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Research Article

Design of an Annular Disc Subject to Thermomechanical Loading

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Two solutions to design a thin annular disc of variable thickness subject to thermomechanical loading are proposed. It is assumed that the thickness of the disc is everywhere sufficiently small for the stresses to be averaged through the thickness. The state of stress is plane. The initiation of plastic yielding is controlled by Mises yield criterion. The design criterion for one of the solutions proposed requires that the distribution of stresses is uniform over the entire disc. In this case there is a relation between optimal values of the loading parameters at the final stage. The specific shape of the disc corresponds to each pair of such parameters. The other solution is obtained under the additional requirement that the distribution of strains is uniform. This solution exists for the disc of constant thickness at specific values of the loading parameters.

1. Introduction

Thin annular discs subject to various loading conditions are a class of commonly used structures in mechanical engineering. Therefore, there is a vast amount of literature on this topic. These studies can be conveniently divided into two groups, namely, analysis and design. The present paper deals with the design of thin discs. Therefore, previous works solely related to the analysis of discs are not considered here. Reviews of some results on the design of thin discs are provided in [1, 2]. An analytical solution to design an annular disc of variable thickness under internal and external pressures has been proposed in [1]. It has been assumed that the initiation of plastic yielding is controlled by Tresca yield criterion. The goal of the design has been to find conditions under which the yield criterion is simultaneously satisfied at all points of the disc. A disc rotating in a temperature field has been considered in [2]. The objective function in the problem is the disc weight. A numerical method has been used to solve the problem. The present paper concerns with an annular disc inserted into a rigid container and subject to thermomechanical loading. One of the loading parameters is uniform temperature and the other is pressure over the inner radius of the disc. Both vary with the time. Two design criteria are adopted. The requirement of one of the design criteria is that the distribution of stresses is uniform over the entire disc at the final stage. Using this criterion, a relation between optimal loading parameters is obtained. The solution is given in analytical form. The additional requirement of the other design criterion is that the state of strain is uniform. The same requirement is adopted to design Michell structures (see [3]). It is shown that in the case of the disc under consideration there is the unique solution for the second design problem. In particular, the thickness of the disc is constant.

2. Statement of the Problem

Consider a thin annular disc of outer radius b and inner radius a inserted into a rigid container of radius a. It is convenient to introduce a cylindrical coordinate system (r, θ, z) with its z-axis coinciding with the axis of symmetry of the disc. The initial thickness of the disc, h, is a function of r. The disc is subject to thermal loading by a uniform temperature field varying with the time. The disc has no stress at the initial temperature. Uniform pressure varying with the time is applied over the inner radius of the disc. The outer radius is fixed to the container. It is evident that the problem is axisymmetric. In particular, the solution is independent of θ. Moreover, the normal stresses in the cylindrical coordinates, σ_r, σ_θ, and σ_z are the principal stresses. It is also assumed that the state of stress is two-dimensional, $\sigma_z = 0$. The pressure applied, thermal expansion caused by a rise of temperature, and the constraints imposed on the disc affect the initial zero-stress state. It is assumed that the rise of temperature above the reference state, T, and the pressure over the inner radius are monotonically nondecreasing functions of the time, t. The boundary conditions are

$$\sigma_r = -\beta\sigma_0, \tag{2.1}$$

at $r = a$ and

$$u = 0, \tag{2.2}$$

at $r = b$. Here u is the radial displacement, β is a function of the time, and σ_0 is a constant introduced for further convenience. The circumferential displacement vanishes everywhere.

It is assumed that the thickness of the disc is everywhere sufficiently small for the stresses to be averaged through the thickness. In this case the only nontrivial equilibrium equation becomes

$$\frac{\partial}{\partial r}(hr\sigma_r) = h\sigma_\theta. \tag{2.3}$$

The total radial, ε_r, and circumferential, ε_θ, strains are defined by

$$\varepsilon_r = \varepsilon_r^T + \varepsilon_r^e + \varepsilon_r^p, \qquad \varepsilon_\theta = \varepsilon_\theta^T + \varepsilon_\theta^e + \varepsilon_\theta^p, \tag{2.4}$$

where the superscript T denotes the thermal portions of the total strains, the superscript e the elastic portions of the total strains, and the superscript p the plastic portions of the total strains. It follows from Hooke's law that

$$\varepsilon_r^e = \frac{\sigma_r - \nu\sigma_\theta}{E}, \qquad \varepsilon_\theta^e = \frac{\sigma_\theta - \nu\sigma_r}{E}, \tag{2.5}$$

where E is Young's modulus and ν is Poisson's ratio. The thermal portions of the total strains are given by

$$\varepsilon_r^T = \varepsilon_\theta^T = \alpha T, \tag{2.6}$$

where α is the thermal coefficient of linear expansion. In the plastic range, Mises yield criterion is adopted. For the problem under consideration this criterion reduces to

$$\sigma_r^2 + \sigma_\theta^2 - \sigma_\theta\sigma_r = \sigma_0^2, \tag{2.7}$$

where σ_0 is the yield stress in tension, a material constant for perfectly plastic materials. This quantity is also involved in (2.1). The associated flow rule is written in terms of the strain rate components. A consequence of this rule is

$$\frac{\dot{\varepsilon}_r^p}{\dot{\varepsilon}_\theta^p} = \frac{2\sigma_r - \sigma_\theta}{2\sigma_\theta - \sigma_r}, \tag{2.8}$$

where $\dot{\varepsilon}_r^p$ and $\dot{\varepsilon}_\theta^p$ are the plastic portions of the total radial and circumferential strain rates. Another essential equation following from the associated flow rule expresses plastic incompressibility, $\dot{\varepsilon}_r^p + \dot{\varepsilon}_\theta^p + \dot{\varepsilon}_z^p = 0$, where $\dot{\varepsilon}_z^p$ is the plastic portion of the total axial strain rate. This equation serves to determine $\dot{\varepsilon}_z^p$ and is not important for the present solution. At small strains,

$$\dot{\varepsilon}_r^p = \frac{\partial \varepsilon_r^p}{\partial t}, \qquad \dot{\varepsilon}_\theta^p = \frac{\partial \varepsilon_\theta^p}{\partial t}. \tag{2.9}$$

According to the design criterion proposed by Michell (see [3]), all of the structural elements must be strained by exactly the same strain magnitude in either simple tension or pure compression. This criterion can be too restrictive for the structure under consideration. Therefore, in the present paper two design criteria are adopted. First, it is required that an equistressed state occurs in the entire disc. Then, the possibility to obtain a uniform distribution of strains is explored.

3. Restriction on Thickness Variation

The same magnitude of the elastic portion of strains can be obtained if and only if the distribution of stress components is uniform. Then, it follows from (2.3) that

$$\frac{\partial}{h\partial r}(hr) = n + 1, \tag{3.1}$$

where n is constant. Let h_0 be the thickness of the disc at $r = b$. Then, the solution of (3.1) satisfying this condition is

$$h = h_0\left(\frac{r}{b}\right)^n. \tag{3.2}$$

Note that this function is often adopted in studies devoted to analysis of thin discs, for example [4–6]. The uniform distribution of stresses is only required in the final stage of loading. Using (3.2) the equation of equilibrium (2.3) for intermediate stages becomes

$$r\frac{\partial\sigma_r}{\partial r} + (1+n)\sigma_r = \sigma_\theta. \tag{3.3}$$

4. Thermoelastic Solution

At the beginning of the process of loading the entire disc is elastic. At this stage,

$$\frac{\partial u}{\partial r} = \varepsilon_r^T + \varepsilon_r^e, \qquad \frac{u}{r} = \varepsilon_\theta^T + \varepsilon_\theta^e. \tag{4.1}$$

Eliminating u between these two equations, using (2.5) and (2.6), and taking into account that T is independent of r yield

$$r\left(\frac{\partial\sigma_\theta}{\partial r} - \nu\frac{\partial\sigma_r}{\partial r}\right) + (1+\nu)(\sigma_\theta - \sigma_r) = 0. \tag{4.2}$$

Eliminating the stress σ_θ in (4.2) by means of (3.3) gives

$$r^2\frac{\partial^2\sigma_r}{\partial r^2} + (3+n)r\frac{\partial\sigma_r}{\partial r} + n(1+\nu)\sigma_r = 0. \tag{4.3}$$

It is convenient to introduce the dimensionless radius ρ by $\rho = r/b$. Then, the general solution of (4.3) is

$$\frac{\sigma_r}{\sigma_0} = A\rho^{s_1} + B\rho^{s_2}, \qquad \frac{\sigma_\theta}{\sigma_0} = A(1+n+s_1)\rho^{s_1} + B(1+n+s_2)\rho^{s_2} \tag{4.4}$$

where A and B are constants of integration and

$$s_1 = -\left(1+\frac{n}{2}\right) - \frac{1}{2}\sqrt{(2-n)^2 + 4n(1-\nu)},$$
$$s_2 = -\left(1+\frac{n}{2}\right) + \frac{1}{2}\sqrt{(2-n)^2 + 4n(1-\nu)}. \tag{4.5}$$

Substituting (4.4) into (2.5) determines ε_θ^e. Then, using this expression for ε_θ^e and (2.6) the radial displacement can be found from the equation $\varepsilon_\theta^e + \varepsilon_\theta^T = u/r$. As a result,

$$\frac{u}{rq} = A(1+n-\nu+s_1)\rho^{s_1} + B(1+n-\nu+s_2)\rho^{s_2} + \tau, \tag{4.6}$$

where $q = \sigma_0/E$ and $\tau = \alpha T/q$. Substituting the boundary conditions (2.1) and (2.2) into (4.4) and (4.6) leads to

$$A = A^e, \qquad A^e = \frac{\beta(1+n-\nu+s_2) - \tau\omega^{s_2}}{(1+n-\nu+s_1)\omega^{s_2} - (1+n-\nu+s_2)\omega^{s_1}},$$
$$B = B^e, \qquad B^e = \frac{\tau\omega^{s_1} - \beta(1+n-\nu+s_1)}{(1+n-\nu+s_1)\omega^{s_2} - (1+n-\nu+s_2)\omega^{s_1}}, \tag{4.7}$$

where $\omega = a/b$.

5. Thermoelastic-Plastic Solution for Design

The yield criterion (2.7) is satisfied by the following substitution:

$$\frac{\sigma_r}{\sigma_0} = -\frac{2\sin\gamma}{\sqrt{3}}, \qquad \frac{\sigma_\theta}{\sigma_0} = -\frac{\left(\sin\gamma + \sqrt{3}\cos\gamma\right)}{\sqrt{3}}, \tag{5.1}$$

where γ is a function of ρ and τ. Substituting (5.1) into (3.3) results in

$$\rho\frac{\partial\gamma}{\partial\rho} = \frac{\sqrt{3}\cos\gamma - (1+2n)\sin\gamma}{2\cos\gamma}. \tag{5.2}$$

The zone where the yield criterion is satisfied should occupy the entire disc at the final stage. As it has been mentioned before, the design criterion chosen is satisfied if and only if the distribution of stress is uniform at this stage. Therefore, it should be uniform over the domain where (5.2) is valid. It follows from (5.1) that the condition that the distribution of the stresses in the plastic zone is uniform is equivalent to the condition that γ is independent of ρ. It is evident that the general solution of (5.2) does not satisfy this requirement. However, this equation has a special solution in the form $\gamma = \gamma_0$, where

$$\tan\gamma_0 = \frac{\sqrt{3}}{1+2n}. \tag{5.3}$$

It is seen from (2.1), (5.1) and (5.3) that this special solution takes place if and only if

$$\beta = \frac{2}{\sqrt{3}} \sin \gamma_0. \tag{5.4}$$

Let R be the dimensionless radius of the elastic/plastic boundary. The general solution (4.4) and (4.6) is valid in the elastic zone. However, A and B are not given by (4.7). The stresses σ_r and σ_θ are continuous across the elastic/plastic boundary. Therefore, it follows from (4.4) and (5.1) that

$$\begin{gathered} -\frac{2}{\sqrt{3}} \sin \gamma_0 = AR^{s_1} + BR^{s_2}, \\ -\frac{\left(\sin \gamma_0 + \sqrt{3} \cos \gamma_0\right)}{\sqrt{3}} = A(1+n+s_1)R^{s_1} + B(1+n+s_2)R^{s_2}. \end{gathered} \tag{5.5}$$

The boundary condition (2.2) combined with (4.6) gives

$$A(1+n-\nu+s_1) + B(1+n-\nu+s_2) + \tau = 0. \tag{5.6}$$

Solving (5.5) for AR^{s_1} and BR^{s_2} results in

$$\begin{aligned} AR^{s_1} &= A_0 = \frac{\sqrt{3}\cos \gamma_0 - (2n + 2s_2 + 1)\sin \gamma_0}{\sqrt{3}s}, \\ BR^{s_2} &= B_0 = \frac{(2n + 2s_1 + 1)\sin \gamma_0 - \sqrt{3}\cos \gamma_0}{\sqrt{3}s}. \end{aligned} \tag{5.7}$$

Thus, the quantities AR^{s_1} and BR^{s_2} are independent of τ. Eliminating A and B in (5.6) by means of (5.7) leads to

$$A_0 R^{-s_1}(1+n-\nu+s_1) + B_0 R^{-s_2}(1+n-\nu+s_2) + \tau = 0. \tag{5.8}$$

It is convenient to introduce the following quantities $\xi_r^p = \partial \varepsilon_r^p / \partial \tau$ and $\xi_\theta^p = \partial \varepsilon_\theta^p / \partial \tau$. Then, (2.8) becomes

$$\xi_r^p = \xi_\theta^p \left(\frac{2\sigma_r - \sigma_\theta}{2\sigma_\theta - \sigma_r} \right). \tag{5.9}$$

Since the stresses are constant in the plastic zone, the elastic strain rates vanish. Moreover, it follows from (2.6) that $\xi_r^T = \partial \varepsilon_r^T / \partial \tau = q$ and $\xi_\theta^T = \partial \varepsilon_\theta^T / \partial \tau = q$. Therefore, the total strain rates in the equation of compatibility can be replaced with their plastic portions. Then, using the definition for ξ_r^p and ξ_θ^p this equation is reduced to

$$\rho \frac{\partial \xi_\theta^p}{\partial \rho} + \xi_\theta^p - \xi_r^p = 0. \tag{5.10}$$

Substituting (5.1) at $\gamma = \gamma_0$ into (5.9) and then eliminating ξ_r^p in (5.10) yield

$$\rho \frac{\partial \xi_\theta^p}{\partial \rho} = -\frac{3n\xi_\theta^p}{1+2n}. \tag{5.11}$$

Here, (5.3) has been used to eliminate $\tan\gamma_0$. Equation (5.11) can be immediately integrated to give

$$\xi_\theta^p = q\xi_0(\tau)\rho^m, \quad m = -\frac{3n}{1+2n}, \tag{5.12}$$

where $\xi_0(\tau)$ is a function of integration. Introduce the notation $w = du/d\tau$. Note that w is proportional to the radial velocity. Using (5.12) the value of w on the plastic side of the elastic/plastic boundary is determined as

$$w_R^p = qR + q\xi_0(\tau)R^{m+1}. \tag{5.13}$$

Differentiating (4.6) with respect to τ yields

$$\frac{w}{rq} = \frac{dA}{d\tau}(1+n-\nu+s_1)\rho^{s_1} + \frac{dB}{d\tau}(1+n-\nu+s_2)\rho^{s_2} + 1. \tag{5.14}$$

Thus, the value of w on the elastic side of the elastic/plastic boundary is

$$w_R^e = \left[\frac{dA}{d\tau}(1+n-\nu+s_1)R^{s_1} + \frac{dB}{d\tau}(1+n-\nu+s_2)R^{s_2} + 1\right]Rq. \tag{5.15}$$

Since $w_R^p = w_R^e$, it follows from (5.13) and (5.15) that

$$\xi_0(\tau) = \left[\frac{dA}{d\tau}(1+n-\nu+s_1)R^{s_1} + \frac{dB}{d\tau}(1+n-\nu+s_2)R^{s_2}\right]R^{-m}. \tag{5.16}$$

This equation can be rewritten in the following equivalent form:

$$\xi_0(\tau) = \left[\frac{dA}{dR}(1+n-\nu+s_1)R^{s_1} + \frac{dB}{dR}(1+n-\nu+s_2)R^{s_2}\right]\frac{dR}{d\tau}R^{-m}. \tag{5.17}$$

Eliminating here the derivatives dA/dR and dB/dR by means of (5.7) gives

$$\begin{aligned}\xi_0(\tau) = -\Big[&s_1(1+n-\nu+s_1)\left[\sqrt{3}-(2n+2s_2+1)\tan\gamma_0\right] \\ &+s_2(1+n-\nu+s_2)\left[(2n+2s_1+1)\tan\gamma_0-\sqrt{3}\right]\Big]\frac{dR}{d\tau}\frac{R^{-m-1}}{\sqrt{3}s}\cos\gamma_0.\end{aligned} \tag{5.18}$$

Eliminating here $\tan\gamma_0$ by means of (5.3) yields

$$\xi_0(\tau) = -2\frac{dR}{d\tau}\frac{s_1 s_2 R^{-m-1}}{(1+2n)}\cos\gamma_0. \tag{5.19}$$

Substituting (5.19) into (5.12) gives

$$\xi_\theta^p = -\frac{2q s_1 s_2 \cos\gamma_0 \rho^m}{(1+2n)}\frac{R^{-m-1}dR}{d\tau}. \tag{5.20}$$

Integrating with respect to τ determines the circumferential plastic strain as

$$\varepsilon_\theta^p = \frac{2q s_1 s_2 \cos\gamma_0 \rho^m}{(1+2n)m}R^{-m} + \varepsilon_0(\rho), \tag{5.21}$$

where $\varepsilon_0(\rho)$ is an arbitrary function of ρ. This function should be found using the condition $\varepsilon_\theta^p = 0$ at the elastic/plastic boundary. Then, it follows from (5.21) that

$$\varepsilon_\theta^p = \frac{2q s_1 s_2 \cos\gamma_0}{(1+2n)m}\left(\frac{\rho^m}{R^m} - 1\right). \tag{5.22}$$

Substituting (5.1) at $\gamma = \gamma_0$ into (5.9), eliminating $\tan\gamma_0$ by means of (5.3), and integrating with the respect to τ using the condition that $\varepsilon_\theta^p = 0$ when $\varepsilon_r^p = 0$ give

$$\varepsilon_r^p = \varepsilon_\theta^p \frac{(1-n)}{(1+2n)}. \tag{5.23}$$

6. Design of the Disc

The solution found can be used to search for two kinds of optimal conditions. In particular, it is possible to search for a uniform distribution of stresses at the final stage of loading. This kind of design requires that the plastic zone occupies the entire disc. The stresses at any point of the disc are given by (5.1), where γ should be replaced with γ_0. Putting $R = 1$ in (5.8) determines the value of $\tau = \tau_p$ at which the entire disc becomes plastic

$$\tau_p = -A_0(1+n-\nu+s_1) - B_0(1+n-\nu+s_2). \tag{6.1}$$

Using (4.5), (5.3), (5.4), (5.7), and (6.1) it is possible to find a relation between the two optimal loading parameters at the final stage, τ_p and β, at a given value of n. The distribution of the elastic and thermal portions of the strain tensor is uniform at these values of the parameters. However, the plastic portion of the strain tensor varies with the radius according to (5.22) and (5.23). Therefore, not all of the requirements of Michell structures are satisfied. The following solution enables the total strain distribution to be uniform. It is evident from (5.12) that it is possible if and only if $\xi_0(\tau) = 0$. Then, it follows from (5.19) that $s_1 = 0$ or $s_2 = 0$. These conditions along with (4.5) provide two equations for n. The equation corresponding to $s_1 = 0$

has no solution. The other equation gives $n = 0$. In this case $s_1 = -2$. Thus, the distribution of strains in the disc of constant thickness is uniform if $\beta = 1$, as follows from (5.3) and (5.4). Moreover, it is seen from (5.3) and (5.7) that $A_0 = 0$ at $n = 0$ and $s_2 = 0$. Therefore, (5.8) does not provide any relation between R and τ. The physical meaning of this feature of the solution is that the plastic zone simultaneously occupies the entire disc of any size. Thus, this design satisfies the criterion adopted in [1]. The corresponding value of τ can be found from the thermoelastic solution. In particular, since β is given, A_e and B_e in (4.7) are solely dependent of τ. Therefore, replacing A and B in (4.4) with A_e and B_e, respectively, and putting $\rho = \omega$ determine the stresses as functions of τ. Finally, substituting these functions into the yield criterion (2.7) gives the equation for the value of τ at which the entire disc becomes plastic.

7. Conclusions

New solutions for design of a thin annular disc subject to thermomechanical loading have been proposed. Two design criteria have been adopted. One of the criteria requires that the state of stress is uniform at the final stage of loading. This criterion leads to a relation between optimal values of the loading parameters for each specific shape of the disc. A more restrictive criterion additionally requires, by analogy to Michell structures, that the state of strain is uniform at the final stage. Application of this criterion has shown that the state of stress and strain required appears in the disc of constant thickness at certain values of the loading parameters. This design also satisfies the requirements formulated in [1]. Possible developments of the approach proposed include plastic anisotropy, pressure-dependency of the yield criterion, and variation of some material properties along the radius.

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