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Research Article

The Inverse Problem for a General Class of Multidimensional Hyperbolic Equations

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Inverse problems for hyperbolic equations are found in geophysical prospecting and seismology, and their multidimensional analogues are especially important for applied work. However, whereas results have been established for the some narrow classes of hyperbolic equations, no results exist for more general classes. This paper proves the solvability of the inverse problem for a general class of multidimensional hyperbolic equations. Our approach consists of properly choosing the shape of the overidentifying condition that is needed to determine the right-hand side of the hyperbolic PDE and then applying the Fourier series method. We are then able to establish the results of the existence of solution for the cases when the unknown right-hand side is time-independent or space independent.

1. Introduction

One key applied informational problem with heavy involvement of advanced mathematical modelling is the inverse problem. Generally speaking, it can be described as follows: a physical, biological, or an economic model relates some parameters, via the knowledge of certain fundamental laws, to observational data. An inverse problem then consists of recovering the (unknown) parameters of the model from known observational data. Such problems often emerge and are intensively studied in geophysics and seismology (in physical sciences, [1–3]), tomography (in biological sciences, [1, 4–6]) and economic geography, econometrics, and macroeconomics (in economic sciences, [7–10]).

The mathematical problems that emerge in such studies often reduce to the analysis of linear second-order partial differential equations. For instance, in recent analysis of spatial economic development, economists construct models with continuous time and space dimensions. Capital accumulation in different points of space is modelled via an equation of motion that becomes a parabolic partial differential equation [11, 12]. An important (inverse) problem is then recovering the unknown parameters of the aggregate production function of the economy from observations about the level of capital stock at some moment in time [7].

Instead, inverse problems for hyperbolic equations emerge in the study of the problems of geophysics and seismology [1–3]. For instance, two major types of problems in geophysical prospecting are (i) the structural problems that are concerned with the structure of sedimentary deposits and the Earth's crust (the structures that have to do with oil and gas resources) and (ii) the location problems that are concerned with establishing local non-homogeneities in sedimentary deposits and crust (caused by mineral deposits, e.g., ore). Both problems basically consists of recovering the Earth's inner structure from the surface measurements of physical fields.

In seismology, one of the most important problems is determining the velocity of propagation of seismic waves within the Earth, given the information about the seismic wave fronts on the surface (coming from different earthquakes). The velocities of longitudinal and transverse waves and the elastic properties of the medium are linked through some known formulae. The determination of the propagation velocities given these formulae reduces mathematically to an inverse problem known as the Herglotz-Wiechert inverse kinematic problem of seismics.

In most practical applications, the parameters of the physical model are a multidimensional function (e.g. three-dimensional space plus time). Therefore, considerable attention has been devoted to the analysis of multidimensional inverse problems. Chapter 8 of [3] provides a good survey of the current state of the literature for hyperbolic problems, together with a fine discussion of the mathematical difficulties involved.

So far, however, most of the results have been obtained for relatively narrow classes of equations: the so-called model equations, that is, including only the wave operator, or only some of the lower-order derivatives. After the classic study of the inverse problem for the wave equation [1], some results have been established (and applied techniques developed) for the hyperbolic equations of the following classes: $u_{tt} = \Delta u - p(x)u$ in [13], $u_{tt} = \Delta u + p(x,t)u$ in [14], and $c(x)u_{tt} = \Delta u$ in [15]. To the best of our knowledge, no results have been yet found for more general classes of hyperbolic equations. This paper tries to close this gap by establishing the existence result for a general class of multidimensional hyperbolic equations.

Our approach consists of properly choosing the shape of the overidentifying condition that is needed to determine the right-hand side of the hyperbolic PDE, and then applying the Fourier series method. We are then able to establish the results of the existence of solution for the cases when the unknown right-hand side is time independent or space-independent.

2. Setup of the Problem

Let D_ε be a finite domain of the Euclidean space of E_{m+1} points $(x_1, \ldots, x_m, t)$ bounded by the surfaces $|x| = t + \varepsilon$, $|x| = 1 - t$ and the plane $t = 0$, where $|x|$ denotes the length of the vector $x = (x_1, \ldots, x_m)$, $0 \leq t \leq (1 - \varepsilon)/2$, and $0 < \varepsilon < 1$. Moreover, let us denote the parts of these surfaces that form the boundary ∂D_ε of the domain D_ε as S_ε, S_1, and S, respectively.

In this domain, a number of applied problems described in the Introduction can be described mathematically as some special cases of the following general inverse problem.

Problem 2.1. Find the functions $u(x,t) \in C(\overline{D}_\varepsilon) \cap C^2(D_\varepsilon)$ and $g(x,t)$ that are linked in the domain D_ε by the following equation:

$$Lu \equiv \Delta_x u - u_{tt} + \sum_{i=1}^{m} a_i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u = g(x,t), \tag{2.1}$$

where the functions $u(x,t)$ should satifsy the conditions

$$u|_S = \tau(x), \qquad u_t|_S = v(x), \tag{2.2}$$

$$u|_{S_\beta} = \sigma(x), \tag{2.3}$$

Δ_x being the Laplace operator defined over the variables $x_1, \ldots, x_m$, $m \geq 2$ and S_β being the cone $\beta|x| = t + \varepsilon$, $0 < \beta = \text{const} < 1$.

Note that in this inverse problem the conditions (2.2) are the conditions of the standard Cauchy problem, while the condition (2.3) is the overidentification condition, which is needed to determine the unknown function $g(x,t)$. The appropriate choice of this condition would allow us to establish the results below.

Before deriving our main results, however, it is useful to switch from the Cartesian coordinates $x_1, \ldots, x_m, t$ to the spherical ones $r, \theta_1, \ldots, \theta_{m-1}, t$, with $0 \leq \theta_1 < 2\pi$, $r \geq 0$, $0 \leq \theta_i \leq \pi$, $i = 2, 3, \ldots, m-1$.

We need further some additional notation. Let $\{Y_{n,m}^k(\theta)\}$ be a system of linearly independent spherical functions of degree n, $1 \leq k \leq k_n$, $(m-2)!n!k_n = (n+m-3)!(2n+m-2)$, $\theta = (\theta_1, \ldots, \theta_{m-1})$, $W_2^l(D_\varepsilon)$, $l = 0, 1, \ldots$—are Sobolev spaces, and $\widetilde{S}_\beta = \{(r,\theta) \in S, \varepsilon < r < (1+\varepsilon)/(1+\beta)\}$.

Also, let us denote as $\tilde{a}_{in}^k(r,t)$, $a_n^k(r,t)$, $\tilde{b}_n^k(r,t)$, $\tilde{c}_n^k(r,t)$, $\overline{g}_n^k(r)$, ρ_n^k, $\overline{\tau}_n^k(r)$, $\overline{v}_n^k(r)$, and $\overline{\sigma}_n^k(r)$ the coefficients of the decomposition of the series (2.4), respectively, of the functions $a_i(r,\theta,t)\rho(\theta)$, $a_i(x_i/r)\rho$, $b(r,\theta,t)\rho$, $c(r,\theta,t)\rho$, $g(r,\theta)$, $\rho(\theta)$, $i = 1, \ldots, m$, $\tau(r,\theta)$, $v(r,\theta)$, and $\sigma(r,\theta)$, and moreover, $\rho(\theta) \in C^\infty(\Gamma)$, Γ being the unit sphere in E_m.

For our analysis, we will exploit the following lemma that has been shown in [16].

Lemma 2.2. *Let $f(r,\theta) \in W_2^l(S)$. If $l \geq m-1$, then the series*

$$f(r,\theta) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} f_n^k(r) Y_{n,m}^k(\theta), \tag{2.4}$$

as well as the series obtained through its differentiation of order $p \leq l - m + 1$, converges absolutely and uniformly.

Next, one introduces the set of functions

$$B^l(S) = \left\{ f(r,\theta) : f \in W_2^l(S), \sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\left(\left\| f_n^k(r) \right\|^2_{C^2(\varepsilon,1)} + \left\| f_n^k(r) \right\|_{C([\varepsilon,1])} \right) \right.$$
$$\left. \times \exp 2\left(n^2 + n(m-2)\right) < \infty,\ l \geq m-1 \right\}. \tag{2.5}$$

Finally, let $a_i(x,t),\ \ b(x,t),\ \ c(x,t) \in W_2^p(D_\varepsilon) \subset C(\overline{D}_\varepsilon),\ \ i = 1,\dots,m,\ \ p \geq m+1$, and $\tau(r,\theta),\ \upsilon(r,\theta) \in B^l(S)$, $\sigma(r,\theta) \in B^I(\widetilde{S}_\beta)$.

3. Main Results

We are now ready to establish our main results. These are theorems of the existence of solutions of Problem 1, when the unknown right-hand side of (2.1) is time independent or space independent.

Theorem 3.1. *If $g(r,\theta,t) = g(r,\theta) \in W_2^l(D_\varepsilon) \cap C^1(\overline{D}_\varepsilon)$, then the functions $u(r,\theta,t)$ and $g(r,\theta)$ always exist.*

Theorem 3.2. *If $g(r,\theta,t) = g(t) \in C^1([0,(1-\varepsilon)/2])$, then the functions $u(r,\theta,t)$ and $g(t)$ always exist.*

Proof of Theorem 1. The uniqueness of the solution of the (direct) Cauchy problem (2.1), (2.2) is well known (see, for instance, [17]). We will search for the solution of this problem in the form of the series

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \overline{u}_n^k(r,t) Y_{n,m}^k(\theta), \tag{3.1}$$

where $\overline{u}_n^k(r,t)$ are the functions to be determined later.

Substituting (3.1) into (2.1), and first multiplying the resulting expression by $\rho(\theta) \neq 0$ and then integrating over the unit sphere Γ, we get for $\overline{u}_n^k$ (see [18, 19])

$$\rho_0^1 \overline{u}_{0rr}^1 - \rho_0^1 \overline{u}_{0tt}^1 + \left(\frac{m-1}{r}\rho_0^1 + \sum_{i=1}^{m} a_{i0}^1 \right) \overline{u}_{0r}^1 + \widetilde{b}_0^1 \overline{u}_{0t}^1 + \widetilde{c}_0^1 \overline{u}_0^1 - \widetilde{c}_0^1 \overline{g}_0^1(r)$$
$$+ \sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\left\{ \rho_n^k \overline{u}_{nrr}^k - \rho_n^k \overline{u}_{ntt}^k + \left(\frac{m-1}{r}\rho_n^k + \sum_{i=1}^{m} a_{in}^k \right) \overline{u}_{nr}^k + \widetilde{b}_n^k \overline{u}_{nt}^k \right.$$
$$\left. + \left[\widetilde{c}_n^k - \frac{\lambda_n}{r^2}\rho_n^k + \sum_{i=1}^{m}\left(\widetilde{a}_{in-1}^k - n a_{in}^k \right) \right] \overline{u}_n^k - \rho_n^k \overline{g}_n^k(r) \right\} = 0, \tag{3.2}$$
$$\lambda_n = n(n+m-2).$$

Next, let us analyze the infinite system of differential equations

$$\rho_0^1\overline{u}_{0rr}^1 - \rho_0^1\overline{u}_{0tt}^1 + \frac{m-1}{r}\rho_0^1\overline{u}_{0r}^1 = \rho_0^1 g_0^1(r), \tag{3.3}$$

$$\begin{aligned}\rho_1^k u_{1rr}^k - \rho_1^k\overline{u}_{1tt}^k + \frac{m-1}{r}\rho_1^k u_{1r}^k - \frac{\lambda_1}{r^2}\rho_1^k\overline{u}_1^k &= \rho_1^k g_1^k(r)\\ &- \frac{1}{k_1}\left(\sum_{i=1}^{m} a_{i0}^1\overline{u}_{0r}^1 + \tilde{b}_0^1\overline{u}_{0t}^1 + \tilde{c}_0^1\overline{u}_0^1\right), \quad n=1,\ k=\overline{1,k_1},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\rho_n^k\overline{u}_{nrr}^k - \rho_n^k\overline{u}_{ntt}^k + \frac{m-1}{r}\rho_n^k\overline{u}_{nr}^k - \frac{\lambda_n}{r^2}\rho_n^k\overline{u}_n^k &= \rho_n^k g_n^k(r)\\ - \frac{1}{k_n}\sum_{k=1}^{k_{n-1}}\Bigg\{\sum_{i=1}^{m} a_{in-1}^k\overline{u}_{n-1r}^k &+ \tilde{b}_{n-1}^k\overline{u}_{n-1t}^k\\ &+\left[\tilde{c}_{n-1}^k + \sum_{i=1}^{m}\left(\tilde{a}_{in-2}^k - (n-1)a_{in-1}^k\right)\right]\overline{u}_{n-1}^k\Bigg\}, \quad k=\overline{1,k_n},\ n=2,3,\ldots.\end{aligned} \tag{3.5}$$

Note that if we sum the expression (3.4) from 1 to k_1, the expression (3.5) from 1 to k_n, and then add the obtained expressions to (3.3), we obtain (3.2). Therefore, if $\{\overline{u}_n^k\}$, $k = \overline{1,k_n}$, $n = 0,1,\ldots$ is the solution of the system (3.3)–(3.5), then it is also the solution of (3.2). The opposite is not true, however; therefore, we cannot show the uniqueness of the solution of Problem 1.

Now, taking into account the orthogonality of the spherical functions $Y_{n,m}^k(\theta)$ ([16]) and given expression (3.1), we obtain, from the boundary-value conditions (2.2), (2.3):

$$\begin{aligned}\overline{u}_n^k(r,0) = \overline{\tau}_n^k(r), \quad \overline{u}_{nt}^k(r,0) = \overline{\upsilon}_n^k(r), \quad \varepsilon \le r \le 1,\\ \overline{u}_n^k(r,\beta r+\varepsilon) = \overline{\sigma}_n^k(r), \quad \varepsilon \le r \le \frac{1-\varepsilon}{1+\beta}, \quad k=\overline{1,k_n},\ n=0,1,\ldots.\end{aligned} \tag{3.6}$$

Thus, we have shown that our inverse problem (2.1)–(2.3) reduces to a system of inverse problems for the equations (3.3)–(3.5). We can now look for the solution of these problems.

It is easy to see that each equation of the system (3.3)–(3.5) can be represented in the form

$$\overline{u}_{nrr}^k - \overline{u}_{ntt}^k + \frac{m-1}{r}\overline{u}_{nr}^k - \frac{\lambda_n}{r^2}\overline{u}_n^k = \overline{g}_n^k(r) + \overline{f}_n^k(r,t), \tag{3.7}$$

where $\overline{f}_n^k(r,t)$ are determined from the previous equations of this system and, moreover, $\overline{f}_0^1(r,t) \equiv 0$. From (3.7), having done the substitution of variables $\overline{u}_n^k(r,t) = r^{(1-m)/2}u_n^k(r,t)$, and then setting $\xi = (r+t)/2$, $\eta = (r-t)/2$, we obtain

$$Mu \equiv u_{n\xi\eta}^k + \frac{[(m-1)(3-m)-4\lambda_n]}{4(\xi+\eta)^2}u_n^k = g_n^k(\xi+\eta) + f_n^k(\xi,\eta),$$
$$g_n^k(\xi+\eta) = (\xi+\eta)^{(m-1)/2}\overline{g}_n^k(\xi+\eta), \qquad f_n^k(\xi,\eta) = (\xi+\eta)^{(m-1)/2}\overline{f}_n^k(\xi+\eta,\ \xi-\eta), \tag{3.8}$$

and, given this result, the boundary-value conditions (3.6) will take the form

$$u_n^k(\xi,\xi) = \tau_n^k(\xi), \quad \left(\frac{\partial u_n^k}{\partial \xi} - \frac{\partial u_n^k}{\partial \eta}\right)\bigg|_{\xi=\eta} = \upsilon_n^k(\xi), \quad \frac{\varepsilon}{2} \le \eta < \xi \le \frac{1}{2},$$
$$u_n^k(\xi,\alpha\xi+\gamma) = \sigma_n^k(\xi), \quad \frac{\varepsilon}{2\beta} \le \xi \le \frac{1}{2},$$
$$\tau_n^k(\xi) = (2\xi)^{(m-1)/2}\overline{\tau}_n^k(2\xi),$$
$$\upsilon_n^k(\xi) = \sqrt{2}(2\xi)^{(m-1)/2}\overline{\upsilon}_n^k(2\xi), \tag{3.9}$$
$$\sigma_n^k(\xi) = \left[(1+\alpha)\xi+\gamma\right]^{(m-1)/2}\overline{\sigma}_n^k \times ((1+\alpha)\xi+\gamma),$$
$$0 < \alpha = \frac{1-\beta}{1+\beta} < 1,\ \gamma = \frac{\varepsilon}{1+\beta},\ k = \overline{1,k_n},\ n = 0,1,\ldots.$$

Using the general solution of the equation (3.8) (see [20]), it is easy to show that the solution of the Cauchy problem for the equation (3.8) takes the form

$$u_n^k(\xi,\eta) = \frac{1}{2}\tau_n^k(\eta)R(\eta,\eta;\xi,\eta) + \frac{1}{2}\tau_n^k(\xi)R(\xi,\xi;\xi,\eta)$$
$$+ \frac{1}{\sqrt{2}}\int_\eta^\xi \left[\upsilon_n^k(\xi_1)R(\xi_1,\xi_1;\xi,\eta) - \tau_n^k(\xi_1)\frac{\partial}{\partial N}R(\xi_1,\eta_1;\xi,\eta)\bigg|_{\xi_1=\eta_1} d\xi_1\right] \tag{3.10}$$
$$+ \int_{1/2}^\xi \int_{\varepsilon/2}^\eta \left[g_n^k(\xi_1+\eta_1) + f_n^k(\xi_1,\eta_1)\right]R(\xi_1,\eta_1;\xi,\eta)\,d\xi_1 d\eta_1,$$

where $R(\xi_1,\eta_1;\xi,\eta) = P_\mu[((\xi_1-\eta_1)(\xi-\eta) + (\xi_1\eta_1+\xi\eta))/(\xi_1+\eta_1)(\xi+\eta)] = P_\mu(z)$ is the Riemann function of the equation $Mu = 0$ (see [21]) and $P_\mu(z)$ is the Legendre function, $\mu = n + (m-3)/2$, $(\partial/\partial N)|_{\xi=\eta} = (1/\sqrt{2})((\partial/\partial\xi) - (\partial/\partial\eta))|_{\xi=\eta}$.

From (3.10), for $\eta = \alpha\xi+\gamma$, using the boundary-value conditions (3.9), we obtain

$$\varphi_n^k(\xi) = \int_{1/2}^\xi \int_{\varepsilon/2}^{\alpha\xi+\gamma} g_n^k(\xi_1+\eta_1)R(\xi_1,\eta_1;\xi,\alpha\xi+\gamma)\,d\xi_1 d\eta_1, \quad \frac{\varepsilon}{2} \le \xi \le \frac{1}{2}, \tag{3.11}$$

where

$$\varphi_n^k(\xi) = \sigma_n^k(\xi) - \frac{\tau_n^k(\alpha\xi+\gamma)}{2} R(\alpha\xi+\gamma, \alpha\xi+\gamma; \xi, \alpha\xi+\gamma) - \frac{\tau_n^k(\xi)}{2} R(\xi, \xi; \xi, \alpha\xi+\gamma)$$
$$- \frac{1}{\sqrt{2}} \int_{\alpha\xi+\gamma}^{\xi} \left[\upsilon_n^k(\xi_1) R(\xi_1, \xi_1; \xi, \alpha\xi+\gamma) - \tau_n^k(\xi_1) \frac{\partial}{\partial N} R(\xi_1, \eta_1; \xi, \alpha\xi+\gamma)\bigg|_{\xi_1=\eta_1} d\xi_1 \right] \quad (3.12)$$
$$- \int_{1/2}^{\xi} \int_{\varepsilon/2}^{\alpha\xi+\gamma} f_n^k(\xi_1+\eta_1) R(\xi_1, \eta_1; \xi, \alpha\xi+\gamma)\, d\xi_1\, d\eta_1,$$

which, after the double differentiation reduces to the following loaded Volterra integral equation of the second kind (see [22]):

$$\frac{1}{2\alpha} \frac{d^2\varphi_n^k}{d\xi^2} g_n^k((\alpha+1)\xi+\gamma) + \frac{1}{2\alpha} \int_{\varepsilon/2}^{\alpha\xi+\gamma} \frac{\partial}{\partial\xi} \left[g_n^k(\xi_1+\xi) R(\xi_1, \xi; \xi, \alpha\xi+\gamma)\, d\xi_1 \right.$$
$$+ \frac{1}{2\alpha} \int_{\varepsilon/2}^{\alpha\xi+\gamma} g_n^k(\xi_1+\xi) \frac{\partial}{\partial\xi} R(\xi_1, \eta_1; \xi, \alpha\xi+\gamma)\bigg|_{\eta_1=\xi} d\xi_1$$
$$+ \frac{1}{2} \int_{1/2}^{\xi} \frac{\partial}{\partial\xi} \left[g_n^k(\alpha\xi+\gamma+\eta_1) R(\alpha\xi+\gamma, \eta_1; \xi, \alpha\xi+\gamma)\, d\eta_1 \right. \quad (3.13)$$
$$+ \frac{1}{2} \int_{1/2}^{\xi} g_n^k(\alpha\xi+\gamma+\eta_1) \frac{\partial}{\partial\xi} R(\xi_1, \eta_1; \xi, \alpha\xi+\gamma)\bigg|_{\xi_1=\alpha\xi+\gamma} d\eta_1$$
$$+ \frac{1}{2\alpha} \int_{1/2}^{\xi} \int_{\varepsilon/2}^{\alpha\xi+\gamma} g_n^k(\xi_1+\eta_1) \frac{\partial^2}{\partial\xi^2} R(\xi_1, \eta_1; \xi, \alpha\xi+\gamma)\, d\xi_1\, d\eta_1.$$

It is handy to rewrite (3.13) as

$$g_n^k(\xi) = F\left(g_n^k\right). \quad (3.14)$$

Given that $|P_\mu(z)| \le C,\ \ |P'_\mu(z)| \le C,\ \ |P''_\mu(z)| \le C$ [23], $C = \text{const}$, the integral operator F maps the full metric space $C^1(\overline{J})$ with the norm $\|g_n^k\| = \max_{\overline{J}}|g_n^k(\xi)| + \max_{\overline{J}}|dg_n^k/d\xi|$ into itself, where J is the interval $(\varepsilon/2, 1/2)$.

Next, let g_{1n}^k and g_{2n}^k be arbitrary elements of the space $C^1(\overline{J})$. It is easy to see that for $g_n^k = g_{1n}^k - g_{2n}^k$, the following estimate is valid:

$$\begin{aligned}
\left|F\left(g_n^k\right)\right| = \Bigg| &-\frac{1}{2\alpha}\int_{\varepsilon/2}^{\alpha\xi+\gamma}\left[\left(g_n^k(\xi_1+\xi)\right)' R(\xi_1,\xi;\xi,\alpha\xi+\gamma) + g_n^k(\xi_1+\xi)\frac{\partial}{\partial\xi}R(\xi_1,\xi;\xi,\alpha\xi+\gamma)\right]d\xi_1 \\
&-\frac{1}{2\alpha}\int_{\varepsilon/2}^{\alpha\xi+\gamma} g_n^k(\xi_1+\xi)\frac{\partial}{\partial\xi}R(\xi_1,\eta_1;\xi,\alpha\xi+\gamma)\Big|_{\eta_1=\xi}d\xi_1 \\
&+\frac{1}{2}\int_{\xi}^{1/2}\Big[\alpha\left(g_n^k(\alpha\xi+\gamma+\eta_1)\right)' R(\alpha\xi+\gamma,\eta_1;\xi,\alpha\xi+\gamma) \\
&\qquad + g_n^k(\alpha\xi+\gamma+\eta_1)\frac{\partial}{\partial\xi}R(\alpha\xi,\eta_1;\xi,\alpha\xi+\gamma)\Big]d\eta_1 \\
&+\frac{1}{2}\int_{\xi}^{1/2} g_n^k(\alpha\xi+\gamma+\eta_1)\frac{\partial}{\partial\xi}R(\xi_1,\eta_1;\xi,\alpha\xi+\gamma)\Big|_{\xi_1=\alpha\xi+\gamma}d\eta_1 \\
&+\frac{1}{2\alpha}\int_{1/2}^{\xi}\int_{\varepsilon/2}^{\alpha\xi+\gamma} g_n^k(\xi_1+\eta_1)\frac{\partial^2}{\partial\xi^2}R(\xi_1,\eta_1;\xi,\alpha\xi+\gamma)d\xi_1 d\eta_1\Bigg| \\
\le{}& \frac{3M}{2\alpha}\left\|g_n^k\right\|\left[\left(\alpha\xi+\gamma-\frac{\varepsilon}{2}\right)+\left(\frac{1}{2}-\xi\right)+\left(\frac{1}{2}-\xi\right)\left(\alpha\xi+\gamma-\frac{\varepsilon}{2}\right)\right], \\
M ={}& \max\left(\max_{\overline{J}*\overline{J}}|R|, \max_{\overline{J}*\overline{J}}\left|\frac{\partial R}{\partial\xi}\right|, \max_{\overline{J}*\overline{J}}\left|\frac{\partial^2 R}{\partial\xi^2}\right|\right).
\end{aligned} \tag{3.15}$$

Furthermore, it is evident that

$$\left|F^2\left(g_n^k\right)\right| \le \left(\frac{3}{2\alpha}M\right)^2\left\|g_n^k\right\|\left[\frac{(\alpha\xi+\gamma-(\varepsilon/2))^2}{2}+\frac{((1/2)-\xi)^2}{2}+\frac{((1/2)-\xi)^2}{2}\frac{(\alpha\xi+\gamma-(\varepsilon/2))^2}{2}\right]. \tag{3.16}$$

Continuing this process, we obtain

$$\left|F^n\left(g_n^k\right)\right| \le \left(\frac{3}{2\alpha}M\right)^n\left\|g_n^k\right\|\left[\frac{(\alpha\xi+\gamma-(\varepsilon/2))^n}{n!}+\frac{((1/2)-\xi)^n}{n!}+\frac{((1/2)-\xi)^n}{n!}\frac{(\alpha\xi+\gamma-(\varepsilon/2))^n}{n!}\right], \tag{3.17}$$

where F^n is the nth degree of the operator F. From here, one can see that we can find such n that

$$\left|F^n\left(g_n^k\right)\right| \le C\|f\|, \quad C = \text{const} < 1. \tag{3.18}$$

The inequality (3.18) implies that the operator F^n is a contraction.

Thus, the operator F has a fixed point [24]. This fixed point is the solution of (3.14), that is, (3.13).

Therefore, having first solved the problem (3.3), (3.6), (3.7) (for $n = 0$), and then (3.4), (3.6), (3.7) (for $n = 1$) and so on, we can find sequentially all $\overline{u}_n^k(r,t)$, $k = \overline{1,k_n}$, $n = 0,1,\ldots$.

Thus, we have shown that

$$\int_{\Gamma} \rho(\theta)\left(Lu - g(x)\right) d\Gamma = 0. \tag{3.19}$$

Let $f(r,\theta,t) = R(r)\rho(\theta)T(t)$, and, moreover, $R(r) \in V_0$ is dense in $L_2(t+\varepsilon, 1-t)$, $\rho(\theta) \in C^{\infty}(\Gamma)$ is dense in $L_2(\Gamma)$, and $T(t) \in V_1$ is dense in $L_2(0,(1-\varepsilon)/2)$. Then, $f(r,\theta,t) \in V$, $V = V_0 \otimes \Gamma \otimes V_1$ is dense in $L_2(D_\varepsilon)$ (see, e.g., [25]).

Thus, from (3.19), it follows that

$$\int_{D_\varepsilon} f(r,\theta,t)\left(Lu - g(x)\right) dD_\varepsilon = 0 \qquad Lu = g(x),\ \forall (x,t) \in D_\varepsilon. \tag{3.20}$$

Therefore, the problem (2.1)–(2.3) has the solutions of the form

$$u(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} r^{(1-m)/2} u_n^k(r,t) Y_{n,m}^k(\theta), \tag{3.21}$$

where $u_n^k(r,t)$ are determined from (3.10), in which $g_n^k(\xi,\eta)$ are found from (3.13).

Taking into account the restrictions on the functions $\tau(r,\theta)$, $\upsilon(r,\theta)$, $\sigma(r,\theta)$, one can analogously prove (as shown, e.g., in [18, 19]) that the obtained solution $u(r,\theta,t)$ in the form (3.21) belongs to the required class.

This completes the proof of Theorem 3.1. Theorem 3.2 is proven analogously. □

4. Conclusion

In this paper, we have proven the existence theorems for the solution of the inverse problem for a general class of multidimensional hyperbolic PDEs, for the cases of time-independent and space-independent unknown right-hand sides of the equation. The potential importance of this work comes from the fact that multidimensional inverse problems for hyperbolic equations are extremely important in applied work, but so far, only the results for relatively narrow classes of equations have been established.

Turning to the limitations of our work, we should note that so far, we have not been able show the uniqueness or stability of the solution. Given the importance of the well-posedness problem in applied fields, we believe that this is an important avenue for future work.

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References

[1] M. Lavrentiev, V. Romanov, and V. Vasiliev, *Multidimensional Inverse Problems for Differential Equations*, Springer, New York, NY, USA, 1970.

[2] V. Romanov, *Some Inverse Problems for the Equations of Hyperbolic Type*, Nauka, Novosibirsk, Russia, 1972.

[3] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer, New York, NY, USA, 2006.

[4] S. Y. Chen, Y. F. Li, Q. Guan, and G. Xiao, “Real-time three-dimensional surface measurement by color encoded light projection,” *Applied Physics Letters*, vol. 89, no. 11, Article ID 111108, 2006.

[5] S. Y. Chen, H. Tong, and C. Cattani, “Markov models for image labeling,” *Mathematical Problems in Engineering*, vol. 2011, Article ID 814356, 18 pages, 2011.

[6] S. Y. Chen and Q. Guan, “Parametric shape representation by a deformable NURBS model for cardiac functional measurements,” *IEEE Transactions on Biomedical Engineering*, vol. 58, no. 3, pp. 480–487, 2011.

[7] R. Engbers, M. Burger, and V. Capasso, *Inverse Problems in Geographical Economics: Parameter Identification in the Spatial Solow Model*, University of Muenster, Muenster, Germany, 2011.

[8] M. Carrasco, J.-P. Florens, and E. Renault, “Linear inverse problems in structural econometrics estimation based on spectral decomposition and regularization,” in *Handbook of Econometrics*, J. Heckman and E. Leamer, Eds., vol. 7, chapter 77, pp. 5633–5751, North-Holland, Amsterdam, The Netherlands, 2007.

[9] C. A. Favero and R. Rovelli, “Macroeconomic stability and the preferences of the Fed: a formal analysis, 1961-98,” *Journal of Money, Credit and Banking*, vol. 35, no. 4, pp. 545–556, 2003.

[10] E. Castelnuovo and P. Surico, “Model uncertainty, optimal monetary policy and the preferences of the fed,” *Scottish Journal of Political Economy*, vol. 51, no. 1, pp. 105–126, 2004.

[11] P. Brito, *The Dynamics of Growth and Distribution in a Spatially Heterogeneous World. Working Paper*, ISEG, Technical University of Lisbon, Lisbon, Portugal, 2004.

[12] R. Boucekkine, C. Camacho, and B. Zou, “Bridging the gap between growth theory and the new economic geography: the spatial Ramsey model,” *Macroeconomic Dynamics*, vol. 13, no. 1, pp. 20–45, 2009.

[13] J.-P. Puel and M. Yamamoto, “Generic well-posedness in a multidimensional hyperbolic inverse problem,” *Journal of Inverse and Ill-Posed Problems*, vol. 5, no. 1, pp. 55–83, 1997.

[14] A. G. Ramm and Rakesh, “Property C and an inverse problem for a hyperbolic equation,” *Journal of Mathematical Analysis and Applications*, vol. 156, no. 1, pp. 209–219, 1991.

[15] C. Daveau, D. Douady, and A Khelifi, “On a hyperbolic coefficient inverse problem via partial dynamic boundary measurements,” *Journal of Applied Mathematics*, Article ID 561395, 14 pages, 2010.

[16] S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Pergamon, New York, NY, USA, 1965.

[17] V. Smirnov, *A Course in Higher Mathematics*, vol. 4, part 2, Addison-Wesley, Boston, Mass, USA, 1964.

[18] S. Aldashev, *Boundary-Value Problems for Multi-Dimensional Hyperbolic and Mixed Equations*, Gylym Press, Almaty, Kazakhstan, 1994.

[19] S. A. Aldashev, “On Darboux problems for a class of multidimensional hyperbolic equations,” *Differentsialnye Uravneniya*, vol. 34, no. 1, pp. 64–68, 1998.

[20] A. V. Bitsadze, *Equations of the Mixed Type*, Pergamon, New York, NY, USA, 1964.

[21] E. T. Copson, “On the Riemann-Green function,” *Archive for Rational Mechanics and Analysis*, vol. 1, pp. 324–348, 1958.

[22] A. Nakhushev, *The Equations of Mathematical Biology*, Nauka Press, Moscow, Russia, 1995.

[23] H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, vol. 1, McGraw–Hill, New York, NY, USA, 1955.

[24] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Pergamon, New York, NY, USA, 2nd edition, 1982.

[25] A. Kolmogorov and S. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Dover Publications, Mineola, NY, USA, 1999.