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Research Article

Existence and Uniqueness for Stochastic Age-Dependent Population with Fractional Brownian Motion

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A model for a class of age-dependent population dynamic system of fractional version with Hurst parameter $h \in (1/2, 1]$ is established. We prove the existence and uniqueness of a mild solution under some regularity and boundedness conditions on the coefficients. The proofs of our results combine techniques of fractional Brownian motion calculus. Ideas of the finite-dimensional approximation by the Galerkin method are used.

1. Introduction

Stochastic differential equations have been found in many applications in areas such as economics, biology, finance, ecology, and other sciences [1–3]. In recent years, existence, uniqueness, stability, invariant measures, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by many authors. For example, it is well known that these topics have been developed mainly by using two different methods, that is, the semigroup approach [4, 5] (e.g., Taniguchi et al. [4] using semigroup methods discussed existence, uniqueness, pth moment, and almost sure Lyapunov exponents of mild solutions to a class of stochastic partial functional differential equations with finite delays) and the variational one (e.g., Krylov and Rozovskii [6] and Pardoux [7]). On the other hand, although stochastic partial functional differential equations also seem very important as stochastic models of biological, chemical, physical, and economical systems, the corresponding properties of these systems have not been studied in great detail (cf. [8, 9]). As a matter of fact, there exists extensive literature on the related topics for deterministic age-dependent population dynamic system. There has been much recent interest in application of deterministic age-structures mathematical models with diffusion. For example, Cushing [10] investigated hierarchical age-dependent populations with intraspecific competition or predation.

There has been much recent interest in application of stochastic population dynamics. For example, Qimin and Chongzhao gave a numerical scheme and showed the convergence of the numerical approximation solution to the true solution to stochastic age-structured population system with diffusion [11]. In papers [12, 13], Qi-Min et al. discussed the existence and uniqueness for stochastic age-dependent population equation, when diffusion coefficient $k = 0$ and $k \neq 0$, respectively. Numerical analysis for stochastic age-dependent population equation has been studied by Zhang and Han [14]. In papers [11–14], the random disturbances are described by stochastic integrals with respect to Wiener processes.

However, the Wiener process is not suitable to replace a noise process if long-rang dependence is modeled. It is then desirable to replace the Wiener process by fractional Brownian motion. But this process is not a semimartingale, so that it is not possible to apply the Itô calculus. A stochastic analysis with respect to fractional Brownian motion is faced with difficulties.

Next, the stochastic continuous time age-dependent model is derived. In [12], the nonlinear age-dependent population dynamic with diffusion can be written in the following form:

$$
\begin{gathered}
\frac{\partial P(r,t,x)}{\partial t} + \frac{\partial P(r,t,x)}{\partial r} - k_1(r,t)\Delta P(r,t,x) \\
= -\mu_1(r,t,x)P(r,t,x) + f_1(r,t,x) + g_1(r,t,x)\frac{dw(t)}{dt}, \quad \text{in } Q_A = (0,A)\times Q, \\
P(0,t,x) = \int_0^A \beta_1(r,t,x)P(r,t,x)dr, \quad \text{in } (0,T)\times\Gamma, \\
P(r,0,x) = P_0(r,x), \quad \text{in } (0,A)\times\Gamma, \\
P(r,t,x) = 0, \quad \text{on } \Sigma_A = (0,A)\times(0,T)\times\partial\Gamma, \\
y(t,x) = \int_0^A P(r,t,x)dr, \quad \text{in } Q,
\end{gathered}
\tag{1.1}
$$

where $t \in (0,T)$, $r \in (0,A)$, $x \in \Gamma \subset R^N$ $(1 \le N \le 3)$, $Q = (0,T)\times\Gamma$, $P(r,t,x)$ denotes the population density of age r at time t in spatial position, x, $\beta_1(r,t,x)$ denotes the fertility rate of females of age r at time t, in spatial position x, $\mu_1(r,t,x)$ denotes the mortality rate of age r at time t, in spatial position x, Δ denotes the Laplace operator with respect to the space variable, and $k_1(r,t) > 0$ is the diffusion coefficient. $f_1(r,t,x) + g_1(r,t,x)(dw(t)/dt)$ denotes effects of external environment for population system, such as emigration and earthquake have. The effects of external environment the deterministic and random parts which depend on r, t, and x. $w(t)$ is a standard Wiener process.

In this paper, suppose that $f_1(r,t,x)$ is stochastically perturbed, with

$$
f_1(r,t,x) \longrightarrow f_1(r,t,x) + g_1(r,t,x)dB^h(t),
\tag{1.2}
$$

where $B^h(t)$ is fractional Brownian motions with the Hurst constant h. Then this environmentally perturbed system may be described by the Itô equation

$$\frac{\partial P(r,t,x)}{\partial t} + \frac{\partial P(r,t,x)}{\partial r} - k_1(r,t)\Delta P(r,t,x) \\ = -\mu_1(r,t,x)P(r,t,x) + f_1(r,t,x) + g_1(r,t,x)dB^h(t), \quad \text{in } Q_A = (0,A) \times Q, \tag{1.3}$$

$$P(0,t,x) = \int_0^A \beta_1(r,t,x)P(r,t,x)dr, \quad \text{in } (0,T) \times \Gamma, \tag{1.4}$$

$$P(r,0,x) = P_0(r,x), \quad \text{in } (0,A) \times \Gamma, \tag{1.5}$$

$$P(r,t,x) = 0, \quad \text{on } \Sigma_A = (0,A) \times (0,T) \times \partial\Gamma, \tag{1.6}$$

$$y(t,x) = \int_0^A P(r,t,x)dr, \quad \text{in } Q, \tag{1.7}$$

new stochastic differential equations (1.3)-(1.7) for an age-dependent population are derived. It is an extension of (1.1).

Our work differs from these references [11–14]. In papers [11–14], the random disturbances are described by stochastic integrals with respect to Wiener processes. In this paper, we study a stochastic age-dependent population dynamic system with an additive noise in the form of a stochastic integral with respect to a Hilbert space-valued fractional Borwnian motion. It is well known that a fractional Brownian motion B^h is a semimartingale if and only if $h = 1/2$, that is, in the case of a classical Brownian motion. For $h = 1/2$, Qimin and Chongzhao discussed the existence and uniqueness for stochastic age-dependent population equation [12]. In this paper, we shall discuss the existence and uniqueness for a stochastic age-dependent population equation with fractional Brownian motions with $h \in [1/2, 1]$. The discussion uses ideas of the finite-dimensional approximation by the Galerkin method.

In Section 2, we begin with some preliminary results which are essential for our analysis and introduce the definition of a solution with respect to stochastic age-dependent populations. In Section 3, we shall prove existence and uniqueness of solution for stochastic age-dependent population equation (1.3).

2. Preliminaries

Consider stochastic age-structured population system with diffusion (1.3). A is the maximal age of the population species, so

$$P(r,t,x) = 0, \quad \forall r \geq A. \tag{2.1}$$

By (1.7), integrating on $[0, A]$ to (1.3) and (1.5) with respect to r, we obtain the following system

$$\begin{gathered}\frac{\partial y}{\partial t} - k(t)\Delta y + \mu(t,x)y - \beta(t,x)y \\ = f(t,x) + g(t,x)\frac{dB^h(t)}{dt}, \quad \text{in } Q = (0,T)\times\Gamma, \\ y(0,x) = y_0(x), \quad \text{in } \Gamma, \\ y(t,x) = 0, \quad \text{on } \Sigma = (0,T)\times\partial\Gamma,\end{gathered} \tag{2.2}$$

where

$$\beta(t,x) \equiv \left(\int_0^A \beta_1(r,t,x)P(r,t,x)dr\right)\left(\int_0^A P(r,t,x)dr\right)^{-1}, \tag{2.3}$$

where $\int_0^A P(r,t,x)dr = y(t,x)$ is the total population, and the birth process is described by the nonlocal boundary conditions $\int_0^A \beta_1(r,t,x)P(r,t,x)dr$ clearly, $\beta(t,x)$ denotes the fertility rate of total population at time t and in spatial position x.

$$\mu(t,x) \equiv \left(\int_0^A \mu_1(r,t,x)P(r,t,x)dr\right)\left(\int_0^A P(r,t,x)dr\right)^{-1}, \tag{2.4}$$

where $\mu(t,x)$ denotes the mortality rate at time t and in spatial position x

$$\begin{gathered}f(t,x) \equiv \int_0^A f_1(r,t,x)dr, \\ g(t,x) \equiv \int_0^A g_1(r,t,x)dr.\end{gathered} \tag{2.5}$$

Let

$$V = H^1(\Gamma) \equiv \left\{\varphi \mid \varphi \in L^2(\Gamma), \frac{\partial\varphi}{\partial x_i} \in L^2(\Gamma), \text{ where } \frac{\partial\varphi}{\partial x_i} \text{ are generalized partial derivatives}\right\}. \tag{2.6}$$

Then $V' = H^{-1}(\Gamma)$ the dual space of V. We denote by $|\cdot|$ and $\|\cdot\|$ the norms in V and V' respectively, by $\langle\cdot,\cdot\rangle$ the duality product between V, V', and by $(\cdot,\cdot)$ the scalar product in H.

We consider stochastic age-structured population system with diffusion of the form

$$\begin{gathered}d_t y(t) - k\Delta y(t)dt + \mu(t,x)y(t)dt - \beta(t,x)y(t)dt \\ = f(t,x)dt + g(t,x)dB^h(t), \quad \text{in } Q = (0,T)\times\Gamma, \\ y(0,x) = y_0(x), \quad \text{in } \Gamma, \\ y(t,x) = 0, \quad \text{on } \Sigma = (0,T)\times\partial\Gamma,\end{gathered} \tag{2.7}$$

where $d_t y(t)$ is the differential of $y(t,x)$ relative to t, that is, $(d_t y(t) = \partial y(t)/\partial t)dt$, $y(t) := y(t,x)$. $T > 0$, $A > 0$.

The integral version of (2.7) is given by the equation

$$y(t) - y(0) - \int_0^t k\Delta y(s)ds - \int_0^t (\beta(s,x) - \mu(s,x))y(s)ds = \int_0^t f(s,x)ds + \int_0^t g(s,x)dB^h(s), \tag{2.8}$$

here $y(t,x) = 0$, on $\Sigma = (0,T) \times \partial\Gamma$.

Let $B_j^h(t)_{t\geq 0}$ $(j = 1,2,\ldots)$ be independent centered Gaussian processes with $B_j^h(0) = 0$ on a given probability space $(\Omega, \mathcal{F}, P)$, where we assume that

$$\begin{gathered} E\left(B_j^h(t) - B_j^h(s)\right)^2 = |t-s|^{2h}\mu_j \quad (j = 1,2,\ldots), \\ \mu_j > 0, \qquad \sum_{j=1}^{\infty}\mu_j < \infty, \end{gathered} \tag{2.9}$$

and $h \in [1/2, 1]$.

The processes $B_j^h(t)_{t\geq 0}$ are independent fractional Brownian motions with the Hurst constant h and $E(B_j^h(1))^2 = \mu_j$ $(j = 1,2,\ldots)$.

It follows from Kleptsyna et al. (cf. [15]) that

$$B_j^h(t) = \left(\int_{-\infty}^0 \left(|t-r|^{h-1/2} - |r|^{h-1/2}\right)dW_j(r) \ + \int_0^t |t-r|^{h-1/2}dW_j(r)\right), \tag{2.10}$$

where $(W_j(t))_{t\geq 0}$ $(j = 1, 2, \ldots)$ are real independent Wiener processes with $EW_j^2(t) = \mu_j t$.

Let K be a separable Hilbert space with the scalar product $(\cdot,\cdot)_K$, and $(e_j)_{j=1,2,\ldots}$ denotes a complete orthogonal system in K, Then

$$\sum_{j=1}^{\infty} E\|B_j^h(t)e_j\|_K^2 = t^{2h}\sum_{j=1}^{\infty}\mu_j < \infty, \tag{2.11}$$

and $B^h(t) = \sum_{j=1}^{\infty} B_j^h(t)e_j$ is called a K-valued fractional Brownian motion where the sum is defined mean square.

Definition 2.1. A H-valued continuous stochastic process $(y(t))_{t\in[0,T]}$ with $y(t) \in V$ (P-a.s) is a solution of (2.7) if it holds for $v \in V$ and all $t \in [0,T]$ that

$$\begin{aligned} (y(t), v)_H = (y(0), v)_H &+ \int_0^t \langle k\Delta y(s), v\rangle ds + \int_0^t (\beta(s,x)y(s) - \mu(s,x)y(s), v)_H ds \\ &+ \int_0^t (f(s,x), v)_H ds + \int_0^t \left(g(s,x)dB^h(s), v\right)_H, \qquad P\text{-a.s.} \end{aligned} \tag{2.12}$$

The objective in this paper is that we hopefully find a unique process $y(t)$ such that (2.7) holds For this objective, we assume that the following conditions are satisfied:

(1) $\mu(t,x)$, $\beta(t,x)$ and $k(r,t)$ are nonnegative measurable, and

$$\begin{aligned} 0 \le k_0 \le k(t) < \infty \quad & \text{in } (0,A)\times(0,T), \\ 0 \le \mu_0 \le \mu(t,x) < \infty \quad & \text{in } (0,A)\times\Gamma, \\ 0 \le \beta(t,x) \le \beta_0 < \infty \quad & \text{in } (0,A)\times\Gamma. \end{aligned} \tag{2.13}$$

(2) Let $f(t,x)$ and $g(t,x)$ be measurable functions which are defined on Q with

$$|f(t,x)| \bigvee |g(t,x)| \le K, \tag{2.14}$$

where K is a positive constant.

3. Existence and Uniqueness of Solutions

Consider also the K-valued fractional Brownian motion $B^{h,n}(t) = \sum_{i=1}^{n} B_i^h(t)e_i$. Obviously, the following lemma holds.

If the process $(y(t))_{t\in[0,T]}$ is a solution of (2.7), then the process $Z(t) = y(t) - \int_0^t g(s)dB^h(s)$ solves

$$\begin{aligned} & d_tZ(t) - k\Delta Z(t)dt + \mu(t,x)\left(Z(t) + \int_0^t g(s)B^h(s)\right)ds - \beta(t,x)\left(Z(t) + \int_0^t g(s)dB^h(s)\right)dt \\ & \quad = f(t,x)dt + k\Delta\int_0^t g(s)dB^h(s)dt, \quad \text{in } Q = (0,T)\times\Gamma, \\ & Z(0,x) = Z_0(x), \quad \text{in } \Gamma, \\ & Z(t,x) = 0, \quad \text{on } \Sigma = (0,T)\times\partial\Gamma, \end{aligned} \tag{3.1}$$

where $Z(t) := Z(t,x)$. If $Z(t)$ is a solution of (3.1), then exists a process $y(t)_{t\in[0,T]}$ so that $Z(t)$ can be written as $Z(t) = y(t) - \int_0^t g(s,x)dB^h(s)$, and consequently $y(t)$ solves (2.7).

As a result, we shall consider (3.1) instead of (2.7). It is noted that, for fixed $\omega \in \Omega$, (3.1) is a deterministic problem.

Lemma 3.1. *Problem* (3.1) *has, for fixed $\omega \in \Omega$, a unique solution $Z(t)$, and there exists a nonnegative random variable η with finite expectation such that*

$$\sup_{0\le s\le T}|Z(s)|^2 + k_0\int_0^T \|Z(s)\|^2 ds \le \eta, \tag{3.2}$$

where for fixed $\omega \in \Omega$, $Z(t)$ is continuous with respect to t in H.

Proof. The Galerkin approximations are defined by $Z_n(t) = \sum_{i=1}^n Z_{n,i}(t)v_i$, where $Z_{n,i}(t)$ solves the stochastic equations

$$\begin{aligned} Z_{n,i}(t) = {} & \big(y(0), v_i\big)_H + \int_0^t \left\langle k\Delta\left(\sum_{k=1}^n Z_{n,k}(s)v_k\right), v_i\right\rangle ds \\ & + \int_0^t \left\langle \big(\beta(s,x) - \mu(s,x)\big)\sum_{k=1}^n Z_{n,k}(s)v_k, v_i\right\rangle ds \\ & + \int_0^t \big(f(s,x), v_i\big)_H ds + \int_0^t \left\langle \big(\beta(s,x) - \mu(s,x)\big)\int_0^s g(u,x)dB^{h,n}(u), v_i\right\rangle ds \\ & + \int_0^t \left\langle k\Delta\left(\int_0^s g(u,x)dB^{h,n}(u)\right), v_i\right\rangle ds. \quad (i = 1, 2, \ldots, n). \end{aligned} \tag{3.3}$$

It follows from the assumption (2) that (3.3) can be solved for every ω by the method of successive approximation, and the iterates are measurable with respect to ω. Consequently, $(Z_{n,i}(t))_{t\in[0,T]}$ $(i = 1, 2, \ldots, n)$ are stochastic processes since y_0 is a random H-valued variable and $(B^{h,n}(t))_{t\in[0,T]}$ is a stochastic process. It follows from (3.3) that

$$\begin{aligned} Z_n(t) = {} & \sum_{i=1}^n \big(y(0), v_i\big)_H v_i + \int_0^t \sum_{i=1}^n \langle k\Delta Z_n(s), v_i\rangle v_i ds \\ & + \int_0^t \sum_{i=1}^n \left(\big(\beta(s,x) - \mu(s,x)\big)\sum_{i=1}^n Z_n(s), v_i\right) v_i ds \\ & + \int_0^t \big(f(s,x), v_i\big)_H v_i ds + \int_0^t \left\langle \big(\beta(s,x) - \mu(s,x)\big)\int_0^s g(u,x)dB^{h,n}(u), v_i\right\rangle v_i ds \\ & + \int_0^t \left\langle k\Delta\left(\int_0^s g(u,x)dB^{h,n}(u)\right), v_i\right\rangle v_i ds. \end{aligned} \tag{3.4}$$

Using the chain rule, we get the following

$$\begin{aligned} |Z_n(t)|^2 = {} & \sum_{j=1}^n \big(y(0), v_j\big)_H^2 + 2\int_0^t k\langle \Delta Z_n(s), Z_n(s)\rangle ds \\ & + 2\int_0^t \langle (\beta(s,x) - \mu(s,x))Z_n(s), Z_n(s)\rangle ds + 2\int_0^t (f(s,x), Z_n(s))ds \\ & + 2\int_0^t \left(\big(\beta(s,x) - \mu(s,x)\big)\int_0^s g(u,x)dB^{h,n}(u)ds, Z_n(s)\right) ds \\ & + 2\int_0^t k\left\langle \Delta\left(\int_0^s g(u,x)dB^{h,n}(u)\right), Z_n(s)\right\rangle ds. \end{aligned} \tag{3.5}$$

If we set $B(t) \equiv 0$ in of Qimin and Chongzhao [11], under assumptions (1)-(2), then this result implies that

$$\sup_{0\le s\le T} |Z_n(s)|^2 + k_0\int_0^t \|Z_n(s)\|^2 ds \le \eta. \tag{3.6}$$

The following result is an analogous to that of Theorem 4 in [1]. In the Galerkin approximation, we have

$$E|Z_n(t) - Z(t)|^2 + E\int_0^t \|Z_n(s) - Z(s)\|^2 ds \longrightarrow 0 \tag{3.7}$$

for all $t \in [0,T]$ and for $n \to \infty$. $Z(t)$ is a H-valued continuous process with $Z(t) \in V$ for all $t \in [0,T]P$-a.s., and $Z(t)$ is a P-a.s. unique solution.

Now let $(B^h(t))_{t\in[0,T]}$ be a H-valued fractional Brownian motion with $\sum_{j=1}^{\infty} \lambda_j \mu_j < \infty$ and $\sum_{j=1}^{\infty} \lambda_j \mu_j^{1/2} < \infty$. We consider the finite-dimensional approximation

$$\sum_{j=1}^{n} \int_0^t g(s,x) dB_j^h(s) v_j ds \tag{3.8}$$

in mean square of the stochastic integral $\int_0^t g(u,x)dB^h(u)$. Obviously this is a stochastic integral with respect to the V-valued Brownain motion $B^{h,n}(u) = \sum_{j=1}^{n} B_j^h(u) v_j$. Consequently, the corresponding Galerkin equations for (2.7) are given by

$$\begin{aligned}
&d_t y^m(t) - k\Delta y^m(t)dt + \mu(t,x)y^m(t)dt - \beta(t,x)y^m(t)dt \\
&\qquad = f(t,x)dt + g(t,x)dB^{h,m}(t), \quad \text{in } (0,T)\times\Gamma, \\
&y^m(0) = \sum_{j=1}^{m} (y_0, v_j) v_j, \quad \text{in } \Gamma, \\
&y^m(t,x) = 0, \quad \text{on } \Sigma = (0,T)\times\partial\Gamma,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&d_t y^n(t) - k\Delta y^n dt + \mu(t,x)y^n(t)dt - \beta(t,x)y^n(t)dt \\
&\qquad = f(t,x)dt + g(t,x)dB^{h,n}(t), \quad \text{in } (0,T)\times\Gamma, \\
&y^n(0) = \sum_{j=1}^{n} (y_0, v_j) v_j, \quad \text{in } \Gamma, \\
&y^n(t,x) = 0, \quad \text{on } \Sigma = (0,T)\times\partial\Gamma.
\end{aligned} \tag{3.10}$$

Lemma 3.1 shows that these problems have solutions. □

Theorem 3.2. *If $\sum_{j=1}^{\infty} \lambda_j \mu_j < \infty$ and $\sum_{j=1}^{\infty} \lambda_j \mu_j^{1/2} < \infty$, then there exists a P-a.s unique solution $(y(t))_{t\in[0,T]}$ of* (2.7) *with*

$$E\left|y(t)\right|^2 + k_0 E\int_0^t \|Z(s)\|^2 ds \le M_{t,h}, \quad \forall t \in [0,T], \tag{3.11}$$

where $M_{t,h}$ is a positive constant.

Proof. We choose $n > m$ with $n = m + p$ and define

$$Z^{m,p}(t) = y^{m+p}(t) - y^{m}(t) - \int_0^t g(u,x)dB^{h,m}(u) + \int_0^t g(u,x)dB^{h,m+p}(u). \tag{3.12}$$

Then

$$\begin{aligned}|Z^{m,p}(t)|^2 \leq{}& \left|y^{m+p}(0) - y^{m}(0)\right|^2 + 2\int_0^t k\langle \Delta Z^{m,p}(s), Z^{m,p}(s)\rangle ds \\ &+ 2\int_0^t \left|\left(\left(\beta(s,x) - \mu(s,x)\right)\left(y^{m+p}(s) - y^{m}(s)\right), Z^{m,p}(s)\right)\right| ds \\ &+ 2\int_0^t \left|\left(f(s,x), Z^{m,p}\right)(s)\right| ds \\ &+ 2\int_0^t \left|\left(k\Delta \int_0^s g(u,x)d\left(B^{h,m+p}(u) - B^{h,m}(u)\right), Z^{m,p}(s)\right)\right| ds.\end{aligned} \tag{3.13}$$

However, by Lemma 2.2 [14] and assumptions (1)-(2), we have

$$\begin{aligned}&\int_0^t \left|\left(k\Delta \int_0^s g(u,x)d\left(B^{h,m+p}(u) - B^{h,n}(u)\right), Z^{m,p}\right)\right| ds \\ &\quad\leq k_0 \sum_{j=m+1}^{n} \lambda_j E \int_0^t \left|\int_0^s g(u,x)dB_j^h(u)\left(v_j, Z^{m,p}(s)\right)\right| ds \\ &\quad\leq \frac{1}{2}k_0 \sum_{j=m+1}^{n} \lambda_j \int_0^t E\left\|\int_0^s g(u,x)v_j dB_j^h(u)\right\|^2 ds + \frac{1}{2}\int_0^t E|Z^{m,p}(s)|^2 ds \\ &\quad\leq \frac{1}{2}k_0 T^{2h} K^2 T \sum_{j=m+1}^{n} \lambda_j \mu_j + \frac{1}{2}\int_0^t E|Z^{m,p}(s)|^2 ds.\end{aligned} \tag{3.14}$$

Further,

$$\begin{aligned}&E\left(\left(\beta(s,x) - \mu(s,x)\right)\left(y^{m+p}(s) - y^{m}(s)\right), Z^{m,p}(s)\right)ds \\ &\quad\leq E\left(\left|\beta_0 - \mu_0\right||Z^{m,p}(s)|^2 + \left|\beta_0 - \mu_0\right|\left|\int_0^s g(u,x)d\left(B^{h,m+p}(u) - B^{h,n}(u)\right)\right||Z^{m,p}(s)|\right) \\ &\quad\leq 2\left|\beta_0 - \mu_0\right|E|Z^{m,p}(s)|^2 + \left|\beta_0 - \mu_0\right|K^2 T \sum_{j=m+1}^{m+p} \mu_j.\end{aligned} \tag{3.15}$$

Consequently, in view of (3.13),

$$\begin{aligned}E|Z^{n}(s)|^2 + 2k_0 E\int_0^t \|Z^{n}(s)\|^2 ds \leq{}& \left(2\left|\beta_0 - \mu_0\right| + K^2 + 1\right)\int_0^t E|Z^{n}(s)|^2 ds \\ &+ k_0 C_h K^2 T \sum_{j=m+1}^{m+p} \lambda_j \mu_j + 2\left|\beta_0 - \mu_0\right| T K^2 \sum_{j=m+1}^{m+p} \mu_j.\end{aligned} \tag{3.16}$$

Then, the Gronwall's lemma implies that

$$E|Z^{m,p}(s)|^2 \longrightarrow 0, \qquad E\int_0^t \|Z^{m,p}(s)\|^2 \longrightarrow 0 \tag{3.17}$$

for $m,p \to \infty$ for all $t \in [0,T]$. In particular, there exists a process $(Z(t))_{t\in[0,T]}$ with $E|Z^m(t) - Z(t)|^2 \to 0$ for $m \to \infty$, and consequently, there exists a process $y(t)$ with $E|y^m(t)-y(t)|^2 \to 0$ for $m \to \infty$. We must now show that $(y(t))_{t\in[0,T]}$ is solution of (2.7). We have

$$\begin{aligned}
&E\left|y^n(t) - y^m(t) + \int_0^s g(u,a)d\left(\overline{B}^{h,m+p}(u) - \overline{B}^{h,n}(u)\right)\right|^2 + 2k_0E\int_0^t \left\|y^n(s) - y^m(s)\right\|^2 ds \\
&\quad\leq 2E\int_0^t k\left\langle \Delta\left(y^n(s) - y^m(s)\right), \int_0^s g(u,x)d\left(B^{h,n}(u) - B^{h,m}(u)\right)\right\rangle ds \\
&\qquad + 2E\int_0^t\left(\left(\beta(s,x) - \mu(s,x)\right)y^n - y^m(s), y^n(s) - y^m(s) + \int_0^s g(u,x)d\left(B^{h,n}(u) - B^{h,m}(u)\right)ds\right. \\
&\qquad + 2E\int_0^t\left(f(s,x), y^n(s) - y^m(s) + \int_0^s g(u,x)d\left(B^{h,m+p}(u) - B^{h,n}(u)\right)\right)ds.
\end{aligned} \tag{3.18}$$

Let $\varepsilon > 0$ be chosen arbitrary. Then there exists $p_0 > 0$ so that $\sum_{j=p+1}^{m+p} \lambda_j\mu_j^{1/2} < \varepsilon$ for all $p > p_0$. Let $y^{n,r}(t) = \sum_{j=1}^r y_j^{n,r}(t)v_j$ and $y^{m,r}(t) = \sum_{j=1}^r y_j^{m,r}(t)v_j$ be the rth Galerkin approximation of $y^n(t)$ and $y^m(t)$, respectively. For $r = m+p$, we have

$$\begin{aligned}
&\left|E\int_0^t k\left\langle \Delta\left(y^{n,r}(s) - y^{m,r}(s)\right), \int_0^s g(u,x)d\left(B^{h,n}(u) - B^{h,m}(u)\right)\right\rangle ds\right| \\
&\quad\leq k_0\left|E\sum_{i=p+1}^{m+p}\int_0^t \lambda_i\left(y_i^{n,r}(s) - y_i^{m,r}(s)\right)\int_0^s g(u,x)dB_i^h(u)ds\right| \\
&\quad\leq k_0E\sum_{i=p+1}^{m+p}\left(\int_0^t \left|y_i^{n,r}(s) - y_i^{m,r}(s)\right|^2 ds\right)^{1/2}\lambda_i\left(E\int_0^t\left|\int_0^s g(u,x)dB_i^h(u)\right|^2 ds\right)^{1/2} \\
&\quad\leq \text{const. } k_0E\sum_{i=p+1}^{m+p}\lambda_i\mu_i^{1/2} \\
&\quad< \text{const. } \times\varepsilon.
\end{aligned} \tag{3.19}$$

Consequently, the first term on the right-hand side of (3.10) is also less than const. $\times\,\varepsilon$. It is clear that the second term and third term on the right-hang side of (3.18) tends to zero. Then (3.18) gives

$$E\int_0^t \|y^{m+p}(s) - y^p(s)\|^2 \longrightarrow 0 \tag{3.20}$$

for $m, p \to \infty$, there is $(y^m(t))$ is also a Cauchy sequence in $L^2_V(\Omega \times [0,T])$ for all $t \in [0,T]$. Let $\overline{y}$ be the limit a of this sequence. Then it follows from the properties of a Gelfand triple that

$$E\int_0^t \left|y_n(s) - \overline{y}(s)\right|^2 \leq ME\int_0^t \left\|y_n(s) - \overline{y}(s)\right\|^2 \longrightarrow 0 \tag{3.21}$$

for $n \to \infty$, where M is a positive constant. Consequently, $\overline{y}(s) = y(s)$(a.s) and it follows from (3.9) that

$$d_t y(t) - k\Delta y(t)ds - \left(\beta(s,x) - \mu(s,x)\right)y(t)ds = f(s,x)ds + g(s,x)dB^h(t), \tag{3.22}$$

hence, we have proved Theorem 3.2. □

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