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Research Article

The Extended Fractional Subequation Method for Nonlinear Fractional Differential Equations

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An extended fractional subequation method is proposed for solving fractional differential equations by introducing a new general ansätz and Bäcklund transformation of the fractional Riccati equation with known solutions. Being concise and straightforward, this method is applied to the space-time fractional coupled Burgers' equations and coupled MKdV equations. As a result, many exact solutions are obtained. It is shown that the considered method provides a very effective, convenient, and powerful mathematical tool for solving fractional differential equations.

1. Introduction

In recent years, nonlinear fractional differential equations (NFDEs) have been attracted great interest. It is caused by both the development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, engineering, and biology [1–7]. For better understanding the mechanisms of the complicated nonlinear physical phenomena as well as further applying them in practical life, the solution of fractional differential equation [8–15] is much involved. In the past, many analytical and numerical methods have been proposed to obtain solutions of NFDEs, such as finite difference method [16], finite element method [17], differential transform method [18, 19], Adomian decomposition method [20–22], variational iteration method [23–25], and homotopy perturbation method [26–28].

Recently, He [29] introduced a new method called fractional sub-equation method to look for traveling wave solutions of NFDEs. The method is based on the homogeneous balance principle [30] and Jumarie's modified Riemann-Liouville derivative [31–33].

By using fractional sub-equation method, Zhang et al. successfully obtained traveling wave solutions of nonlinear time fractional biological population model and (4 + 1)-dimensional space-time fractional Fokas equation. More recently, Guo et al. [34] and Lu [35] improved Zhang et al.'s work [30] and obtained exact solutions of some nonlinear fractional differential equations.

The present paper is motivated by the desire to improve the work made in [30, 34, 35] by proposing a new and more general ansätz so that it can be used to construct more general exact solutions which contain not only the results obtained by using the method in [30, 34, 35] as special cases but also a series of new and more general exact solutions. To illustrate the validity and advantages of the method, we will apply it to the space-time fractional coupled Burgers' equations and coupled MKdV equations.

The rest of this paper is organized as follows. In Section 2, we will describe the Modified Riemann-Liouville derivative and give the main steps of the method here. In Section 3, we illustrate the method in detail with space-time fractional coupled Burgers' equations and coupled MKdV equations. In Section 4, some conclusions are given.

2. Description of Modified Riemann-Liouville Derivative and the Proposed Method

The Jumarie's modified Riemann-Liouville derivative is defined as

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x (x-\xi)^{-\alpha-1}\left(f(\xi)-f(0)\right)d\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x (x-\xi)^{-1}\left(f(\xi)-f(0)\right)d\xi, & 0 < \alpha < 1 \\ \left(f^{(n)}(x)\right)^{\alpha-n}, & n \le \alpha < n+1,\ n \ge 1. \end{cases} \tag{2.1}$$

Some properties for the proposed modified Riemann-Liouville derivative are listed in [32, 33, 36] as follows:

$$D_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0, \tag{2.2}$$

$$D_x^\alpha (u(x)v(x)) = v(x) D_x^\alpha u(x) + u(x) D_x^\alpha v(x), \tag{2.3}$$

$$D_x^\alpha f[u(x)] = f'_u[u(x)] D_x^\alpha u(x), \tag{2.4}$$

$$D_x^\alpha f[u(x)] = D_u^\alpha f[u(x)] \left(u'(x)\right)^\alpha. \tag{2.5}$$

Remark 2.1. In the above formulas (2.3)–(2.5), $u(x)$ is nondifferentiable function in (2.3) and (2.4) and differentiable in (2.5), $v(x)$ is non-differentiable, and $f(u)$ is differentiable in (2.4) and non-differentiable in (2.5). We should use the formulas (2.3)–(2.5) carefully. In this paper, we use formulas (2.3) and (2.5) to obtain the solutions of fractional differential equations. By using the formulas (2.3) and (2.4) to search solutions, one can refer to [37, 38]. In [37, 38], He et al. introduced the fractional complex transform to convert an FDE into its differential partner easily. This transform is accessible to those who know advanced calculus.

We present the main steps of the extended fractional sub-equation method as follows.

Step 1. For a given NFDEs with independent variables $X = (x_1, x_2, x_3, \ldots, x_m, t)$ and dependent variable u,

$$P\left(u, u_t, u_{x_1}, u_{x_2}, u_{x_3}, \ldots, D_t^{\alpha} u, D_{x_1}^{\alpha} u, D_{x_2}^{\alpha} u, D_{x_3}^{\alpha} u, \ldots\right) = 0, \quad 0 < \alpha \leq 1, \tag{2.6}$$

where $D_t^{\alpha} u$, $D_{x_1}^{\alpha} u$, $D_{x_2}^{\alpha} u$, and $D_{x_3}^{\alpha} u$ are the modified Riemann-Liouville derivatives of u with respect to t, x_1, x_2, and x_3, and P is a polynomial in $u = u(x_1, x_2, x_3, \ldots, x_m, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 2. By means of the traveling wave transformation,

$$u(x,t) = u(\xi), \quad \xi = x_1 + k_1 x_2 + \cdots + k_{m-1} x_m + ct + \xi_0, \tag{2.7}$$

where k_i and c are constants to be determined later; (2.6) becomes the following form:

$$P\left(u(\xi), u'(\xi), u''(\xi), D_{\xi}^{\alpha} u(\xi), \ldots\right) = 0. \tag{2.8}$$

Step 3. We suppose that (2.6) has the following solution:

$$u(\xi) = \sum_{i=-m}^{-1} a_i \varphi^i(\xi) + a_0 + \sum_{i=1}^{m} a_i \varphi^i(\xi), \tag{2.9}$$

where a_i $(i = -m, -m+1, \ldots, m-1, m)$ are constants to be determined later, and

$$\varphi(\xi) = \frac{-\sigma B + D\phi(\xi)}{D + B\phi(\xi)}. \tag{2.10}$$

Here B, D are arbitrary parameters, and $\phi(\xi)$ satisfies the following fractional Riccati equation:

$$D_{\xi}^{\alpha} \phi(\xi) = \sigma + \phi^2(\xi), \tag{2.11}$$

where σ is a constant. Recently, Zhang et al.'s [30] first obtained the following solutions of (2.11):

$$\phi(\xi) = \begin{cases} -\sqrt{-\sigma} \tanh_{\alpha}\left(\sqrt{-\sigma}\xi\right), & \sigma < 0, \\ -\sqrt{-\sigma} \coth_{\alpha}\left(\sqrt{-\sigma}\xi\right), & \sigma < 0, \\ \sqrt{\sigma} \tan_{\alpha}\left(\sqrt{\sigma}\xi\right), & \sigma > 0, \\ -\sqrt{\sigma} \cot_{\alpha}\left(\sqrt{\sigma}\xi\right), & \sigma < 0, \\ -\dfrac{\Gamma(1+\alpha)}{\xi^{\alpha}}, \ \omega \text{ is a const}, & \sigma = 0, \end{cases} \tag{2.12}$$

where the generalized hyperbolic and trigonometric functions are defined as

$$\begin{aligned}
\cosh_\alpha(z) &= \frac{E_\alpha(z^\alpha)+E_\alpha(-z^\alpha)}{2}, \qquad \sinh_\alpha(z) = \frac{E_\alpha(z^\alpha)+E_\alpha(-z^\alpha)}{2},\\
\cos_\alpha(z) &= \frac{E_\alpha(iz^\alpha)+E_\alpha(-iz^\alpha)}{2}, \qquad \sin_\alpha(z) = \frac{E_\alpha(iz^\alpha)+E_\alpha(-iz^\alpha)}{2},\\
\tanh_\alpha(z) &= \frac{\sinh_\alpha(z)}{\cosh_\alpha(z)}, \qquad \coth_\alpha(z) = \frac{\cosh_\alpha(z)}{\sinh_\alpha(z)},\\
\tan_\alpha(z) &= \frac{\sin_\alpha(z)}{\cos_\alpha(z)}, \qquad \cot_\alpha(z) = \frac{\cos_\alpha(z)}{\sin_\alpha(z)}.
\end{aligned} \tag{2.13}$$

Here, $E_\alpha(z) = \sum_{k=0}^{\infty} z^k/\Gamma(1+k\alpha)$ $(\alpha > 0)$ is the Mittag-Leffler function in one parameter.

Step 4. Substituting (2.9) into (2.8) and collecting all terms with the same order of $\psi(\xi)$, then setting each coefficient of $\psi^k(\xi)$ to zero, yields a set of overdetermined nonlinear algebraic system for a_i $(i = -m,\ldots,m)$, k_j $(j = 1,2,\ldots,m-1)$ and c.

Step 5. Using the results obtained in the above steps and the solutions of (2.11) into (2.9), we can finally obtain exact solutions of (2.6).

Remark 2.2. As we know, the choice of an appropriate ansätz is very important when using the direct method to look for exact solutions. It can be easily found that the transformation (2.9) is more general than that introduced in [30, 34, 35]. To be more precise, if $B = 0$, then (2.9) becomes (15) in [34]. If we set a_i $(i = -m, -m+1,\ldots,-1) = 0$, then (2.9) becomes (10) in [35]. It shows that taking full advantage of the transformation (2.9), we may obtain new and more general exact solutions including not only all solutions obtained by the methods [30, 34, 35] but also other new solutions. It should be noted that the method can also be extended to other similar sub-equations [38] easily.

3. Application of the Proposed Method

Example 3.1. We first consider the space-time fractional coupled Burgers' equations [39, 40]:

$$\begin{aligned}
D_t^\alpha u - D_x^{2\alpha} u + 2uD_x^\alpha u + pD_x^\alpha(uv) &= 0,\\
D_t^\alpha v - D_x^{2\alpha} v + 2vD_x^\alpha v + qD_x^\alpha(uv) &= 0.
\end{aligned} \tag{3.1}$$

Using the traveling wave transformations $\xi = x + ct + \xi_0$, (3.1) can be reduced to the following nonlinear fractional ODEs:

$$\begin{aligned}
c^\alpha D_\xi^\alpha u - D_\xi^{2\alpha} u + 2uD_\xi^\alpha u + pD_\xi^\alpha(uv) &= 0,\\
c^\alpha D_\xi^\alpha v - D_\xi^{2\alpha} v + 2vD_\xi^\alpha v + qD_\xi^\alpha(uv) &= 0.
\end{aligned} \tag{3.2}$$

According to the method described in Section 2, we suppose that (3.1) has the following formal solution:

$$u(\xi) = a_0 + a_1\varphi(\xi) + \frac{a_{-1}}{\varphi(\xi)},$$
$$v(\xi) = b_0 + b_1\varphi(\xi) + \frac{b_{-1}}{\varphi(\xi)}. \tag{3.3}$$

Substituting (3.3) into (3.2), and setting each coefficients of $\varphi^k(\xi)$ to zero, we get a system of underdetermined equations for a_i, b_i $(i = -1, 0, 1)$ and c. To avoid tediousness, we omit the overdetermined nonlinear equations. Solving the system, we get the following solution sets.

Case 1. Consider

$$a_0 = \frac{(p+1)c^\alpha}{2-2pq}, \qquad a_1 = \frac{p+1}{pq-1}, \qquad a_{-1} = \frac{(p+1)\sigma}{1-pq},$$
$$b_0 = \frac{(q+1)c^\alpha}{2-2pq}, \qquad b_1 = \frac{q+1}{pq-1}, \qquad b_{-1} = \frac{(q+1)\sigma}{1-pq}. \tag{3.4}$$

Case 2. Consider

$$a_0 = \frac{(p+1)c^\alpha}{2-2pq}, \qquad a_1 = \frac{p+1}{pq-1}, \qquad a_{-1} = 0,$$
$$b_0 = \frac{(q+1)c^\alpha}{2-2pq}, \qquad b_1 = \frac{q+1}{pq-1}, \qquad b_{-1} = 0. \tag{3.5}$$

Case 3. Consider

$$a_0 = \frac{(p+1)c^\alpha}{2-2pq}, \qquad a_1 = 0, \qquad a_{-1} = \frac{(p+1)\sigma}{1-pq},$$
$$b_0 = \frac{(q+1)c^\alpha}{2-2pq}, \qquad b_1 = 0, \qquad b_{-1} = \frac{(q+1)\sigma}{1-pq}. \tag{3.6}$$

From Case 1, we obtain the following solutions of (3.1):

$$u = \frac{(p+1)c^\alpha}{2-2pq} + \frac{p+1}{pq-1}\frac{\sigma B + D\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)}{D - B\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)}$$
$$+ \frac{p+1}{pq-1}\frac{D\sqrt{-\sigma} + B\sigma\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)}{B\sqrt{-\sigma} + D\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)},$$

$$v=\frac{(q+1)c^{\alpha}}{2-2pq}+\frac{q+1}{pq-1}\frac{\sigma B+D\sqrt{-\sigma}\tanh_{\alpha}(\sqrt{-\sigma}\xi)}{D-B\sqrt{-\sigma}\tanh_{\alpha}(\sqrt{-\sigma}\xi)}+\frac{q+1}{pq-1}\frac{D\sqrt{-\sigma}+B\sigma\tanh_{\alpha}(\sqrt{-\sigma}\xi)}{B\sqrt{-\sigma}+D\tanh_{\alpha}(\sqrt{-\sigma}\xi)}, \tag{3.7}$$

where $\sigma<0$ and $\xi=x+ct+\xi_0$.

Consider

$$\begin{aligned}u&=\frac{(p+1)c^{\alpha}}{2-2pq}+\frac{p+1}{pq-1}\frac{\sigma B+D\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\xi)}{D-B\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\xi)}+\frac{p+1}{pq-1}\frac{D\sqrt{-\sigma}+B\sigma\coth_{\alpha}(\sqrt{-\sigma}\xi)}{B\sqrt{-\sigma}+D\coth_{\alpha}(\sqrt{-\sigma}\xi)},\\ v&=\frac{(q+1)c^{\alpha}}{2-2pq}+\frac{q+1}{pq-1}\frac{\sigma B+D\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\xi)}{D-B\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\xi)}+\frac{q+1}{pq-1}\frac{D\sqrt{-\sigma}+B\sigma\coth_{\alpha}(\sqrt{-\sigma}\xi)}{B\sqrt{-\sigma}+D\coth_{\alpha}(\sqrt{-\sigma}\xi)},\end{aligned} \tag{3.8}$$

where $\sigma<0$ and $\xi=x+ct+\xi_0$.

Consider

$$\begin{aligned}u&=\frac{(p+1)c^{\alpha}}{2-2pq}-\frac{p+1}{pq-1}\frac{D\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\xi)-\sigma B}{D+B\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\xi)}+\frac{p+1}{1-pq}\frac{D\sqrt{\sigma}+B\sigma\tan_{\alpha}(\sqrt{\sigma}\xi)}{D\tan_{\alpha}(\sqrt{\sigma}\xi)-B\sqrt{\sigma}},\\ v&=\frac{(q+1)c^{\alpha}}{2-2pq}-\frac{q+1}{pq-1}\frac{D\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\xi)-\sigma B}{D+B\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\xi)}+\frac{q+1}{1-pq}\frac{D\sqrt{\sigma}+B\sigma\tan_{\alpha}(\sqrt{\sigma}\xi)}{D\tan_{\alpha}(\sqrt{\sigma}\xi)-B\sqrt{\sigma}},\end{aligned} \tag{3.9}$$

where $\sigma>0$ and $\xi=x+ct+\xi_0$.

Consider

$$u=\frac{(p+1)c^{\alpha}}{2-2pq}+\frac{p+1}{pq-1}\frac{\sigma B+D\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\xi)}{D-B\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\xi)}+\frac{p+1}{1-pq}\frac{B\sigma\cot_{\alpha}(\sqrt{\sigma}\xi)-D\sqrt{\sigma}}{B\sqrt{\sigma}+D\cot_{\alpha}(\sqrt{\sigma}\xi)},$$

$$v=\frac{(q+1)c^{\alpha}}{2-2pq}-\frac{q+1}{pq-1}\frac{\sigma B+D\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\xi)}{D-B\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\xi)}+\frac{q+1}{1-pq}\frac{B\sigma\cot_{\alpha}(\sqrt{\sigma}\xi)-D\sqrt{\sigma}}{B\sqrt{\sigma}+D\cot_{\alpha}(\sqrt{\sigma}\xi)}, \tag{3.10}$$

where $\sigma>0$ and $\xi=x+ct+\xi_0$.

Consider

$$\begin{aligned}u&=\frac{(p+1)c^{\alpha}}{2-2pq}+\frac{p+1}{pq-1}\frac{D\Gamma(1+\alpha)}{D(\xi^{\alpha}+\omega)-B\Gamma(1+\alpha)},\\ v&=\frac{(q+1)c^{\alpha}}{2-2pq}+\frac{q+1}{pq-1}\frac{D\Gamma(1+\alpha)}{D(\xi^{\alpha}+\omega)-B\Gamma(1+\alpha)},\end{aligned} \tag{3.11}$$

where $\sigma=0$ and $\xi=x+ct+\xi_0$.

From Cases 2 and 3, we can obtain many other exact solutions of (3.1). Here, we omit them for simplicity.

Example 3.2. We next consider the following space-time fractional coupled MKdV equations [15, 28]:

$$\begin{aligned}D_t^{\alpha}u&=\frac{1}{2}D_x^{3\alpha}u-3u^2D_x^{\alpha}u+\frac{3}{2}D_x^{2\alpha}u+3D_x^{\alpha}(uv)-3\lambda D_x^{\alpha}u,\\ D_t^{\alpha}v&=-D_x^{3\alpha}u-3vD_x^{\alpha}v-3D_x^{\alpha}uD_x^{\alpha}v+3u^2D_x^{\alpha}v+3\lambda D_x^{\alpha}v.\end{aligned} \tag{3.12}$$

Using the traveling wave transformations $\xi=x+ct+\xi_0$, (3.12) can be reduced to the following nonlinear fractional ODEs:

$$\begin{aligned}c^{\alpha}D_{\xi}^{\alpha}u&=\frac{1}{2}D_{\xi}^{3\alpha}u-3u^2D_{\xi}^{\alpha}u+\frac{3}{2}D_{\xi}^{2\alpha}u+3D_{\xi}^{\alpha}(uv)-3\lambda D_{\xi}^{\alpha}u,\\ c^{\alpha}D_{\xi}^{\alpha}v&=-D_{\xi}^{3\alpha}u-3vD_{\xi}^{\alpha}v-3D_{\xi}^{\alpha}uD_{\xi}^{\alpha}v+3u^2D_{\xi}^{\alpha}v+3\lambda D_{\xi}^{\alpha}v.\end{aligned} \tag{3.13}$$

According to the method described in Section 2, we suppose that (3.12) has the following two formal solutions:

$$\begin{aligned}u(\xi)&=a_0+a_1\varphi(\xi)+\frac{a_{-1}}{\varphi(\xi)},\\ v(\xi)&=b_0+b_1\varphi(\xi)+\frac{b_{-1}}{\varphi(\xi)},\end{aligned} \tag{3.14}$$

$$u(\xi) = a_0 + a_1\psi(\xi) + \frac{a_{-1}}{\psi(\xi)},$$
$$v(\xi) = b_0 + b_1\psi(\xi) + b_2\psi^2(\xi) + \frac{b_{-1}}{\psi(\xi)} + \frac{b_{-2}}{\psi^2(\xi)}. \tag{3.15}$$

Substituting (3.14) into (3.12), and setting each coefficients of $\psi^k(\xi)$ to zero, we get a system of underdetermined equations for a_i, b_i $(i = -1, 0, 1)$ and c. To avoid tediousness, we omit the overdetermined, highly nonlinear equations. Solving the system, we get the following solution sets.

Case 1. Consider

$$a_1 = -1, \qquad a_{-1} = \sigma, \qquad b_0 = \lambda, \qquad b_1 = -2a_0,$$
$$b_{-1} = 2\sigma a_0, \qquad c = \left(4\sigma + 3a_0^2\right)^{1/\alpha}, \tag{3.16}$$

where a_0 is an arbitrary constant.

Case 2. Consider

$$a_1 = -1, \qquad a_{-1} = 0, \qquad b_0 = \lambda, \qquad b_1 = -2a_0,$$
$$b_{-1} = 0, \qquad c = \left(\sigma + 3a_0^2\right)^{1/\alpha}, \tag{3.17}$$

where a_0 is an arbitrary constant.

Case 3. Consider

$$a_1 = 0, \qquad a_{-1} = \sigma, \qquad b_0 = \lambda, \qquad b_1 = 0,$$
$$b_{-1} = 2\sigma a_0, \qquad c = \left(\sigma + 3a_0^2\right)^{1/\alpha}, \tag{3.18}$$

where a_0 is an arbitrary constant.

From Case 1, we obtain the following solutions of (3.12):

$$u = a_0 + \frac{\sigma B + D\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)}{D - B\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)} - \frac{D\sqrt{-\sigma} + B\sigma\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)}{B\sqrt{-\sigma} + D\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)},$$
$$v = \lambda + 2a_0\frac{\sigma B + D\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)}{D - B\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)} - 2a_0\frac{D\sqrt{-\sigma} + B\sigma\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)}{B\sqrt{-\sigma} + D\tanh_\alpha\left(\sqrt{-\sigma}\xi\right)}, \tag{3.19}$$

where $\sigma < 0$ and $\xi = x + (4\sigma + 3a_0^2)^{1/\alpha}t + \xi_0$.

Consider

$$u = a_0 + \frac{\sigma B + D\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}{D - B\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)} - \frac{D\sqrt{-\sigma} + B\sigma\coth_\alpha(\sqrt{-\sigma}\xi)}{B\sqrt{-\sigma} + D\coth_\alpha(\sqrt{-\sigma}\xi)},$$
$$v = \lambda + 2a_0\frac{\sigma B + D\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}{D - B\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)} - 2a_0\frac{D\sqrt{-\sigma} + B\sigma\coth_\alpha(\sqrt{-\sigma}\xi)}{B\sqrt{-\sigma} + D\coth_\alpha(\sqrt{-\sigma}\xi)}, \quad (3.20)$$

where $\sigma < 0$ and $\xi = x + (4\sigma + 3a_0^2)^{1/\alpha}t + \xi_0$.

Consider

$$u = a_0 + \frac{\sigma B - D\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}{D + B\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)} + \frac{D\sqrt{\sigma} + B\sigma\tanh_\alpha(\sqrt{\sigma}\xi)}{D\tan_\alpha(\sqrt{\sigma}\xi) - B\sqrt{\sigma}},$$
$$v = \lambda + 2a_0\frac{\sigma B - D\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}{D + B\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)} + 2a_0\frac{D\sqrt{\sigma} + B\sigma\tanh_\alpha(\sqrt{\sigma}\xi)}{D\tan_\alpha(\sqrt{\sigma}\xi) - B\sqrt{\sigma}}, \quad (3.21)$$

where $\sigma > 0$ and $\xi = x + (4\sigma + 3a_0^2)^{1/\alpha}t + \xi_0$.

Consider

$$u = a_0 + \frac{\sigma B + D\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}{D - B\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)} - \frac{D\sqrt{\sigma} - B\sigma\cot_\alpha(\sqrt{\sigma}\xi)}{B\sqrt{\sigma} + D\cot_\alpha(\sqrt{\sigma}\xi)},$$
$$v = \lambda + 2a_0\frac{\sigma B + D\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}{D - B\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)} - 2a_0\frac{D\sqrt{\sigma} - B\sigma\cot_\alpha(\sqrt{\sigma}\xi)}{B\sqrt{\sigma} + D\cot_\alpha(\sqrt{\sigma}\xi)}, \quad (3.22)$$

where $\sigma > 0$ and $\xi = x + (4\sigma + 3a_0^2)^{1/\alpha}t + \xi_0$.

Consider

$$u = a_0 + \frac{D\Gamma(1+\alpha)}{D(\xi^\alpha + \omega) - B\Gamma(1+\alpha)},$$
$$v = \lambda + 2a_0\frac{D\Gamma(1+\alpha)}{D(\xi^\alpha + \omega) - B\Gamma(1+\alpha)}, \quad (3.23)$$

where $\sigma = 0$ and $\xi = x + (3a_0^2)^{1/\alpha}t + \xi_0$.

From Cases 2 and 3, we can obtain many other exact solutions of (3.12). Here, we do not list them for simplicity.

It should be noted that when using ansätz (3.15), we can also obtain more exact solutions of (3.12) by the same procedure. For the sake of simplicity, we omit them too.

Remark 3.3. It seems that the Exp-function method [30] is more general than the extend sub-equation method. The adopted sub-equation method actually uses the same idea as the one using an FDE $D_\xi^\alpha v = a + bv^2$ with constants a and b. You can choose either of these methods for analyzing a new equation or a previously unstudied (or partially studied) problem.

4. Conclusion

In this paper, based on a new general ansätz and Bäcklund transformation of the fractional Riccati equation with known solutions, we propose a new method called extended fractional sub-equation method to construct exact solutions of fractional differential equations. In order to illustrate the validity and advantages of the algorithm, we apply it to space-time fractional coupled Burgers' equations and coupled MKdV equations. As a result, many exact solutions are obtained. The results show that the extended fractional sub-equation method is direct, effective, and can be used for many other fractional differential equations in mathematical physics.

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