Research Article

Stabilization of the Ball on the Beam System by Means of the Inverse Lyapunov Approach

Carlos Aguilar-Ibañez,[1] Miguel S. Suarez-Castanon,[2] and José de Jesús Rubio[3]

[1] CIC-IPN, Unidad Profesional Adolfo López Mateos, Avendia Juan de Dios Bátiz S/N,
  Casi Esquire Miguel Othón de Mendizábal, Colonia Nueva Industrial Vallejo, Delegación
  Gustavo A. Madero, 07738 Mexico City, DF, Mexico
[2] ESCOM-IPN, 07738 Mexico City, DF, Mexico
[3] SEPI-ESIME Azcapotzalco, 02250 Mexico City, DF, Mexico

Correspondence should be addressed to Carlos Aguilar-Ibañez, caguilar@cic.ipn.mx

A novel inverse Lyapunov approach in conjunction with the energy shaping technique is applied to derive a stabilizing controller for the ball on the beam system. The proposed strategy consists of shaping a candidate Lyapunov function as if it were an inverse stability problem. To this purpose, we fix a suitable dissipation function of the unknown energy function, with the property that the selected dissipation divides the corresponding time derivative of the candidate Lyapunov function. Afterwards, the stabilizing controller is directly obtained from the already shaped Lyapunov function. The stability analysis of the closed-loop system is carried out by using the invariance theorem of LaSalle. Simulation results to test the effectiveness of the obtained controller are presented.

1. Introduction

The ball and the beam system (**BBS**) is a popular and important nonlinear system due to its simplicity and easiness to understand and implement in the laboratory. It is also an unstable system, and for this reason it has been widely used not only as a test bed for the effectiveness of control design techniques offered by modern control theory [1, 2] but also to avoid the danger that usually accompanies real unstable systems when brought to the laboratory. In fact, the dynamics of this system are very similar to those found in aerospace systems.

The **BBS** system consists of a beam, which is made to rotate in a vertical plane by applying a torque at the center of rotation, and a ball that is free to roll along the beam. Since

the system does not have a well-defined relative degree at the origin, the exact input-output linearization approach cannot be directly applied to stabilize it around the origin; that is, this system is not feedback linearizable by means of static or dynamic state feedback. This obstacle makes it difficult to design either a stabilizable or a tracking controller [2, 3]. Fortunately, the system is locally controllable around the origin. Hence, it is possible to control it, if it is initialized close enough to the origin by using the direct pole placement method.

Due to its importance several works related to the control of the **BBS** can be found in the literature. A control strategy based on an approximate feedback linearization was proposed by Hauser et al. in [2]. The main idea consists of discarding certain terms to avoid singularities. The drawback of this strategy is that the closed-loop system behaves properly in a small region, but it fails in a large one. In the same spirit, combined with suitable intelligent switches, we mention the works of [4, 5]. In the first work the authors present a control scheme that switches between exact and approximate input-output linearization control laws; in the other work the use of exact input-output linearization in combination with fuzzy dynamic control is proposed. A constructive approach based on the Lyapunov theory was developed in [6], where a numerical approximation for solving one **PDE** was considered. In the similar works of [7–10], energy matching conditions were used for the stabilization of the **BBS**. They also used some numerical approximations in order to solve approximately two matching conditions required to derive the candidate Lyapunov function. A major contribution, rather similar to the matching energy-based approach, was considered in [11, 12]. In these works, the authors solved the two matching conditions related with the potential and kinetic energies of the closed-loop system. In [13], a nested saturation design was proposed in order to bring the ball and the beam to the unstable equilibrium position. Following the same idea, a global asymptotic stabilization was developed with state-dependent saturation levels [14]. A novel work based on a modified nonlinear **PD** control strategy, tested in the laboratory, was presented in [15]. Finally, many control strategies for the stabilization of the **BBS** can be found in the literature, but most of them manage the physical model by introducing some nonlinear approximations or switching through singularities (see [1, 3]).

In this paper we propose a novel inverse Lyapunov-based procedure in combination with the energy shaping method to stabilize the **BBS**. Intuitively, the Lyapunov function is found as if it were an inverse stability problem; that is, we first choose the dissipation rate function of the time derivative of the unknown candidate Lyapunov function. For that purpose, we shape a suitable candidate Lyapunov function, which is locally strictly positive definite inside an admissible set of attraction. Afterwards, the control is proposed in such a way that the time derivative of the obtained Lyapunov function is forced to be equal to the proposed dissipation rate function. The proposed Lyapunov function is formed by adding a kinetic energy function and a particular function, which can be considered as the corresponding potential energy function. To carry this out, we found two restriction equations related to the potential and kinetic energies. The main characteristic of our control strategy is that we do not need to force the closed-loop system to follow another stable Euler-Lagrange or Hamiltonian system, contrary to what was previously proposed in [7–11, 16, 17].

The rest of this paper is organized as follows. In Section 2 we present the control model of the **BBS**. In Section 3 we briefly introduce the inverse Lyapunov method for solving the stabilization of the **BBS**; we also discuss the asymptotic convergence of the closed-loop system. In Section 4 we present some numerical simulations to assess the effectiveness of our control strategy. In Section 5 some conclusions are given.

2. System Dynamics

Consider the **BBS** shown in Figure 1, which consists of a beam that rotates in a vertical plane when a torque is applied to its rotational center and a ball that freely moves forwards and backwards along the beam with only a single degree of freedom. The **BBS** nonlinear model is described by the following set of differential equations (see [1, 2]):

$$\left(m + \frac{J_B}{R^3}\right)\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta + \beta\dot{r} = 0,$$
$$(mr^2 + J_B + J)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\theta = \tau, \tag{2.1}$$

where r is the ball position along the beam, θ is the beam angle, J is the moment of inertia of the beam around the rotating pivot, J_B is the moment of inertia of the ball with respect to its center, R is the radius ball, m is the ball mass, $\beta > 0$ is the friction coefficient, and τ is the torque of the system. After applying the following feedback

$$\tau = u\left(mr^2 + J_B + J\right) + 2Mr\dot{r}\dot{\theta} + mgr\cos\theta \tag{2.2}$$

into system (2.1), it can be rewritten as

$$\ddot{r} = dr\dot{\theta}^2 - n\sin\theta - b\dot{r},$$
$$\ddot{\theta} = u, \tag{2.3}$$

where

$$b = \frac{\beta}{m + J_B/R^3}, \qquad d = \frac{m}{m + J_B/R^3}, \qquad n = \frac{mg}{m + J_B/R^3}. \tag{2.4}$$

Note that u can be seen as a virtual controller that acts directly on the actuated coordinate θ. Naturally, the latter system equations can be written as

$$\ddot{q} = S(x) + Fu, \tag{2.5}$$

where $q^T = (r, \theta)$ and $x^T = (q, \dot{q})$.[1]

3. Control Strategy

The control objective is to bring all the states of system (2.3) to the unstable equilibrium point $x = 0$, restricting both the beam angle and the ball position to inside the admissible set $Q \in R^2$, defined by

$$Q = \left\{q = (r, \theta) : |r| \le L \wedge |\theta| \le \overline{\theta} < \frac{\pi}{2}\right\}, \tag{3.1}$$

where the positive constants L and $\overline{\theta}$ are known. To this end, a suitable candidate Lyapunov function is constructed by using the *Inverse Lyapunov Approach*.
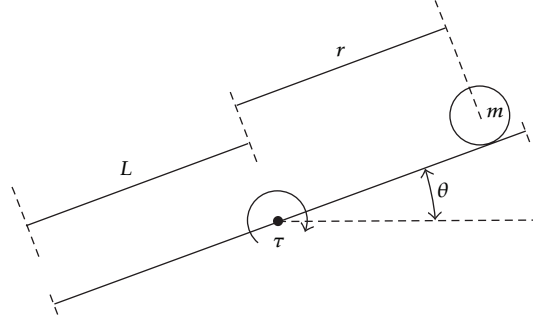
Figure 1: The ball and beam system.

3.1. Inverse Lyapunov Approach

A brief description of the inverse Lyapunov method, inspired in the previous work of Ortega and García-Canseco [18], is introduced next.

Let us propose a candidate Lyapunov function for the closed-loop system energy function, of the form

$$V(x) = \frac{1}{2}\dot{q}^T K_c(r)\dot{q} + V_p(q), \tag{3.2}$$

where the closed-loop inertia matrix $K_c(r) = K_c^T(r) > 0$, and the closed-loop potential energy functions $V_p(q) > 0$, will be defined in the forthcoming developments. A straightforward calculation shows that, along with the solutions of (2.5), $\dot{V}$ is given by

$$\dot{V}(x) = \left(\nabla_q V\right)^T \dot{q} + \left(\nabla_{\dot{q}} V^T\right)(S(x) + Fu(x)). \tag{3.3}$$

Comment 1. In fact, $V_p(q)$ is selected such that $\nabla_x V_p(x)|_{x=0}$ and $\nabla_x^2 V_p(x)|_{x=0} > 0$; that is, we require that V_p be strictly locally convex around the origin.

Fixing the following auxiliary variable as $\eta(x) = \dot{\theta} + \alpha(r)\dot{r}$, with $\alpha(r) \neq 0$, for all $r \in Q$ (it is given in advance), we want to find $u(x) \in R$, $V_p(q) \in R^+$ and $K_c(r) > 0$, with $x \in D \subset Q \times R^2$.[2] Such that $\dot{V}$ can be rewritten as

$$\dot{V}(x) = \eta(x)(\beta(x) + u(x)) + R(x), \tag{3.4}$$

where $\beta(x)$ and $R(x)$ are continuous functions. Then, we propose the control law as

$$u(x) = -k_d\eta(x) - \beta(x), \tag{3.5}$$

for some $k_d > 0$, which evidently leads to

$$\dot{V}(x) = -k_d\eta^2(x) + R(x). \tag{3.6}$$

In order to guarantee that $\dot{V}$ be semidefinite negative, we require that a k_d exists, such that

$$-k_d \eta^2(x) + R(x) \leq 0. \tag{3.7}$$

Physically, we are choosing a convenient dissipation function, $\eta(x)$, of the unknown closed-loop energy function $V(x)$, with the property that n divides $(\dot{V} - R)(x)$. We must underscore that the fixed η relies on the unactuated coordinate r, in agreement with the structure of the closed-loop energy function.

On the other hand, $R(x)$ is the work of the friction forces, which act over the unactuated coordinate.

Closed-Loop System Stability

If we are able to shape the candidate Lyapunov function (3.2), such that its time derivative, along the trajectories of system (2.3), can be expressed as (3.4) under the assumption that (3.7) holds, then V qualifies as a Lyapunov function, because it is a nonincreasing and positive definite function in the neighborhood of the origin and proper on its sub-level (i.e., there exists a $c > 0$, such that $V(x) \leq c$ defines a compact set, with closed level curves). Consequently, x is stable in the Lyapunov sense.

3.2. Solving the BBS Stabilization Problem by Applying the Inverse Lyapunov Approach

In this section we explain how to take the original expression of $\dot{V}$, defined in (3.3), to the desired form (3.4) for the particular case of the **BBS**.

Defining, K_c, as[3]

$$K_c = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}, \tag{3.8}$$

and according to (3.3), we have that $\dot{V}$ can be expressed as

$$\dot{V}(x) = \dot{q}^T \left(v_p(q) + v_d(x) + K_c F u \right) + R_v(x), \tag{3.9}$$

where

$$v_d(x) = \{v_{d_i}\}_{i=1}^2 = \frac{1}{2} \nabla_q \left(\dot{q}^T K_c \dot{q} \right) + K_c \begin{bmatrix} dr\dot{\theta}^2 \\ 0 \end{bmatrix}, \tag{3.10}$$

$$v_p(q) = \{v_{p_i}\}_{i=1}^2 = \nabla_q V_p(q) + K_c \begin{bmatrix} -n\sin\theta \\ 0 \end{bmatrix}. \tag{3.11}$$

Equating equation (3.9) with (3.3), we obtain, after some simple algebraic manipulations, the following:

$$\overbrace{R_v(x) - R(x)}^{T_0} + \overbrace{\left(\dot{q}^T K_c F - \eta(x)\right)u}^{T_1} + \overbrace{\dot{q}^T\left(v_p(q) + v_d(x)\right) - \eta(x)\beta(x)}^{T_2} = 0. \tag{3.12}$$

From the above we have that $R_v(x) = R(x)$. Now, as the matrix K_c and functions $v_p(q)$ and $v_d(x)$ can be seen as free control parameters, we can select them, such that the following equalities hold:

$$\dot{q}^T K_c F = n(x), \tag{3.13}$$

$$\dot{q}^T\left(v_p(q) + v_d(x)\right) = \eta(x)\beta(x). \tag{3.14}$$

This implies that $T_i = 0$, with $i = 0, 1, 2$. Indeed, it is justified because K_c is constituted by three free parameters.

Note that this equation has two unknown parameters, given by k_1 and V_p. Hence, in order to solve it, we require that

$$\dot{q}^T v_p(q) = \zeta_p(q)n(x), \qquad \dot{q}^T v_d(x) = \zeta_d(x)n(x), \tag{3.15}$$

where the continuous functions ζ_p and ζ_d will be computed later by using simple polynomials factorization. Consequently, $\beta(x)$ is directly computed by:

$$\beta(x) = \zeta_p(q) + \zeta_d(x). \tag{3.16}$$

Finally, ζ_p and ζ_d are obtained according to the following remark.

Remark 3.1. Notice that $\dot{q}^T v_p(q)$ and $\dot{q}^T v_d(q)$ are polynomials with respect to variables $(\dot{r}, \dot{\theta})$. Consequently, thefollowing equalities

$$\dot{q}^T v_p(q)\Big|_{\dot{\theta}=-\alpha(r)\dot{r}} = 0, \tag{3.17}$$

$$\dot{q}^T v_d(x)\Big|_{\dot{\theta}=-\alpha(r)\dot{r}} = 0 \tag{3.18}$$

imply that functions $\zeta_p(q)$ and $\zeta_d(x)$ satisfy the restrictions in (3.15). In other words, the selected η must divide the two scalar functions $\dot{q}^T v_p(q)$ and $\dot{q}^T v_d(q)$.

3.2.1. Computing the Needed Candidate Lyapunov Function

In this section we obtain the unknown control variables K_c and V_p. We begin by solving the restriction equation (3.13). For simplicity, we set $\alpha(r) = 1$ and $\eta = -\dot{r} + \dot{\theta}$. Therefore, from (3.13) and (3.8) we evidently have that

$$\dot{q}^T K_c F = k_2 \dot{r} + k_3 \dot{\theta} = -\dot{r} + \dot{\theta}, \tag{3.19}$$

which leads to $k_2 = -1$ and $k_3 = 1$. Now, substituting the fixed values k_2 and k_3 (3.10), we have that

$$\begin{bmatrix} v_{d_1} \\ v_{d_2} \end{bmatrix} = \frac{1}{2}\nabla_q\left(\dot{q}^T K_c \dot{q}\right) + K_c \begin{bmatrix} dr\dot{\theta}^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{k_1'}{2}\dot{r}^2 + dk_1 r\dot{\theta}^2 \\ -dr\dot{\theta}^2 \end{bmatrix}. \tag{3.20}$$

Next, substituting the above v_{d_1} and v_{d_2} in (3.17), we obtain

$$v_{d_2}(x) + v_{d_1}(x)|_{\dot{\theta}=\dot{r}} = \dot{r}^2\left(\frac{k_1'}{2} + dr(k_1 - 1)\right) = 0, \tag{3.21}$$

which produces the following equation; $k_1' = -2rd(k_1 - 1)$ and whose solution is given by $k_1 = 1 + \overline{k}_1 e^{-dr^2}$, where $\overline{k}_1 > 0$. Hence, matrix K_c can be taken as

$$K_c = \begin{bmatrix} 1 + \overline{k}_1 e^{-dr^2} & -1 \\ -1 & 1 \end{bmatrix}. \tag{3.22}$$

According with (3.22), we have that $\det(K_c) = \overline{k}_1 e^{-dr^2} > 0$; that is, $K_c > 0$, when r is finite.

Now, to obtain $V_p(q)$, we proceed to substitute the obtained K_c into the relation (3.11), having

$$\begin{bmatrix} v_{p_1} \\ v_{p_2} \end{bmatrix} = \nabla_q V_p + K_c \begin{bmatrix} -n\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -\left(1 + \overline{k}_1 e^{-dr^2}\right)n\sin\theta + \dfrac{\partial V_p}{\partial r} \\ n\sin\theta + \dfrac{\partial V_p}{\partial \theta} \end{bmatrix}. \tag{3.23}$$

From (3.18), we have that

$$0 = v_{p_2}(q) + v_{p_1}(q) = -n\overline{k}_1 e^{-dr^2}\sin\theta + \frac{\partial V_p}{\partial r} + \frac{\partial V_p}{\partial \theta}, \tag{3.24}$$

whose solution is given by

$$V_p(q) = n\overline{k}_1 \int_0^r \sin(\theta - r + s)e^{-ds^2}ds + \Omega(r - \theta), \tag{3.25}$$

where $\Omega(*)$ must be selected such that V_p has a local minimum at the origin $q = 0$. To assure this condition, it is enough to define $\Omega(s) = k_p s^2/2$, where $k_p > n\overline{k}_1$. Therefore, V_p reads as

$$V_p(q) = n\overline{k}_1 \int_0^r \sin(\theta - r + s)e^{-ds^2}ds + \frac{k_p}{2}(r - \theta)^2, \tag{3.26}$$

where the integral term can be exactly computed as

$$\int_0^r \sin(\theta - r + s)e^{-ds^2}ds = \cos(\theta - r)I_{\sin r} + \sin(\theta - r)I_{\cos r}, \tag{3.27}$$

where

$$I_{\cos r} = \int_0^r \cos(s)e^{-ds^2}ds = \alpha_1\phi_s(r),$$

$$I_{\sin r} = \int_0^r \sin(s)e^{-ds^2}ds = \alpha_1(\alpha_0 + \phi_c(r)), \tag{3.28}$$

and[4]

$$\alpha_0 = 2\operatorname{Im}\left[\operatorname{erf}\left(\frac{i}{2\sqrt{d}}\right)\right], \qquad \alpha_1 = \frac{\sqrt{\pi}\,\exp(-1/4d)}{4\sqrt{d}},$$

$$\phi_s(r) = 2\operatorname{Re}\left[\operatorname{erf}\left(\frac{i+2dr}{2\sqrt{d}}\right)\right], \qquad \phi_c(r) = -2\operatorname{Im}\left[\operatorname{erf}\left(\frac{i+2dr}{2\sqrt{d}}\right)\right]. \tag{3.29}$$

Remark 3.2. Notice that relation (3.7) can be rewritten as $R_d(x) = \dot{q}^T H \dot{q}$, where

$$H = \begin{bmatrix} -k_d & k_d + bk_1/2 \\ k_d + bk_1/2 & -bk_1 - k_d \end{bmatrix}\dot{q}, \tag{3.30}$$

so that $\dot{q}^T H \dot{q} < 0$, if the parameter k_d is selected such that $-b + 4(k_1 - 1)k_d > 0$; recall that $k_1 > 1$.

Hence, the needed controller, defined by (3.5) and (3.16), is given that[5]

$$u = -k_d(-\dot{r} + \dot{\theta}) - \left(n\sin\theta + \frac{\partial V_p}{\partial\theta}\right) - \left(-dr\dot{\theta}^2 + d\overline{k}_1 r\dot{r}(\dot{r} + \dot{\theta})e^{-dr^2}\right), \tag{3.31}$$

where $\overline{k}_1 > 0$ and $k_d > 0$.

We end this section introducing the following important remark.

Remark 3.3. Notice that we can always compute

$$\tilde{c} = \max_{c>0} q \in Q : V_p(q) = c; \text{ such that } V_p(q) = \tilde{c} \text{ is a closed curve.} \tag{3.32}$$
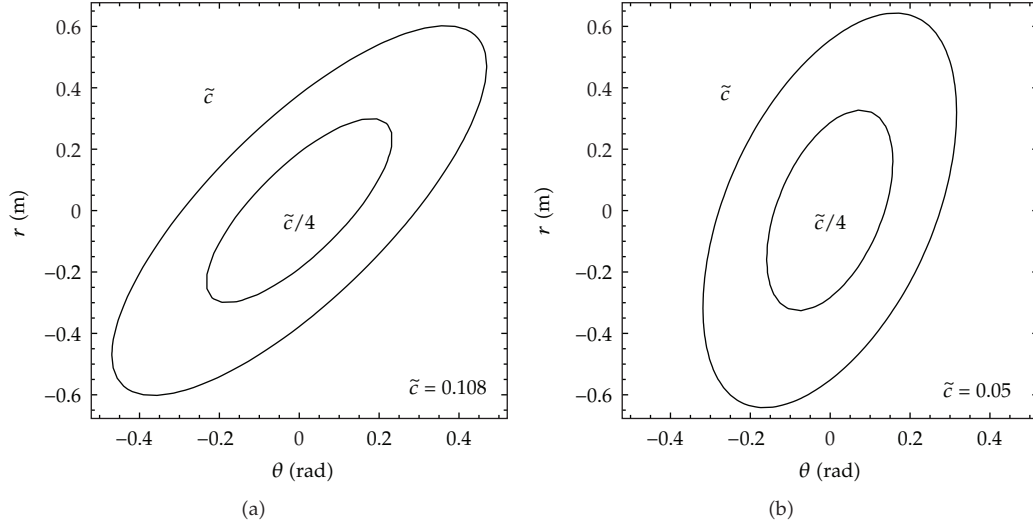
Figure 2: Level curves of the function $\Phi(q)$ around the origin, for two sets of values: $\eta\bar{k} = 1$, $\delta = 0.015$ and $k_p = 2.5$ (a) and $\eta\bar{k} = 1$, $\delta = 0.015$ and $k_p = 1.3$ (b) with restrictions on $|r| \leq 0.6$ (m) and $|\theta| \leq 0.5$ (rad).

To illustrate the geometrical estimation of the bound "$\tilde{c}$", we fix the parameter values $k_p = 2.5$ and $k_p = 1.3$, the other physical parameter values were set as $n\bar{k}_1 = 1$ and $d = 0.015$, and the admissible restricted set was chosen as $|r| \leq 0.6$ [m] and $|\theta| \leq 0.5$ [rad]. This setup allows us to give an estimated c, which evidently guaranties that the level curves of V_p are closed. By a thorough numerical inspection, we found that $\tilde{c} \cong 0.108$ for $k_p = 2.5$ and $\tilde{c} = 0.05$ for $k_p = 1.3$. Figure 2 shows the corresponding level curves.

3.3. Asymptotic Convergence of the Closed-Loop System

Since the obtained V is a non-increasing and positive definite function in some neighborhood that contains the origin, then the closed-loop system is, at least, locally stable in the Lyapunov sense. To assure that the trajectories of the closed-loop system asymptotically converge to the origin, restricted to $q(t) \in Q$, for $t > 0$, we must define the set $\Omega_{\tilde{c}} \in R^4$, where

$$\Omega_{\tilde{c}} = \left\{ (q, \dot{q}) : q \in Q \wedge V(q, \dot{q}) < \tilde{c} \right\}. \tag{3.33}$$

The set $\Omega_{\tilde{c}}$ defines a compact invariant set because, for any initial conditions $x_0 = (q_0, \dot{q}_0)$, with $q_0 \in Q$, provided that $V(x_0) < \tilde{c}$, then $V(x) < \tilde{c}$, with $q \in Q$.

The rest of the stability proof is based on LaSalle's invariant theorem [19, 20]. To apply this theorem we need to define a compact (closed and bounded) set $\Omega_{\tilde{c}}$, which must satisfy that every solution of system (2.3), in closed-loop with (3.31), starting in $\Omega_{\tilde{c}}$ remains in $\Omega_{\tilde{c}}$, for all future time. Then, we define the following invariant set S, as:

$$S = \left\{ x \in \Omega_{\tilde{c}} : \dot{V}(x) = 0 \right\} = \left\{ (q, \dot{q}) \in \Omega_{\tilde{c}} : R_d(x) = \dot{q}^T H \dot{q} = 0 \right\}, \tag{3.34}$$

where $H < 0$. Let M be the largest invariant set in S. Because the theorem of LaSalle claims that every solution starting in a compact set $\Omega_{\tilde{c}}$ approaches M, as $t \rightarrow \infty$, we compute the largest invariant set $M \subset S$. Clearly, we have that $\dot{r} = 0$ and $\dot{\theta} = 0$, on the set S. Therefore, we must have that $\ddot{r} = 0$ and $\ddot{\theta} = 0$, on the set S. Similarly, we must have that $r = r_*$ and $\theta = \theta_*$, with r_* and θ_* being constants. Hence, on the set S, the first equation of (2.3) is written as $0 = -n\sin\theta_*$, then $\theta_* = k\pi$, where k is an integer. However, $\theta_* \in (-\pi/2, \pi/2)$ because $(q, \dot{q}) \in \Omega_{\tilde{c}}$; consequently $\theta = 0$, on the set S. In a similar way, we can show that $r_* = 0$. Then, on the set S, we have that $q = 0$ and $\dot{q} = 0$. Therefore, the largest invariant set M contained inside set S is given by the single point $x = (q = 0, \dot{q} = 0)$. Thus, according to the theorem of LaSalle [19], all the trajectories starting in $\Omega_{\tilde{c}}$ asymptotically converge towards the largest invariant set $M \subset S$, which is the equilibrium point $x = 0$.

We finish this section by presenting the main proposition of this paper.

Proposition 3.4. *Consider system* (2.3) *in closed-loop with* (3.31), *under conditions of Remarks 3.2 and 3.3. Then the origin of the closed-loop system is locally asymptotically stable with the domain of attraction defined by* (3.33).

4. Numerical Simulations

To show the effectiveness of the proposed nonlinear control strategy we have carried out some numerical simulations by means of the Matlab program. The original system p,arameters, with their respective physical restrictions, were set as

$$
\begin{aligned}
&m = 0.1\,\text{kg}, \quad R = 0.015\,\text{m}, \quad J_b = 2.25 \times 10^{-5}\,\text{kg}\cdot\text{m}^2, \quad \overline{\theta} = 0.5\,\text{rad}, \\
&M = 0.2\,\text{kg}, \quad L = 0.6\,\text{m}, \quad\quad J = 0.36\,\text{kg}\cdot\text{m}^2, \quad\quad \beta = 0.2\,\text{New}\cdot\text{m/s}
\end{aligned}
\tag{4.1}
$$

From the above, we have that $b = 0.029$, $d = 0.01477$, and $n = 0.1448$. The physical control parameters were fixed as $k_p = 2.5$, $\overline{k}_1 = 1/n$ and $k_d = 0.5$, while the initial conditions were fixed; as $x_0 = (0.55\,\text{m}; 0; 0.45\,\text{rad}; 0)$. Notice that the proposed set of parameters $\{d, k_p, \overline{k}_1, n\}$ are in agreement with the computation of the restricted stability domain, which has been done in the previous section (see Figure 2(a)); besides the initial conditions satisfy the inequality $V(x_0) < \tilde{c} = 1.08$.[6]

Figure 3 shows the corresponding response of system (2.3) in closed-loop with (3.31), under the conditions in the Remark 3.3. From this figure one can see that both the system position coordinates and the system torque asymptotically converge to the origin, assuring that $|\theta| \leq \overline{\theta}$ and $|r| \leq L$.

In order to provide an intuitive idea of how good our nonlinear control strategy (**ACL**) is in comparison with the control techniques proposed by Yu in [15] and Hauser et al. in [2], here respectively referred as (**YCL**) and (**HCL**), we carried out a second experiment using the same setup as before and assuming that $\beta = 0$. The obtained characteristic polynomial of our control strategy of the linearized system is given by

$$
p(s) = 0.0634 + 0.255s + 0.88s^2 + 1.72s^3 + s^4.
\tag{4.2}
$$

The control parameters of the **YNC** and the **HNC** were selected, such that their corresponding characteristic polynomials coincided with the polynomial (4.2). The initial conditions were
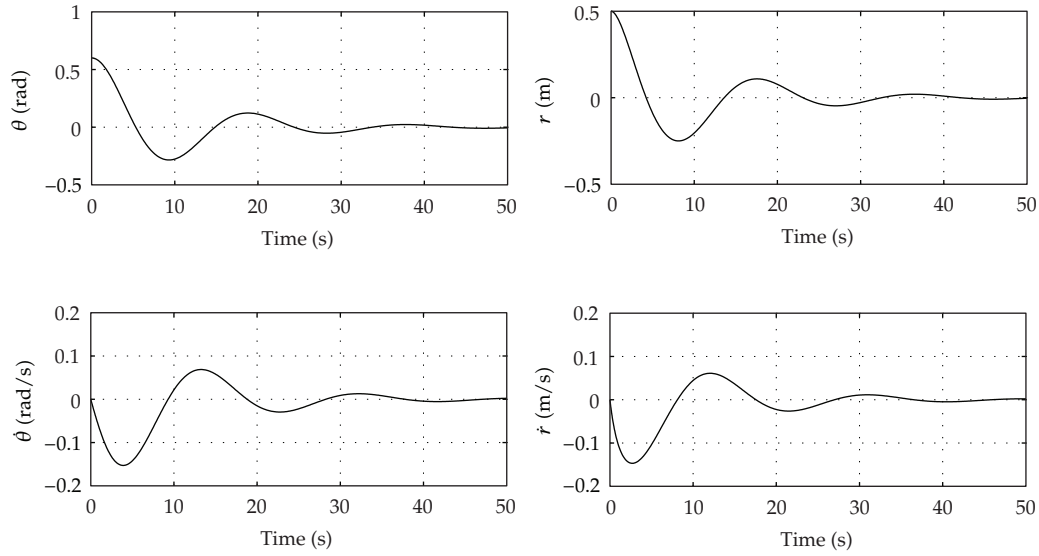
Figure 3: Closed-loop response of the **BBS** to the initial conditions: (a) $x_0 = (0.55 \, \text{m}; \, 0; \, 0.45 \, \text{rad}; \, 0)$.
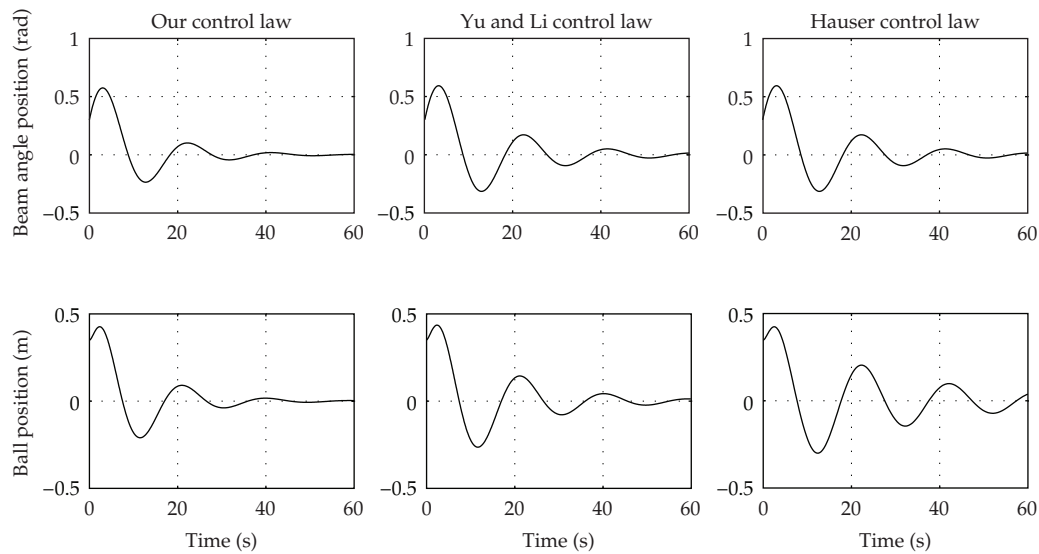

Figure 4: Closed-loop response of the **BBS** to the proposed **ANC** in comparison with **YNC** and **HNC**.

fixed as $(0.3 \, \text{m}; \, 0.18 \, \text{m/s}; \, 0.35 \, \text{rad}; \, 0.18 \, \text{rad/s})$. The simulation results are shown in Figure 4. As we can see, our control strategy outperforms the closed-loop responses of the **YNC** and **HNC** control strategies.

Comment 2. A comparative study between our control strategy and other control strategies presented in the literature for solving the stabilization of the **BBS**, is beyond the scope of this work.

5. Conclusions

In this work we proposed a novel procedure to stabilize the **BBS** by using the inverse Lyapunov approach in conjunction with the energy shaping technique. This procedure consists of finding the candidate Lyapunov function as if it were an inverse stability problem. To carry it out, we chose a convenient dissipation function of the unknown closed-loop energy function. Then, we proceeded to obtain the needed energy function, which is the addition of the positive potential energy and the positive kinetic energy. Afterwards, we directly derived the stabilizing controller from the already-obtained time derivative of the Lyapunov function. The corresponding asymptotic convergence analysis was done by applying the theorem of LaSalle. To assess the performance and effectiveness of the proposed control strategy, we carried out some numerical simulations. The simulation results allow us to conclude that our strategy behaves quite well in comparison with other well-known control strategies. It is worth mentioning that, to our knowledge, the procedure used here to obtain the needed Lyapunov function has not been used before to control the **BBS**.

Acknowledgments

Endnotes

1. Evidently,

$$S(x) = \begin{bmatrix} \delta r \dot{\theta}^2 - n \sin\theta - b\dot{r} & 0 \end{bmatrix}^T \qquad F = \begin{bmatrix} 0 & 1 \end{bmatrix}^T. \tag{5.1}$$

2. The set D is related with the region of attraction of the closed-loop system.

3. For simplicity, we use $K_c = K(r)$, $k_i = k_i(r)$, $k_i' = d/dr k_i(r)$, for $i = \{1, 2, 3\}$.

4. Symbol erf stands for the Gauss error function, defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-s^2\right) ds. \tag{5.2}$$

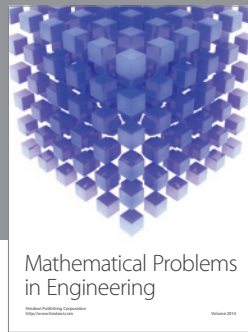5. After some simple algebraic manipulations it is easy to show that

$$\zeta_p = n \sin\theta + \frac{\partial V_p}{\partial\theta}; \qquad \zeta_d = -dr\dot{\theta}^2 + d\bar{k}_1 r\dot{r}(\dot{r} + \dot{\theta})e^{-dr^2} \tag{5.3}$$

6. For this particular case, the condition of **Remark 3.2** is satisfied, because

$$-b + 4(k_1 - 1)k_d = b + 2\exp\left(-d * L^2\right) = 13.7 > 0. \tag{5.4}$$

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