bounded and locally Lipschitz continuous with respect to (x, v) uniformly in $t \in \mathbb{R}_+$.

Definition 2.3 (see [34]). System (2.4) is said to be stochastic integral input-to-state stable (SiISS) using Lyapunov function if there exist functions $V \in C^2(\mathbb{R}^n; \mathbb{R})$, $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$, and a merely positive definite continuous function α such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \mathcal{L}V(x) \leq -\alpha(|x|) + \gamma(|v|). \tag{2.5}$$

The function V satisfying (2.5) is said to be a SiISS-Lyapunov function, and (α, γ) in (2.5) is called the SiISS supply rate of system (2.4).

Lemma 2.4 (see [35]). *For system* (2.4) *satisfying* (2.5), *if there exists a positive definite continuous function* $\tilde{\alpha}$ *such that* $\limsup_{s \rightarrow 0_+} \tilde{\alpha}(s)/\alpha(s) < \infty$, $\limsup_{s \rightarrow \infty} \tilde{\alpha}(s)/\alpha(s) < \infty$, *then there exists a function* $\tilde{\gamma} \in \mathcal{K}_\infty$ *such that* $(\tilde{\alpha}, \tilde{\gamma})$ *is a new SiISS supply rate of system* (2.4). *Moreover, if* $\limsup_{s \rightarrow 0_+} \gamma(s)/s^m < \infty$, *then* $\limsup_{s \rightarrow 0_+} \tilde{\gamma}(s)/s^m < \infty$, *where* m *is any positive integer.*

   The following lemma shows that the condition at infinity can be removed if more prior information on stochastic system is known.

Assumption H. For functions g, V, α given in (2.4), (2.5) with $\liminf_{s \rightarrow \infty} \alpha(s) = \infty$, there exist known smooth positive definite functions ϕ_1, ϕ_2 such that $\|g(x, v, t)\| \leq \phi_1(|x|)$, $|\partial V(x)/\partial x| \leq \phi_2(|x|)$ and $\limsup_{s \rightarrow 0_+} \phi_1^2(s)\phi_2^2(s)/\alpha(s) < \infty$.

Lemma 2.5 (see [35]). *For system* (2.4) *satisfying* (2.5) *and Assumption H, if there exists a positive definite function $\tilde{\alpha}$ such that*

$$\limsup_{s \to 0+} \frac{\tilde{\alpha}(s)}{\alpha(s)} < \infty, \qquad \int_0^\infty e^{-\int_0^s (1/\zeta(\alpha_1^{-1}(\tau)))d\tau} \left[\xi\left(\alpha_1^{-1}(s)\right) \right]' ds < \infty, \tag{2.6}$$

where $\xi(\cdot) \geq 0$, $\zeta(\cdot) > 0$ are smooth increasing functions with $\xi(s)\alpha(s) \geq 2\tilde{\alpha}(s)$, $\zeta(s)\alpha(s) \geq \phi_1^2(s)\phi_2^2(s)$ for any $s \geq 0$, then there exists a function $\tilde{\gamma} \in \mathcal{K}_\infty$ such that $(\tilde{\alpha}, \tilde{\gamma})$ is a new SiISS supply rate of system (2.4). *Moreover, if $\limsup_{s \to 0+} \gamma(s)/s^m < \infty$, then $\limsup_{s \to 0+} \tilde{\gamma}(s)/s^m < \infty$, where m is any positive integer.*

Lemma 2.6. *Let x, y be real variables, then for any positive integers m, n and continuous function $a(\cdot) \geq 0$, $a(\cdot)x^m y^n \leq b|x|^{m+n} + (n/(m+n))((m+n)/m)^{-m/n} a^{(m+n)/n}(\cdot)b^{m/n}|y|^{m+n}$, where $b > 0$ is any real number.*

3. Problem Statement

In this paper, we consider a class of stochastic nonlinear systems described by

$$d\eta = \varphi_0(\eta, x_1)dt + \psi_0(\eta, x_1)dw,$$

$$dx_1 = x_2 dt + \varphi_1(\eta, x)dt + \psi_1(\eta, x)dw,$$

$$\vdots \tag{3.1}$$

$$dx_{n-1} = x_n dt + \varphi_{n-1}(\eta, x)dt + \psi_{n-1}(\eta, x)dw,$$

$$dx_n = udt + \varphi_n(\eta, x)dt + \psi_n(\eta, x)dw,$$

$$y = x_1,$$

where $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$, $u, y \in \mathbb{R}$ represent the unmeasurable state, the control input, and the measurable output, respectively. $\eta \in \mathbb{R}^{n_0}$ is referred to as the stochastic inverse dynamics. The initial value $(\eta^T(0), x_1(0), \ldots, x_n(0))$ can be chosen arbitrarily. w is an m-dimensional standard Wiener process defined as in (2.1). Uncertain functions $\varphi_i : \mathbb{R}^{n_0} \times \mathbb{R}^n \to \mathbb{R}$, $\psi_i : \mathbb{R}^{n_0} \times \mathbb{R}^n \to \mathbb{R}^m$, $1 \leq i \leq n$, are smooth functions. It is assumed that φ_0 and ψ_0 are locally Lipschitz continuous functions.

The research purpose of this paper is to design an adaptive output feedback controller for system (3.1) by using stochastic LaSalle theorem and small-gain type conditions on SiISS, in such a way that, for all initial conditions, the solutions of the closed-loop system are bounded almost surely and the closed-loop systems are globally stable in probability. To achieve the control purpose, we need the following assumptions.

Assumption 3.1. For each $1 \leq i \leq n$, there exist the unknown constant $\theta_i > 0$, the known nonnegative smooth functions φ_{i0}, φ_{i1}, ψ_{i0} and ψ_{i1} with $\varphi_{ij}(0) = 0$, $\psi_{ij}(0) = 0$, $j = 0, 1$, such that $|\varphi_i(\eta, x)| \leq \theta_i(\varphi_{i0}(|\eta|) + \varphi_{i1}(x_1))$, $|\psi_i(\eta, x)| \leq \theta_i(\psi_{i0}(|\eta|) + \psi_{i1}(x_1))$.

For Assumption 3.1, there exist smooth functions $\overline{\varphi}_{i1}$ and $\overline{\psi}_{i1}$ satisfying

$$\varphi_{i1}(x_1) = x_1\overline{\varphi}_{i1}(x_1), \qquad \psi_{i1}(x_1) = x_1\overline{\psi}_{i1}(x_1), \tag{3.2}$$

which will be frequently used in the subsequent sections.

Assumption 3.2. For the η-subsystem, there exists an SiISS-Lyapunov function $V_0(\eta)$. Namely, V_0 satisfies $\underline{\alpha}(|\eta|) \leq V_0(\eta) \leq \overline{\alpha}(|\eta|)$, $\mathcal{L}V_0(\eta) \leq -\alpha(|\eta|) + \gamma(|x_1|)$, where $\underline{\alpha}, \overline{\alpha}, \gamma$ are class $\mathcal{K}_\infty$ functions, and α is merely a continuous positive definite function.

4. Design of an Adaptive Output Feedback Controller

4.1. Reduced-Order Observer Design

Introduce the following reduced-order observer:

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + k_{i+1}y - k_i(\hat{x}_1 + k_1y), \quad 1 \leq i \leq n-2,$$

$$\dot{\hat{x}}_{n-1} = u - k_{n-1}(\hat{x}_1 + k_1y), \tag{4.1}$$

where $k = (k_1, \ldots, k_{n-1})^T$ is chosen such that $A_0 = \begin{bmatrix} -k & I_{n-2} \\ & 0\ldots0 \end{bmatrix}$ is asymptotically stable. Define the error variable

$$e_i = \frac{1}{\theta^*}(x_{i+1} - \hat{x}_i - k_ix_1), \quad 1 \leq i \leq n-1, \ \theta^* = \max\{1, \theta_1, \ldots, \theta_n\}. \tag{4.2}$$

By (3.1), (4.1), and (4.2), one gets

$$de_i = (e_{i+1} - k_ie_1)dt + \frac{1}{\theta^*}(\varphi_{i+1}(\eta, x) - k_i\varphi_1(\eta, x))dt + \frac{1}{\theta^*}(\psi_{i+1}(\eta, x) - k_i\psi_1(\eta, x))dw, \tag{4.3}$$

whose compact form is

$$de = (A_0e + \Phi(\eta, x))dt + \Psi(\eta, x)dw, \tag{4.4}$$

where

$$e = (e_1, \ldots, e_{n-1})^T,$$

$$\Phi(\eta, x) = \frac{1}{\theta^*}(\varphi_2(\eta, x) - k_1\varphi_1(\eta, x), \varphi_3(\eta, x) - k_2\varphi_1(\eta, x), \ldots, \varphi_n(\eta, x) - k_{n-1}\varphi_1(\eta, x))^T,$$

$$\Psi(\eta, x) = \frac{1}{\theta^*}(\psi_2(\eta, x) - k_1\psi_1(\eta, x), \psi_3(\eta, x) - k_2\psi_1(\eta, x), \ldots, \psi_n(\eta, x) - k_{n-1}\psi_1(\eta, x))^T. \tag{4.5}$$

4.2. The Design of Adaptive Backstepping Controller

From (3.1), (4.1), (4.2), and (4.4), the interconnected system is represented as

$$d\eta = \varphi_0(\eta, y)dt + \psi_0(\eta, y)dw,$$

$$de = (A_0 e + \Phi(\eta, x))dt + \Psi(\eta, x)dw,$$

$$dy = (\hat{x}_1 + k_1 y + \theta^* e_1 + \varphi_1(\eta, x))dt + \psi_1(\eta, x)dw,$$

$$d\hat{x}_1 = (\hat{x}_2 + k_2 y - k_1(\hat{x}_1 + k_1 y))dt, \tag{4.6}$$

$$d\hat{x}_2 = (\hat{x}_3 + k_3 y - k_2(\hat{x}_1 + k_1 y))dt,$$

$$\vdots$$

$$d\hat{x}_{n-1} = (u - k_{n-1}(\hat{x}_1 + k_1 y))dt.$$

Next, we will develop an adaptive backstepping controller by using the backstepping method. Firstly, a coordinate transformation is introduced

$$z_1 = y, \qquad z_{i+1} = \hat{x}_i - \alpha_i\left(y, \hat{x}_1, \ldots, \hat{x}_{i-1}, \hat{\theta}\right), \quad i = 1, \ldots, n-1. \tag{4.7}$$

Step 1. By (4.6) and (4.7), one has

$$dz_1 = (\alpha_1 + z_2 + k_1 y + \theta^* e_1 + \varphi_1(\eta, x))dt + \psi_1(\eta, x)dw. \tag{4.8}$$

Since A_0 is asymptotically stable, there exists a positive definite matrix P such that $PA_0 + A_0^T P = -I_{n-1}$. Choose

$$V_1\left(e, z_1, \hat{\theta}\right) = \frac{\delta}{2}\left(e^T P e\right)^2 + \frac{1}{2\Gamma}\tilde{\theta}^2 + \frac{1}{4}z_1^4, \quad \delta > 0, \ \Gamma > 0, \tag{4.9}$$

where $\tilde{\theta} = \hat{\theta} - \theta$, $\hat{\theta}$ is the estimate of $\theta = \max\{\theta^*, \theta^{*(4/3)}, \theta^{*4}\}$. In view of (2.2), (4.4), (4.8), and (4.9), then

$$\mathcal{L}V_1 = 2\delta e^T P e e^T P(A_0 e + \Phi(\eta, x)) + \frac{1}{2}\operatorname{Tr}\left\{\Psi^T(\eta, x)\left(4\delta\left(e^T P\right)^T e^T P + 2\delta e^T P e P\right)\Psi(\eta, x)\right\}$$

$$+ \frac{1}{\Gamma}\tilde{\theta}\dot{\hat{\theta}} + z_1^3(\alpha_1 + z_2 + k_1 y + \theta^* e_1 + \varphi_1(\eta, x)) + \frac{3}{2}z_1^2\operatorname{Tr}\left\{\psi_1^T(\eta, x)\psi_1(\eta, x)\right\}$$

$$= \delta e^T P e e^T \left(P A_0 + A_0^T P \right) e + 2 \delta e^T P e e^T P \Phi(\eta, x)$$

$$+ \operatorname{Tr} \left\{ \Psi^T(\eta, x) \left(2 \delta \left(e^T P \right)^T e^T P \ + \delta e^T P e P \right) \Psi(\eta, x) \right\}$$

$$+ \frac{1}{\Gamma} \widetilde{\theta} \dot{\hat{\theta}} + z_1^3 (\alpha_1 + z_2 + k_1 y + \theta^* e_1 + \varphi_1(\eta, x)) + \frac{3}{2} z_1^2 \operatorname{Tr} \left\{ \psi_1^T(\eta, x) \psi_1(\eta, x) \right\}$$

$$\leq -\delta \lambda_{\min}(P) |e|^4 + 2 \delta \|P\|^2 |e|^3 |\Phi(\eta, x)| + 3 \delta \|P\|^2 |e|^2 \|\Psi(\eta, x)\|^2 + \frac{1}{\Gamma} \widetilde{\theta} \dot{\hat{\theta}}$$

$$+ z_1^3 (\alpha_1 + z_2 + k_1 y + \theta^* e_1 + \varphi_1(\eta, x)) + \frac{3}{2} z_1^2 \operatorname{Tr} \left\{ \psi_1^T(\eta, x) \psi_1(\eta, x) \right\}.$$

(4.10)

Applying Assumption 3.1, (3.2), (4.3), (4.7), and Lemma 2.6, it follows that

$$2 \delta \|P\|^2 |e|^3 \Phi |(\eta, x)| \ \leq a_{01} |e|^4 + \overline{a}_{01} |\Phi(\eta, x)|^4,$$

$$3 \delta \|P\|^2 |e|^2 \|\Psi(\eta, x)\|^2 \leq a_{02} |e|^4 + \overline{a}_{02} \|\Psi(\eta, x)\|^4,$$

$$z_1^3 z_2 \leq a_{10} z_1^4 + \overline{a}_{10} z_2^4, \qquad \theta^* z_1^3 e_1 \leq a_{11} e_1^4 + \theta^{*(4/3)} \gamma_{11}(z_1) z_1^4 \leq a_{11} e_1^4 + \theta \gamma_{11}(z_1) z_1^4,$$

$$z_1^3 \varphi_1(\eta, x) \leq |z_1|^3 \theta^* (\varphi_{10}(|\eta|) + \varphi_{11}(z_1)) \leq a_{12} \varphi_{10}^4(|\eta|) + \theta \gamma_{12}(z_1) z_1^4,$$

$$\frac{3}{2} z_1^2 \operatorname{Tr} \left\{ \psi_1^T(\eta, x) \psi_1(\eta, x) \right\} \leq 3 z_1^2 \theta^{*2} \left(\psi_{10}^2(|\eta|) + \psi_{11}^2(z_1) \right) \leq a_{13} \psi_{10}^4(|\eta|) + \theta \gamma_{13}(z_1) z_1^4,$$

(4.11)

where $\overline{a}_{01}$, $\overline{a}_{02}$, γ_{11}, γ_{12}, and γ_{13} are smooth functions, $a_{01}, a_{02}, a_{10}, \overline{a}_{10}, a_{11}, a_{12}, a_{13} > 0$ are constants.

Using $(a_1 + \cdots + a_n)^2 \leq (x + a)^n = n \sum\limits_{i=1}^{n} a_i^2$, $(a + b)^4 \leq 8a^4 + 8b^4$, $y = x_1$, Assumption 3.1, (3.2), and (4.5), one gets

$$|\Phi(\eta, x)|^4 \leq 64(n-1) \left(\left(k_1^4 + \cdots + k_{n-1}^4 \right) \left(\varphi_{10}^4 + y^4 \overline{\varphi}_{11}^4 \right) + \varphi_{20}^4 + \cdots + \varphi_{n0}^4 + y^4 \left(\overline{\varphi}_{21}^4 + \cdots + \overline{\varphi}_{n1}^4 \right) \right),$$

$$\|\Psi(\eta, x)\|^4 \leq 64(n-1) \left(\left(k_1^4 + \cdots + k_{n-1}^4 \right) \left(\psi_{10}^4 + y^4 \overline{\psi}_{11}^4 \right) + \psi_{20}^4 + \cdots + \psi_{n0}^4 + y^4 \left(\overline{\psi}_{21}^4 + \cdots + \overline{\psi}_{n1}^4 \right) \right).$$

(4.12)

Substituting (4.11)-(4.12) into (4.10) and using $z_1 = y$ lead to

$$\mathcal{L} V_1 \leq -a_{00} |e|^4 + a_{11} e_1^4 + \Delta_1(|\eta|) + \overline{a}_{10} z_2^4 + z_1^3 (\alpha_1 + k_1 z_1 + \Delta_{00}(z_1) z_1 + a_{10} z_1)$$

$$+ \theta (\gamma_{11}(z_1) + \gamma_{12}(z_1) + \gamma_{13}(z_1)) z_1^4 + \frac{1}{\Gamma} \widetilde{\theta} \dot{\hat{\theta}},$$

(4.13)

where

$$a_{00} = \delta\lambda_{\min}(P) - a_{01} - a_{02},$$

$$\Delta_1(|\eta|) = 64(n-1)\overline{a}_{01}\left(\left(k_1^4 + \cdots + k_{n-1}^4\right)\varphi_{10}^4(|\eta|) + \varphi_{20}^4(|\eta|) + \cdots + \varphi_{n0}^4(|\eta|)\right)$$

$$+ 64(n-1)\overline{a}_{02}\left(\left(k_1^4 + \cdots + k_{n-1}^4\right)\psi_{10}^4(|\eta|) + \psi_{20}^4(|\eta|) + \cdots + \psi_{n0}^4(|\eta|)\right)$$

$$+ a_{12}\varphi_{10}^4(|\eta|) + a_{13}\psi_{10}^4(|\eta|),$$

$$\Delta_{00}(z_1) = 64(n-1)\overline{a}_{01}\left(\left(k_1^4 + \cdots + k_{n-1}^4\right)\overline{\varphi}_{11}^4(z_1) + \overline{\varphi}_{21}^4(z_1) + \cdots + \overline{\varphi}_{n1}^4(z_1)\right)$$

$$+ 64(n-1)\overline{a}_{02}\left(\left(k_1^4 + \cdots + k_{n-1}^4\right)\overline{\psi}_{11}^4(z_1) + \overline{\psi}_{21}^4(z_1) + \cdots + \overline{\psi}_{n1}^4(z_1)\right). \tag{4.14}$$

Choosing the virtual control $\alpha_1(\cdot)$ and the tuning function $\tau_1(\cdot)$

$$\alpha_1\left(y,\widehat{\theta}\right) = -z_1\left(c_1 + \nu_1(z_1) + k_1 + \Delta_{00}(z_1) + a_{10} + \widehat{\theta}\big(\gamma_{11}(z_1) + \gamma_{12}(z_1) + \gamma_{13}(z_1)\big)\right),$$

$$\tau_1(y) = \Gamma\big(\gamma_{11}(z_1) + \gamma_{12}(z_1) + \gamma_{13}(z_1)\big)z_1^4, \tag{4.15}$$

one gets

$$\mathcal{L}V_1 \leq -c_1 z_1^4 - \nu_1(z_1)z_1^4 - a_{00}|e|^4 + a_{11}e_1^4 + \Delta_1(|\eta|) + \overline{a}_{10}z_2^4 + \frac{1}{\Gamma}\widetilde{\theta}\big(\dot{\widehat{\theta}} - \tau_1\big), \tag{4.16}$$

where $c_1 > 0$ is design parameter and $\nu_1(z_1) > 0$ is a smooth function to be chosen later.

Step i ($i = 2,\ldots,n$). For notation coherence, denote $u = \alpha_n$, $z_{n+1} = 0$. At this step, we can obtain a property similar to (4.16), which is presented by the following lemma.

Lemma 4.1. *For the ith Lyapunov function* $V_i(e,\overline{z}_i,\widehat{\theta}) = (\delta/2)(e^T Pe)^2 + (1/2\Gamma)\widetilde{\theta}^2 + \sum_{j=1}^{i}(z_j^4/4)$, *there are the virtual control law* $\alpha_i(y,\widehat{x}_1,\ldots,\widehat{x}_{i-1},\widehat{\theta})$ *and the tuning function* τ_i *with the form*

$$\alpha_i\left(y,\widehat{x}_1,\ldots,\widehat{x}_{i-1},\widehat{\theta}\right) = -\Omega_i - z_i\left(c_i + \overline{a}_{i-1,0} + a_{i0} + \widehat{\theta}\big(\gamma_{i1} + \gamma_{i2} + \gamma_{i3} + \gamma_{i4}\big)\right.$$

$$\left. + \Gamma\big(\gamma_{i1} + \gamma_{i2} + \gamma_{i3} + \gamma_{i4}\big)\sum_{j=1}^{i-1}\frac{\partial\alpha_j}{\partial\widehat{\theta}}z_{j+1}^3\right) + \frac{\partial\alpha_{i-1}}{\partial\widehat{\theta}}\tau_i, \tag{4.17}$$

$$\tau_i\left(y,\widehat{x}_1,\ldots,\widehat{x}_{i-1},\widehat{\theta}\right) = \tau_{i-1} + \Gamma\big(\gamma_{i1} + \gamma_{i2} + \gamma_{i3} + \gamma_{i4}\big)z_i^4,$$

such that

$$\mathcal{L}V_i \leq -\sum_{j=1}^{i} c_j z_j^4 + \sum_{j=2}^{i} b_j z_1^4 - \nu_1(z_1)z_1^4 - a_{00}|e|^4 + \sum_{j=1}^{i} a_{j1}e_1^4 + \Delta_i(|\eta|) + \overline{a}_{i0}z_{i+1}^4$$

$$+ \left(\frac{1}{\Gamma}\widetilde{\theta} - \sum_{j=1}^{i-1} \frac{\partial \alpha_j}{\partial \widehat{\theta}} z_{j+1}^3 \right)\left(\dot{\widehat{\theta}} - \tau_i \right),$$

(4.18)

where $c_i > 0$ is the designed parameters, a_{i0}, $\overline{a}_{i-1,0}$, a_{i1}, b_i are some positive constants, γ_{i1}, γ_{i2}, γ_{i3}, γ_{i4} are smooth nonnegative functions, whose choices are given in the proof.

Proof. See the appendix. □

Therefore, at the end of the recursive procedure, the controller can be constructed as

$$u = \alpha_n\left(y, \widehat{x}_1, \ldots, \widehat{x}_{n-1}, \widehat{\theta} \right), \qquad \dot{\widehat{\theta}} = \tau_n\left(y, \widehat{x}_1, \ldots, \widehat{x}_{n-1}, \widehat{\theta} \right).$$

(4.19)

Choosing parameters δ, a_{01}, a_{02}, a_{11}, ..., a_{n1}, c_2, ..., c_n to satisfy

$$a_0 = \delta\lambda_{\min}(P) - a_{01} - a_{02} - \sum_{i=1}^{n} a_{i1} > 0, \qquad c_2, \ldots, c_n > 0,$$

(4.20)

by (4.16) and (4.18), one has

$$\mathcal{L}V_n \leq -\sum_{i=1}^{n} c_i z_i^4 + \sum_{j=2}^{n} b_j z_1^4 - \nu_1(z_1)z_1^4 - a_0|e|^4 + \Delta_n(|\eta|),$$

(4.21)

where

$$V_n\left(e, z, \widehat{\theta} \right) = \frac{\delta}{2}\left(e^T P e \right)^2 + \frac{1}{2\Gamma}\widetilde{\theta}^2 + \frac{1}{4}\sum_{i=1}^{n} z_i^4,$$

$$\Delta_n(|\eta|) = 64(n-1)\overline{a}_{01}\left(\left(k_1^4 + \ldots + k_{n-1}^4 \right)\varphi_{10}^4(|\eta|) + \varphi_{20}^4(|\eta|) + \ldots + \varphi_{n0}^4(|\eta|) \right)$$

$$+ 64(n-1)\overline{a}_{02}\left(\left(k_1^4 + \ldots + k_{n-1}^4 \right)\psi_{10}^4(|\eta|) + \psi_{20}^4(|\eta|) + \ldots + \psi_{n0}^4(|\eta|) \right)$$

$$+ \sum_{i=1}^{n}\left(a_{i2}\varphi_{10}^4(|\eta|) + a_{i3}\psi_{10}^4(|\eta|) \right) + \sum_{j=2}^{n} a_{j4}\psi_{10}^4(|\eta|).$$

(4.22)

5. Stability Analysis

We state the main theorems in this paper. This section is divided into two parts.

5.1. Case I

Theorem 5.1. *Assume that Assumptions 3.1 and 3.2 hold with the following properties:*

$$\limsup_{s\to 0+}\frac{\Delta_n(|\eta|)}{\alpha(s)} < \infty, \qquad \limsup_{s\to\infty}\frac{\Delta_n(|\eta|)}{\alpha(s)} < \infty. \tag{5.1}$$

If $\limsup_{s\to 0+}\gamma(s)/s^4 < \infty$ *in Assumption 3.2, by appropriately choosing the positive smooth function* $v_1(\cdot)$ *in* (4.15) *and the parameters* δ, a_{01}, a_{02}, a_{11}, ..., a_{n1}, c_2, ..., c_n *to satisfy* (4.20), *then*

(i) *the closed-loop system consisting of* (3.1), (4.1), (4.2), (4.7), (4.15), (4.17), *and* (4.19) *has a unique and almost surely bounded strong solution on* $[0,\infty)$;

(ii) *for each initial value* $(\eta(0), x(0), \hat{x}(0), \hat{\theta}(0))$, *the equilibrium* $(\eta, x) = (0,0)$ *is globally stable in probability, where* $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_{n-1})$.

Proof. For any constant $\epsilon > 0$, by (5.1), one has

$$\limsup_{s\to 0+}\frac{(1+\epsilon)\Delta_n(|\eta|)}{\alpha(s)} < \infty, \qquad \limsup_{s\to\infty}\frac{(1+\epsilon)\Delta_n(|\eta|)}{\alpha(s)} < \infty. \tag{5.2}$$

For the η-subsystem, by Lemma 2.4, there exists a positive and radially unbounded Lyapunov function $\tilde{V}_0(\eta) \in C^2$ and $\tilde{\gamma} = \rho\gamma$ such that

$$\mathscr{L}\tilde{V}_0(\eta) \le -(1+\epsilon)\Delta_n(|\eta|) + \tilde{\gamma}(|z_1|), \tag{5.3}$$

$$\limsup_{s\to 0+}\frac{\tilde{\gamma}(s)}{s^4} < \infty, \tag{5.4}$$

where ρ is a positive constant satisfying $(1+\epsilon)\Delta_n(|\eta|) \le \rho\alpha(s)$ for all $s \ge 0$. Choosing the following Lyapunov function for the entire closed-loop system

$$V\left(\eta, e, z, \hat{\theta}\right) = \tilde{V}_0(\eta) + V_n\left(e, z, \hat{\theta}\right) \tag{5.5}$$

and combining (4.21) and (5.3), one obtains

$$\mathscr{L}V\left(\eta, e, z, \hat{\theta}\right) \le -\sum_{i=1}^{n}c_i z_i^4 - \left(v_1(z_1) - \sum_{j=2}^{n}b_j\right)z_1^4 - a_0|e|^4 - \epsilon\Delta_n(|\eta|) + \tilde{\gamma}(|z_1|). \tag{5.6}$$

By (5.4), there always exists a smooth function $v_1(\cdot)$ to satisfy the following two inequalities:

$$z_1^4\left(v_1(z_1) - \sum_{j=2}^{n}b_j\right) \ge \tilde{\gamma}(|z_1|), \qquad v_1(z_1) \ge \sum_{j=2}^{n}b_j. \tag{5.7}$$

Substituting (5.7) into (5.6) leads to

$$\mathscr{L}V\left(\eta, e, z, \widehat{\theta}\right) \leq -\sum_{i=1}^{n} c_i z_i^4 - a_0|e|^4 - \epsilon\Delta_n(|\eta|)$$

$$\triangleq -W(\eta, e, z).$$

(5.8)

Noting that $\varphi_{i0}(\cdot)$ and $\psi_{i0}(\cdot)$ are nonnegative smooth functions and using (4.22), it follows that $\Delta_n(\cdot)$ and $W(\cdot)$ are continuous nonnegative functions. By (5.5), (5.8), and Lemma 2.2, one concludes that all the solutions of the closed-loop system are bounded almost surely, the equilibrium $(\eta, e, z) = (0, 0, 0)$ is globally stable in probability. By (3.1), (4.1), (4.2), (4.7), (4.15), (4.17), (4.19), and the almost sure boundedness of all the signals, it is not difficult to recursively prove that the equilibrium $(\eta, x) = (0, 0)$ is globally stable in probability. $\qquad\square$

5.2. Case II

If more information about α in Assumption 3.2 is known, that is, $\liminf_{s\to\infty}\alpha(s) = \infty$, further results under the weaker conditions is given as follows.

Assumption 5.2. For functions ψ_0 and V_0 given by (4.22) and Assumption 3.2, there exist known smooth nonnegative functions $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ with $\widetilde{\varphi}_1(0) = \widetilde{\varphi}_2(0) = 0$, such that $\|\psi_0(\eta, x_1)\| \leq \widetilde{\varphi}_1(|\eta|)$ and $|\partial V_0(\eta)/\partial\eta| \leq \widetilde{\varphi}_2(|\eta|)$.

Lemma 5.3. *For Δ_n, $\underline{\alpha}$, α, $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ given by (4.22), Assumptions 3.2 and 5.2, if*

$$\limsup_{s\to 0+}\frac{\Delta_n(s) + \widetilde{\varphi}_1^2(s)\widetilde{\varphi}_2^2(s)}{\alpha(s)} < \infty, \qquad \int_0^\infty e^{-\int_0^s (1/\zeta_1(\underline{\alpha}^{-1}(\tau)))d\tau}\left[\xi_1\left(\underline{\alpha}^{-1}(s)\right)\right]' ds < \infty, \qquad (5.9)$$

where $\xi_1(\cdot) \geq 0$ and $\zeta_1(\cdot) > 0$ are smooth increasing functions satisfying

$$\xi_1(s)\alpha(s) \geq 2(1+\epsilon)\Delta_n(s), \qquad \zeta_1(s)\alpha(s) \geq \widetilde{\varphi}_1^2(s)\widetilde{\varphi}_2^2(s), \qquad \forall s \geq 0, \qquad (5.10)$$

ϵ is any positive constant. Then there exists a function $\widetilde{\gamma} \in \mathcal{K}_\infty$ such that $((1 + \epsilon)\Delta_n, \widetilde{\gamma})$ is a new SiISS supply rate of the η-subsystem in (3.1). Moreover, if γ in Assumptions 3.2 satisfies $\limsup_{s\to 0+}\gamma(s)/s^4 < \infty$, then $\limsup_{s\to 0+}\widetilde{\gamma}(s)/s^4 < \infty$.

Since the condition (5.9) is weaker than (5.1), by using Lemma 5.3, we give further results under the weaker condition (5.9).

Theorem 5.4. *Suppose that Assumptions 3.1–5.2 and the conditions of Lemma 5.3 hold. If $\limsup_{s\to 0+}\gamma(s)/s^4 < \infty$ in Assumption 3.2, by appropriately choosing the positive smooth function $\nu_1(\cdot)$ in (4.15) and the parameters δ, a_{01}, a_{02}, a_{11}, ..., a_{n1}, c_2, ..., c_n to satisfy (4.20), then*

(1) *the closed-loop system consisting of (3.1), (4.1), (4.2), (4.7), (4.15), (4.17), and (4.19) has a unique and almost surely bounded strong solution on $[0, \infty)$;*

(2) *for each initial value $(\eta(0), x(0), \widehat{x}(0), \widehat{\theta}(0))$, the equilibrium $(\eta, x) = (0, 0)$ is globally stable in probability.*

6. Conclusions

This paper further considers a more general class of stochastic nonlinear systems with uncertain parameters and SiISS inverse dynamics. By combining the stochastic LaSalle theorem and small-gain type conditions on SiISS, an adaptive output feedback controller is designed to guarantee that all the closed-loop signals are bounded almost surely and the stochastic closed-loop system is globally stable in probability.

There are two remaining problems to be investigated: (1) an essential problem is to find a practical example with explicit physical meaning for system (3.1). A preliminary attempt on high-order stochastic nonlinear system can be found in [28]. (2) How to design an output feedback controller by using this method in this paper for system (3.1) in which the drift and diffusion vector fields depend on the unmeasurable states besides the measurable output?

Appendix

Proof of Lemma 4.1. We prove Lemma 4.1 by induction. Assume that at Step $i - 1$, there are virtual control laws

$$
\alpha_{i-1}\left(y, \widehat{x}_1, \ldots, \widehat{x}_{i-2}, \widehat{\theta}\right) = -\Omega_{i-1} - z_{i-1}
$$

$$
\times \left(c_{i-1} + \overline{a}_{i-2,0} + a_{i-1,0} + \overline{a}_{i-1,1} + \widehat{\theta}\left(\gamma_{i-1,1} + \gamma_{i-1,2} + \gamma_{i-1,3} + \gamma_{i-1,4}\right) \right.
$$

$$
\left. + \Gamma\left(\gamma_{i-1,1} + \gamma_{i-1,2} + \gamma_{i-1,3} + \gamma_{i-1,4}\right) + \sum_{j=1}^{i-2} \frac{\partial \alpha_j}{\partial \widehat{\theta}} z_{j+1}^3 \right) + \frac{\partial \alpha_{i-2}}{\partial \widehat{\theta}} \tau_{i-1},
$$

$$
\tau_{i-1}\left(y, \widehat{x}_1, \ldots, \widehat{x}_{i-2}, \widehat{\theta}\right) = \tau_{i-2} + \Gamma\left(\gamma_{i-1,1} + \gamma_{i-1,2} + \gamma_{i-1,3} + \gamma_{i-1,4}\right) z_{i-1}^4,
$$

$$
\text{(A.1)}
$$

such that $V_{i-1}(e, \overline{z}_{i-1}, \widehat{\theta}) = (\delta/2)(e^T P e)^2 + (1/2\Gamma)\widetilde{\theta}^2 + \sum_{j=1}^{i-1}(z_j^4/4)$ satisfies

$$
\mathcal{L}V_{i-1} \leq -\sum_{j=1}^{i-1} c_j z_j^4 + \sum_{j=2}^{i-1} b_j z_1^4 - \nu_1(z_1)z_1^4 - a_{00}|e|^4 + \sum_{j=1}^{i-1} a_{j1} e_1^4 + \Delta_{i-1}(|\eta|)
$$

$$
+ \overline{a}_{i-1,0} z_i^4 + \left(\frac{1}{\Gamma}\widetilde{\theta} - \sum_{j=1}^{i-2} \frac{\partial \alpha_j}{\partial \widehat{\theta}} z_{j+1}^3 \right)\left(\dot{\widehat{\theta}} - \tau_{i-1} \right),
$$

$$
\text{(A.2)}
$$

where $c_{i-1} > 0$ are the designed parameters, $a_{i-1,0}, \overline{a}_{i-1,0}, a_{i-1,1}, b_{i-1}$ are some positive constants, and $\gamma_{i-1,1}, \gamma_{i-1,2}, \gamma_{i-1,3}, \gamma_{i-1,4}$ are smooth nonnegative functions. In the sequel, we will prove that Lemma 4.1 still holds for Step i. Choosing

$$
V_i\left(e, \overline{z}_i, \widehat{\theta}\right) = V_{i-1}\left(e, \overline{z}_{i-1}, \widehat{\theta}\right) + \frac{1}{4}z_i^4,
$$

$$
\text{(A.3)}
$$

with the use of (4.6), (4.7), (4.15), and (A.1), the ItÔ differential of z_i is given as follows:

$$dz_i = \left(\alpha_i + z_{i+1} + \Omega_i - \frac{\partial \alpha_{i-1}}{\partial y} \theta^* e_1 - \frac{\partial \alpha_{i-1}}{\partial y} \varphi_1(\eta, x) - \frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \dot{\widehat{\theta}} \right) dt$$

$$- \frac{1}{2} \frac{\partial^2 \alpha_{i-1}}{\partial y^2} \psi_1(\eta, x) \psi_1^T(\eta, x) dt - \frac{\partial \alpha_{i-1}}{\partial y} \psi_1(\eta, x) dw,$$

(A.4)

where $\Omega_i = k_i y - k_{i-1}(\widehat{x}_1 + k_1 y) - \sum_{j=1}^{i-2}(\partial \alpha_{i-1}/\partial \widehat{x}_j)(\widehat{x}_{j+1} + k_{j+1} y - k_j(\widehat{x}_1 + k_1 y)) - (\partial \alpha_{i-1}/\partial y)(\widehat{x}_1 + k_1 y)$. Using (2.2) and (A.2)–(A.4), we arrive at

$$\mathcal{L}V_i \leq -\sum_{j=1}^{i-1} c_j z_j^4 + \sum_{j=2}^{i-1} b_j z_1^4 - \nu_1(z_1) z_1^4 - a_{00}|e|^4 + \sum_{j=1}^{i-1} a_{j1} e_1^4 + \Delta_{i-1}(|\eta|) + \overline{a}_{i-1,0} z_i^4$$

$$+ \left(\frac{1}{\Gamma} \widetilde{\theta} - \sum_{j=1}^{i-2} \frac{\partial \alpha_j}{\partial \widehat{\theta}} z_{j+1}^3 \right) \left(\dot{\widehat{\theta}} - \tau_{i-1} \right)$$

$$+ z_i^3 \left(\alpha_i + \Omega_i - \frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \dot{\widehat{\theta}} + z_{i+1} - \frac{\partial \alpha_{i-1}}{\partial y} \theta^* e_1 - \frac{\partial \alpha_{i-1}}{\partial y} \varphi_1(\eta, x) - \frac{1}{2} \frac{\partial^2 \alpha_{i-1}}{\partial y^2} \psi_1(\eta, x) \psi_1^T(\eta, x) \right)$$

$$+ \frac{3}{2} z_i^2 \operatorname{Tr} \left\{ \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 \psi_1^T(\eta, x) \psi_1(\eta, x) \right\}.$$

(A.5)

Now, we estimate the last five terms, respectively, in the right-hand side of (A.5). According to Assumption 3.1, (3.2), (4.7), and Lemma 2.6, there exist positive real numbers a_{i0}, $\overline{a}_{i0}$, a_{i1}, a_{i2}, a_{i3}, a_{i4}, b_{i2}, b_{i3}, b_{i4}, smooth nonnegative functions γ_{i1}, γ_{i2}, γ_{i3}, γ_{i4} such that

$$z_i^3 z_{i+1} \leq a_{i0} z_i^4 + \overline{a}_{i0} z_{i+1}^4, \qquad -z_i^3 \frac{\partial \alpha_{i-1}}{\partial y} \theta^* e_1 \leq a_{i1} e_1^4 + \theta \gamma_{i1}(\overline{z}_i) z_i^4,$$

$$-z_i^3 \frac{\partial \alpha_{i-1}}{\partial y} \varphi_1(\eta, x) \leq |z_i|^3 \left(1 + \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 \right)^{1/2} (\varphi_{10}(|\eta|) + \varphi_{11}(z_1))$$

$$\leq a_{i2} \varphi_{10}^4(|\eta|) + b_{i2} z_1^4 + \theta \gamma_{i2}(\overline{z}_i) z_i^4,$$

$$-\frac{1}{2} \frac{\partial^2 \alpha_{i-1}}{\partial y^2} z_i^3 \psi_1(\eta, x) \psi_1^T(\eta, x) \leq |z_i|^3 \left| \frac{\partial^2 \alpha_{i-1}}{\partial y^2} \right| (\psi_{10}^2(|\eta|) + \psi_{11}^2(z_1))$$

(A.6)

$$\leq a_{i3} \psi_{10}^4(|\eta|) + b_{i3} z_1^4 + \theta \gamma_{i3}(\overline{z}_i) z_i^4,$$

$$\frac{3}{2} z_i^2 \operatorname{Tr} \left\{ \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 \psi_1^T(\eta, x) \psi_1(\eta, x) \right\} \leq 3 z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 (\psi_{10}^2(|\eta|) + \psi_{11}^2(z_1))$$

$$\leq a_{i4} \psi_{10}^4(|\eta|) + b_{i4} z_1^4 + \theta \gamma_{i4}(\overline{z}_i) z_i^4.$$

Choosing α_i and τ_i as (4.17) and substituting (A.6) into (A.5), (4.18) holds, where $c_i > 0$ is a design parameter,

$$\Delta_i(|\eta|) = \Delta_{i-1}(|\eta|) + a_{i2}\varphi_{10}^4(|\eta|) + a_{i3}\psi_{10}^4(|\eta|) + a_{i4}\psi_{10}^4(|\eta|), \qquad b_i = b_{i2} + b_{i3} + b_{i4}. \tag{A.7}$$

$\square$

Acknowledgments

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