MPE — Volume 2, pp. 1-34
Reprints available directly from the publisher
Photocopying permitted by license only

A GENERAL THEORY OF ROTORCRAFT TRIM

DAVID A. PETERS and DINESH BARWEY

Center for Computational Mechanics, Washington University, St. Louis, MO 63130

(Received February 7, 1995)

In this paper we offer a general theory of rotorcraft trim. The theory is set in the context of control theory. It allows for completely arbitrary trim controls and trim settings for multi-rotor aircraft with tests to ensure that a system is trimmable. In addition, the theory allows for "optimal trim" in which some variable is minimized or maximized rather than set to a specified value. The theory shows that sequential trim cannot work for free flight. The theory is not tied to any particular trim algorithm; but, in this paper, it is exercised with periodic shooting to show how free-flying rotorcraft can be trimmed in a variety of ways (zero yaw, zero pitch, zero roll, minimum power, etc.) by use of the general theory. The paper also discusses applications to harmonic balance and auto-pilot trim techniques.

KEYWORDS: *Rotorcraft; trim*

1. INTRODUCTION

1.1. Background

The word "trim" is derived from the Middle English word "trimmen" which means to "prepare" or "put in order" [1]. The term came to be used in a maritime context for adjusting the sails or the cargo of a ship in order to achieve the best attitude and velocity moving through the sea. Later yet, the word "trim" was adopted by the aviation community to imply the correct adjustment of aircraft controls, attitude, and cargo in order to obtain a desired steady flight condition. For rotorcraft, the concept of a trimmed flight condition is quite more complicated than it is for a fixed-wing aircraft. A rotorcraft has aerodynamic components that rotate with respect to each other and with respect to the air mass. Thus, periodic forces and periodic coefficients enter the dynamic equations; and these cannot be eliminated by a time average. (For example, time-averaging blade flapping would eliminate important roll and pitch moments.) Thus, a rotorcraft trim implies a periodic dynamic solution to a system of nonlinear equations with unknown parameters (e.g., controls or airframe attitudes). The parameters must be adjusted such that this periodic solution satisfies some constraints that reflect a desired flight condition. The resulting solution then gives controls, attitudes, and power required for that flight condition.

In rotorcraft dynamic analysis, however, the importance of trim is much more than just the calculation of the controls and power required for a given flight path. The accurate calculation of trim is also crucial to the determination of flight mechanics and handling qualities since conventional stability derivatives are related in a strong, nonlinear way to the flight condition. Furthermore, the aeroelastic stability of a rotorcraft is strongly influenced by the trim settings and periodic trimmed solution [2]. This is because the rotor blade equations are highly nonlinear. Thus, the Floquet perturbation stability analysis depends on the periodic orbit (i.e., solution) that is perturbed. In fact, the entire set of perturbation equations (upon which stability, handling qualities, and control system designs are based) is predicated upon the successful solution of the trim problem in such a way that a set of state-space perturbation equations can be obtained about the trimmed solution. Lastly, the trimmed solution is important in that it provides the vibratory rotor loads and airframe vibrations.

One might wonder why the solution of this complicated problem might not be by-passed by a simple substitution of measured or approximate controls into the equations followed by a time-history solution. The reason that such a procedure does not work is that any mathematical model is only an approximation to the actual system. Thus, the measured controls from a flight test (or approximate controls) will not trim the mathematical model of the rotor. As a result, the time history produced by these approximate controls will drift away from the desired solution. For a free-flight condition, this drifting will be unstable or neutrally stable. Given this uncertainty, one must compute the trim for every specific mathematical model and flight condition that is to be studied.

1.2. Types of Trim

Until now, there has not been a unified theory of rotor trim. One reason for this has been the large number of types of trim that various investigators have utilized. Reference [2] introduces the concept of "moment trim" for a fixed-hub rotor in which collective and cyclic pitch are used to give a specified thrust (or coning angle) and specified pitching and rolling moments (or tip-path-plane tilts), the latter usually being zero. Reference [2] also introduces the concept of "propulsive trim" in which the fixed hub has an unspecified forward tilt which is used to balance the fuselage drag. Reference [3] mentions moment trim (p. 758) but also uses "trim" in other contexts. Page 132 discusses trim as a force equilibrium in order to find an unknown climb rate. Page 182 discusses power required as part of the trim solution. Pages 243–249 discuss force, moment, and power as trim variables to be balanced. Pages 758–759 mention a six-degree-of-freedom trim in which three forces and three moments are balanced by collective pitch, the two cyclic pitches, tail rotor pitch, and body pitch and roll angles (yaw defined).

In general, trim constraints are usually placed on some combination of either forces (to place a body in equilibrium) or displacements (to place a body in a desired position). The unknown "controls" that must be adjusted to effect the trim constraints can involve rotor controls, external aerodynamic surface movements, body attitudes, or unknown velocities (such as climb rate). In simple systems, the constraints can involve forces at a single point or deflections of a single node. In general, however, there may not be, for example, well-defined structural nodes that correspond to given body angles. Similarly, the "control" parameters can be difficult to define. Thus, one of the tasks in unifying trim is to develop a mathematical framework that is independent of the details of the mathematical model (e.g., rigid bodies, modes, finite elements, etc.)

Other variations also exist in definitions of trim. Most analysis codes have trim methodologies in which the user must choose among different types of trim [4]–[8]. Usually, one must choose between a fixed-hub (or wind-tunnel) trim and a so-called "free-flight" trim. In [8], the "free-flight" trim is actually a fixed-hub trim in which the fixed geometric angles of the hub are taken as controls. This is a very useful concept, but it does have disadvantages. First, in a comprehensive analysis, there may not be a well defined hub. Second, it ignores vibrations of the hub which can couple with the periodic coefficients to create steady loads [9]. Third, this approach suppresses the body dynamics. This implies that one cannot do an integrated stability and handling qualities analysis with such a trim. Reference [10] attempts to represent fuselage motions through a sweeping technique in which the rigid-body modes are removed from the finite-element solution at each time step. There is also trim in hover which requires specification of an average yaw position. These many ways of looking at trim reflect the special difficulties and seemingly fundamental differences between trim definitions. In this paper we will try to resolve these.

1.3. Solution Strategies

The purpose of this paper is not to introduce a unified trim solution technique. It is rather to introduce a unified theory of trim around which solution techniques can be designed. Nevertheless, the theory has implications concerning the general applicability of any trim strategy. Thus, it is useful to survey existing strategies in the literature. Perhaps the oldest trim strategy is the harmonic balance [11]–[13]. This methodology has been extended to a general numerical technique [14] and is used in industrial codes [7]. Reference [15] discusses the numerical efficiency of the method. When harmonic balance is used for the states, the controls are usually found in parallel by addition of augmented equations, [3] pages 772–774 and [16]. Harmonic balance has also been used in an iterative fashion between rotor and body substructures [17], [18].

Another popular method of finding trim is to guess controls, time march until transients decay, and then check the trim-constraint errors. A Newton-Raphson (or secant-method) procedure is used to iterate on the controls. This is the method used in C-81 [5], and has been the method of choice on most trim programs [11], [4]. However, when trimming a free-flight case, artificial springs to ground must be added in order for convergence to be reached, such as in 2GCHAS [4], [19]. Furthermore, this method cannot trim unstable systems.

A third type of solution strategy is periodic shooting in which the initial conditions and controls necessary for a trimmed solution are found by an integration through one period (to obtain errors) followed by Newton-Raphson iteration [20]. The method can be applied either sequentially or in parallel and in a "damped" version [21], and it has been used in DYSCO [22]–[23]. Other methods based on the transition matrix [24]–[25] are variations of periodic shooting.

A fourth strategy is finite elements in time [26]–[27]. Actually, finite elements in time is simply an alternative time-marching strategy that could be used in periodic shooting or marching until transients decay. However, when the periodicity constraint is enforced by assembly, the method becomes akin to a harmonic balance approach because periodicity is constrained [28]–[29]; and this is what most authors imply by a finite-element trim. In fact, [30] uses a harmonic balance for the equation solution but Newton-Raphson iteration (in a sequential, nested approach) for controls and unknown attitudes. The rotor and body are uncoupled such that the effects of hub vibration on the rotor are neglected. Thus, it is just like the model and solution strategy of [8] and [29] but with harmonic balance used rather than time finite elements.

The last strategy to be discussed is that of auto-pilots. These can only be used for stable systems. A set of auto-pilot equations augments the system equations such that controls are "flown" to trim [31]–[32].

Later in this paper, we will further categorize these methods based on the theorems and results of the unified trim theory.

1.4. Scope of Work

In this paper, we set out to offer a unified mathematical description of the trim problem that can include a wide range of trim possibilities. In addition, it will include trimming to steady-acceleration maneuvers [33], the new concept of multi-blade trim [34]–[35], and the concept of optimum trim in which a trim parameter is minimized rather than set to zero. This general formulation will allow the development of trimmability conditions to determine if a trim problem is well formulated. It will also allow assessment and categorization of trim techniques on a more solid mathematical basis, based on the theoretical trim structure. Numerical examples will illustrate the theory.

Most important is the fact that the trim theory developed here is not limited to particular mathematical models. Thus, equations can be implicit or explicit; the formulation can be first-order, second-order, mixed, or multi-body; and there can be any number of rotors rotating at various speeds including unsteady RPM due to engine, drive-train dynamics. Included is the case in which RPM is not known a *priori*. Thus, the theory should form a basis for evaluation of trim methodologies.

2. GENERAL FORMULATION

2.1. State Equations

The first element of a trim formulation is the equations that define the states. A general form for these equations is

$$\dot{x}_i = f_i\,(x_j, \lambda_m, \theta_k); \qquad i\,, j = 1, N \tag{1}$$

$$m = 1, J; \qquad k = 1, K$$

$$0 = F_l\,(x_j, \lambda_m, \theta_k); \qquad l = 1, J \tag{2}$$

where x_i are the states, λ_m are Lagrange multipliers associated with multi-body formulations, and the θ_k are controls. The states can include displacements, velocities, rotations, angular velocities, damper states, control states (voltages, etc), and engine states (including temperature, pressure, and other thermodynamic properties).

Aerodynamic quantities can also be states. For example, [36] defines lift coefficients as states, [37] shows how vortex lattice codes can be defined as state models, and [38] defines inflow states. Any CFD analysis code is state-space, and [39] shows how the number of CFD states can be reduced for dynamic analysis. In short, any system property governed by differential equation can be a state.

The Lagrange multipliers, λ_m, are generally the internal forces associated with multi-body formulations, and Eq. (2) represents the internal force balance that must be satisfied at every time step. In displacement methods, the λ_m are eliminated explicitly leaving

$$\dot{x}_i = \hat{f}_i(x_j, \theta_k); \qquad i = 1, N \tag{3}$$

Here, however, we keep the more general form of Eqs. (1)–(2). The θ_k are any unknown parameters (not functions of time) that appear in the equations and that must be chosen so as to satisfy trim constraints. These we call "controls", although they can include unknown velocities, angular rates, body attitudes, throttle position or any other unknown parameter. The controls are constant parameters (not functions of time) and, therefore, cannot depend on the states.

The fact that Eqs. (1)–(2) are written in first-order form does not preclude any solution strategy that capitalizes on the second-order form. However, since velocities (or momenta) in second-order formulations can always be defined as states, Eqs. (1)–(2) are simply the most general form and the form we will use for this development.

Similarly, the above does not preclude equations in completely implicit form

$$F_i\,(x_j,\ \dot{x}_m, \theta_k) = 0; \qquad i, j = 1, N \tag{4}$$

When equations are in this form and the $\dot{x}_j$'s cannot be found explicitly, one can simply define λ_m by

$$\dot{x}_m \equiv \lambda_m\,; \qquad m = 1, N \tag{5}$$

Equation (5) thus takes the form of Eq. (1) with $f_m = \lambda_m$, and Eq. (4) takes the form of Eq. (2) with $J = N$. Thus, we see that Eqs. (1)–(2) are a completely general description of rotorcraft equations that can accommodate first-order forms, second-order forms, implicit forms, explicit forms, displacement versions, and multi-body versions. These equations have a solution that depends on the controls θ_k and the initial conditions $x_i(0)$. No independent initial conditions are allowed for the λ_m (except to distinguish between multi-valued solutions) since Eq. (2) provides the $\lambda_m(0)$ given $x_i\,(0)$.

2.2. Quasi-Periodicity Conditions

As we have seen above, the system solution will require proper choice of the N unknowns, $x_i(0)$. The N equations that help to specify these unknowns are the periodicity conditions. For simple rotor models, these conditions are straightforward.

$$x_i\,(T) = x_i\,(0)\ \text{(simplest)} \tag{6}$$

where $T = 2\pi/\Omega$ is the rotor period. However, for a general formulation, we require a more general definition of this condition. First, there is the definition of T. For systems with many rotors and with drive-train dynamics, T can be defined rigorously in terms of an average rate of rotation at subsystem interfaces, as described in Appendix A. There is always a minimum time T after which every subsystem interface will have completed an integer number of complete revolutions. Of course, one must remember that T (or Ω) may not always be known *a priori*. Thus, T must also be considered to be a function of the unknown controls (i.e., T or Ω can be one of the θ_k).

Appendix A also points out that, when the rotor subsystems have Q_i sectors of symmetry each (i.e., Q_i identical blades), there can be an even smaller common period T after which every rotor has returned to a symmetric position that is an integer multiple of $1/Q_i$ periods. As pointed out in [35], a periodicity condition can be written based on this shorter time interval; and, thus, the shorter period is sufficient for determination of trim and Floquet stability. The multi-blade periodicity conditions can be expressed as

$$x_i\,(T) = P_{ij}x_j(0)\ \text{(multi}-\text{blade)} \tag{7}$$

The matrix $[P]$ is made up of smaller diagonal partitions. For x_i's on nonrotating components, $[P] = [I]$ (or $P_{ij} = \delta_{ij}$). For x_i's on "attached" rotating substructures, the matrix $[P]$ is given by

$$[P] = [p]^{N_i} \tag{8}$$

where $[\,p]$ is the permutation matrix, Appendix B, and N_i is the number of sector passages during the interval 0 to T (also defined in Appendix A). It should be noted that the designations 0 and T are arbitrary. Any two times that differ by T can be used as the reference values.

A final generalization of periodicity must be made because rotor equations can be written in global coordinates. Therefore, since a rotorcraft trim can include average, rigid-body velocities (or angular velocities), the difference in $x_i\,(T)$ and $x_i\,(0)$ can be proportional to a product of velocity and time rather than zero. This we call "quasi-periodic" (uniform motion plus periodic motion). The general, quasi-periodic conditions thus become

$$x_i\,(T) = P_{ij}\,x_j\,(0) + z_i\,(\theta_k)\ \text{(general)} \tag{9}$$

where z_i are the differences between the quasi-periodic states at the beginning and end of one period. The z_i must be written as possible functions of the controls θ_k because the rigid-body velocities can sometimes be unknown parameters. An example would be trimming a rotor with a given RPM and a given collective pitch to an unknown rate of climb, U. Since U is unknown, it must be defined as one of the controls θ_k; and UT (the difference in vertical position) will thus depend on the controls.

Equation (9) is the general form of the quasi-periodicity conditions. However, because of the state equations, Eq. (1), it is possible to eliminate $x_i\,(T)$ and rewrite these as

$$\int_0^T f_i(x_j, \lambda_m, \theta_k)dt + [\delta_{ij} - P_{ij}]\,x_j\,(0) = z_i(\theta_k) \tag{10}$$

where δ_{ij} is the Kronecker delta (or identity matrix). This alternative form of the quasi-periodicity conditions is also useful in the derivations to follow. (Here, and through this paper, repeated subscripts imply summation.)

2.3. Trim Constraints

The final development in the general formulation is the expression of the L trim constraints that will complete the set of $N + L$ equations in $N + K$ unknown, $x_i\,(0)$ and θ_k. Although the quantities to be trimmed might include forces, displacements, or power, these can always be written in terms of the states. Thus, any quantity to be trimmed can be expressed as $g_l(x_j, \lambda_m, \theta_k)$. If the values to be trimmed should depend on a derivative of a state, Eq. (1) can always be used to express that derivative in terms of the states and controls.

Usually, one would be interested in specifying the time average of any g_l. However, it might also be desirable to trim to a specific value of g_l at some reference time $t = 0$. For example, one might want a particular blade to be pointed in a particular direction at $t = 0$. To distinguish quantities trimmed at a point from those to be time-averaged, we call the latter $h_l(x_j, \lambda_m, \theta_k)$ implying that they are evaluated at $t = 0$. Thus, the general trim constraints can be written as

$$\frac{1}{T}\int_0^T g_l(x_j, \lambda_m, \theta_k)dt + h_l(x_i, \lambda_m, \theta_k) = G_l; \quad l = 1, L \tag{11}$$

where G_l is the desired value of the left-hand side. In most cases, either g_l or h_l would be zero for any particular value of l. However, Eq. (11) would also allow for trimming to a linear combination of time-averaged and point values. Of course, in Eq. (11), one could always eliminate G_l by incorporating it into the left-hand-side integral with no loss of generality. Similarly, $g_l = T\delta(t)h_l$ can be used to incorporate h_l into the same integral. Nevertheless, here we leave the constraints in the form of Eq. (11) for convenience of use of the formulation.

It is interesting to note the similarities between Eqs. (10) and (11). One can see that the periodicity constraints , Eq. (10), have the same form as the trim constraints, Eq. (11). In particular, $g_l \equiv T f_l$, $G_l \equiv z_l(\theta_k)$ gives them the same form. Thus, all constraints (both periodicity and trim) have a common form. This correspondence also leads to the conclusion that no g_l can be a linear combination of the f_i. If it were, then the condition in Eq. (11) would become a linear combination of periodicity conditions and not an independent equation. In other words, no g_l should be a combination of the derivatives of the states. Any constraint of that form would already be included in Eq. (9) or (10). An equivalent way of stating this is that no g_l should be the sum of all forces on any free-body since that is equivalent to a state-variable derivative.

2.4. Summary

Equations (1) and (2), (9) or (10), and (11) form the basis of a general trim theory. Of course, one is always free to pursue simplifications. For example, if the λ_m can be eliminated, then Eq. (2) is not necessary. For periodicity, Eq. (9), the change of variable

$\bar{x}_i = x_i + z_i(\theta_k)t/T$ can always eliminate the z_i from the periodicity condition; and one can further choose not to use multiblade trim, leaving Eq. (6) as the simplified periodicity condition. Also, the change of time variable $\bar{t} = \Omega t$ can be used to eliminate the unknown controls from T, ($\bar{T} = 2\pi$). Lastly, G_l and h_l can be incorporated into g_l in the trim constraints, Eq. (11). Although these simplifications can, in general, always be made. It is not always convenient to do so. Thus, we leave them in the most general formulation.

3. TRIMMABILITY

To trim, one must solve Eqs. (1) and (2) subject to the constraints implied by Eqs. (9) and (11). An important point, therefore, is whether or not any such formulation is well posed. That is, can a solution be found? The property that ensures that a solution does exist, we call "trimmability." In this section, we will develop tests on the equations, periodicity constraints, and trim constraints that are necessary for trimmability.

3.1. Solvability

First we address the equations of motion, Eqs. (1) and (2). Given initial conditions, a set of equations like Eq. (1) can always be time-stepped from 0 to T provided that the λ_m can be found from Eq. (2) at each time step. The condition on Eq. (2) that will ensure the ability to move towards an iterative solution is found from the Jacobian.

$$[D_{ij}] = [\, \partial F_i / \partial \lambda_j \,] \tag{12}$$

For the system to be solvable, the matrix D must remain invertible at every time step and every λ iteration. This $J \times J$ matrix we call the solvability matrix. Thus, the number of F's must equal the number of λ's; and D must be nonsingular.

The next step in testing the trimmability of any system is to study the nature of the solution to Eqs. (1) and (2). Virtually all methods of finding a solution to nonlinear equations are based on some type of quasi-linearization about the last guess [24]. Thus, to test the ability to move toward a desired solution, we investigate the perturbation of Eqs. (1) and (2).

$$\{\delta \dot{x}_i\} = [\partial f_i / \partial x_j]\{\delta x_j\} + [\partial f_i / \partial \lambda_m]\{\delta \lambda_m\} + [\partial f_i / \partial \theta_k]\{\delta \theta_k\} \tag{13}$$

$$\{0\} = [\partial F_l / \partial x_j]\{\delta x_j\} + [\partial F_l / \partial \lambda_m]\{\delta \lambda_m\} + [\partial F_l / \partial \theta_k]\{\delta \theta_k\} \tag{14}$$

These can be combined to eliminate $\delta\lambda_m$.

$$\{\delta \dot{x}_i\} = [\partial \hat{f}_i / \partial x_j]\{\delta x_j\} + [\partial \hat{f}_i / \partial \theta_k]\{\partial \theta_k\} \tag{15}$$

where

$$[\partial \hat{f}_i/\partial x_j] = [\partial f_i/\partial x_j] - [\partial f_i/\partial \lambda_m][\partial F_l/\partial \lambda_m]^{-1}[\partial F_l/\partial x_j] \tag{16}$$

$$[\partial \hat{f}_i/\partial \theta_k] = [\partial f_i/\partial \theta_k] - [\partial f_i/\partial \lambda_m][\partial F_l/\partial \lambda_m]^{-1}[\partial F_l/\partial \theta_k] \tag{17}$$

where all matrices are periodic with T. Equations (15)–(17) cannot be formed unless the solvability matrix, D_{ij}, can be inverted. Thus, solvability is a prerequisite both for time marching and for system perturbation, each of which is necessary to trim.

3.2. Periodic Solution

Next, we consider the conditions required to find a periodic solution to the perturbation equations. This section is not offering a solution strategy. It is merely offering proofs about solutions.

Consider, first, the homogeneous solution to Eq. (15), $\delta\theta_k \equiv 0$. This system can be solved by Floquet theory [25] in which one forms the Floquet transition matrix, $\Phi(T)$, finds its eigenvalues Λ_j, forms the characteristic exponents, $\eta_j = \log(\Lambda_j)/T$, and then computes periodic eigenvectors ϕ_{ij}. This homogeneous solution can be used to uncouple the nonhomogeneous Eq. (15) as:

$$\{\dot{r}\} = [\eta_i]\{r\} + [\psi][\partial \hat{f}_i/\partial \theta_k]\{\delta\theta_k\} \tag{18}$$

where

$$\{\delta x\} = [\phi]\{r\}, \qquad [\psi] = [\phi]^{-1} \tag{19}$$

We assume, with no loss of generality, that multi-blade trim is not used and that quasi-periodicity has been reduced to periodicity, Eq. (6), by the appropriate change of variable. Because the right-hand side of Eq. (18) is periodic, we can form its Fourier coefficients and solve Eq. (18) by harmonic balance, provided that no $\eta_i \equiv 0$ (i.e., no $\Lambda_i \equiv 1$) and provided that there are no repeated roots. (We will return to these special cases later.)

The complex Fourier coefficients are

$$C_{jk}^n = \frac{1}{T}\int_0^T \psi_{ji}\partial \hat{f}_i/\partial\theta_k e^{-in\Omega t}dt$$

$$-\infty < n < +\infty \tag{20}$$

where the repeated subscripts imply summation. The periodic solution for $r_j(t)$ follows immediately

$$r_j(t) = \frac{C_{jk}^n}{in\Omega - \eta_j} e^{in\Omega t}\delta\theta_k \equiv \frac{\partial r_j}{\partial \theta_k}\delta\theta_k \tag{21}$$

The initial condition leading to this periodic solution is immediately found from Eq. (21).

$$\delta x_i(0) = \phi_{ij}(0) r_j(0) \tag{22}$$

Thus, when no $\eta_j \neq 0$, the N initial conditions are found uniquely from the N periodicity constraints without consideration of the trim constraints.

When one or more $\eta_j = 0$, however, Eq. (21) has an indeterminacy at $n = 0$. That is, the average part of $r_j(t)$ is singular for any j for which $\eta_j \equiv 0$ unless the secular term at $n = 0$ is set equal to zero by the condition

$$C^0_{jk}\, \delta\theta_k \equiv 0; \qquad j = 1, S \tag{23}$$

(No other integer indeterminacy need be considered since one can always choose $-\Omega/2 < Im(\eta_j) \leq \Omega/2$ based on the $\log(\Lambda_j)$.) Equation (23) implies an extra constraint on the $\delta\theta_k$, but there is a balancing extra undetermined constant, the average portion of $r_j\,(t)$. Thus, the number of equations and unknowns remains balanced; but the periodicity condition becomes cross-coupled with the trim constraints (i.e., $\delta\theta_k$ affects periodicity; and $r_j(0)$, which determines the average $r_j(t)$, affects trim constraints).

For repeated roots with nondistinct eigenvectors, the above conclusions remain unchanged. The Jordan vectors can be used to obtain a Jordan Canonical form of Eq. (18). For $\eta_j \neq 0$, a unique periodic solution is obtained. For $\eta_j = 0$ as a repeated root, only the $C^0_{jk}\delta\theta_k$ due to the original eigenvector must be zero in Eqs. (20) and (23). The equations formed by the Jordan vectors will consequently give a unique solution. Thus, in general, for a periodic solution to exist, Eq. (23) must hold for every independent eigenvector (not Jordan vector) for which $\eta_j \equiv 0$.

There is a direct consequence of this result on trimming techniques. When trimming a free-flight condition or a case with engine drive-train dynamics, there are always rigid-body (zero frequency) modes. Thus, in those cases, the periodic constraints and trim constraints are cross-coupled. It follows that a completely sequential trim strategy cannot be used for a free-flight condition, as will become clear below. This is true for time-marching, harmonic balance, time finite elements, or any other solution technique. This is why springs to ground or fixed hubs must be utilized in the literature for such cases [4].

3.3. Trim Constraints

The last piece of trimmability deals with the trim constraints, Eq. (11). As before, we consider the perturbation of those equations about a periodic guess that may not satisfy the constraints. We also assume (without loss of generality) that all θ_k dependence has been removed from T, and that h_l has been included in g_l. First, consider the problem with no zero eigenvalues. For that case, the entire periodic solution has been found as a function of $\delta\theta_k$, Eq. (21), and the constraint equations become

$$\frac{1}{T}\int_0^T ([\partial \hat{g}_l/\partial x_i]\,[\phi_{ij}]\,\{\partial r_j/\partial\theta_k]\,\delta\theta_k + [\partial \hat{g}_l/\partial\theta_k]\,\delta\theta_k)dt = e_l \tag{24}$$

where e_l is the error in G_k for the periodic guess, and $\hat{g}_l$ is g_l with λ_m removed as it was done for f_i in Eqs. (16)–(17). For Eq. (24) to have a solution that can reduce e_l, the trimmability matrix, A_{lk}, must be invertible, where:

$$A_{lk} = \frac{1}{T}\int_0^T ([\partial \hat{g}_l/\partial x_i]\,[\phi_{ij}]\,[\partial r_j/\partial \theta_k] + [\partial \hat{g}_l/\partial \theta_k])\,dt \tag{25}$$

For A_{lk} to be invertible, it must at least be square. Thus, the number of controls must equal the number of trim constraints, and A_{lk} must not be singular. Because $\partial r_j/\partial \theta_k$ depends upon $\partial \hat{f}_i/\partial \theta_k$, it follows that, if any $\hat{g}_l$ is a linear combination of $\hat{f}_i$, then A_{lk} will be singular.

The physical interpretation of this condition is that of controllability. The θ_k must be capable of controlling the g_l. If $K = L$ and the row and column ranks of A_{lk} are equal to K, then the system is exactly trimmable. If the row rank of A_{lk} is greater than its column rank, it is untrimmable. If the row rank is less than the column rank, it is overtrimmable, the subject of the next section.

Last, we consider the case for which there are S zero eigenvalues, η_j, with independent eigenvectors ϕ_{ij} and corresponding left-hand vectors ψ_{ji}. In that case, for those values of j that give $\eta_j \equiv 0$, the average values of r_j, given by

$$\bar{r}_j \equiv \frac{1}{T}\int_0^T r_j(t)\,dt \tag{26}$$

remain unknowns along with the $\delta\theta_k$; and Eq. (23) must also be satisfied. Thus, the coupled periodicity-trimmability conditions can be written from Eqs. (24) and (26) as

$$\begin{bmatrix} A_{lk} & H_{li} \\ C^0_{jk} & 0 \end{bmatrix} \begin{Bmatrix} \delta\theta_k \\ \bar{r}_i \end{Bmatrix} = \begin{Bmatrix} e_l \\ 0 \end{Bmatrix} \tag{27}$$

where $i, j = 1, S$ and

$$C^0_{jk} = \frac{1}{T}\int_0^T \psi_{ji}\,[\partial \hat{f}_i/\partial \theta_k]\,dt \tag{28}$$

$$H_{li} = \frac{1}{T}\int_0^T [\partial \hat{g}_l/\partial x_k]\,[\phi_{ki}]\,dt \tag{29}$$

The A_{lk} partition is found from Eq. (25), but the $n = 0$ coefficients for $\eta_j = 0$ terms in Eq. (21) are replaced by the corresponding secular terms,

$$\frac{C^0_{jk}}{0}\,\delta\theta_k \rightarrow C^0_{jk}t\,\delta\theta_k$$

The still undetermined constant terms, $\bar{r}_j$, appear on the left-hand side of Eq. (24) as

$$\cdots + \frac{1}{T}\int_0^T [\partial \hat{g}_l/\partial x_i]\,[\phi_{ij}]\,\{\bar{r}_j\}\,dt = \cdots \tag{30}$$

$$J = 1, S$$

which results in Eq. (29).

Equation (27) can be inverted if both A_{lk} and B_{ji} can be inverted, where

$$B_{ji} = C^0_{jk}\,[A_{lk}]^{-1}\,H_{li} \tag{31}$$

Thus, the trimmability condition (that A_{lk} be non-singular) remains as with no zero eigenvalues; but, with zero eigenvalues, an extra matrix (the secular term matrix, B_{ji}) must also be invertible. It is a square matrix of order equal to the number of eigenvalues at the origin, S.

The B_{ji} matrix also has physical significance. In particular, the θ_k must be capable of cancelling the forces on the secular terms (C^0_{jk}); and the undetermined constants, $\bar{r}_i$, must be included in the trim constants H_{li}. This implies that the normal rigid body translations, that usually have zero eigenvalues, should be included as constraints on position in the form of g_l. The same is true for the rotor azimuth angle when there are drive-train dynamics. Rigid-body rotations usually couple with translations as non-zero eigenvalue, flight-mechanics modes so that they do not result in zero eigenvalues. Other zero eigenvalues, associated with *N/rev* parametric instabilities would not be trimmable unless a control and a g_l were placed on the unstable mode.

3.4. Summary

For a system to be trimmable, the following matrices must be non-singular:

1. $L \times L$ Solvability matrix, D_{ij}, Eq. (12), where L = number of Lagrange multipliers.
2. $K \times K$ trimmability matrix, A_{lk}, Eq. (25), where K = both the number of controls and the number of trim constraints.
3. $S \times S$ secular-term matrix, B_{ji}, Eq. (31), where S = number of zero poles with independent eigenvectors, ϕ_{ij}.

Of course, as a nonlinear iteration progresses toward a "solution", these matrices may become singular as physical limits are approached. Thus, we can say that, if a trimmed solution exists, these matrices will be invertible for perturbations about that solution.

Last, it is easily seen from Eq. (27) why a sequential trim cannot converge for free flight. In a sequential scheme, the off-diagonal matrices are neglected, and the upper and lower diagonal partitions are solved in sequence. If any roots exist with zero eigenvalues, the lower-left portion is singular. Therefore, any general trim algorithm must consider periodicity and trim constraints in parallel.

4. OPTIMAL TRIM

4.1. Background

In the previous section, we developed trimmability conditions. Two of those conditions were that the number of controls must equal the number of control constraints and that the trimmability matrix, A_{lk}, must be non-singular. When the number of controls is greater than the number of trim constraints ($K > L$) and the row rank and column rank of A_{lk} are L and K, respectively, the problem is "over-trimmable" and there are an infinite numbers of ways to trim. This opens up the possibility of trimming with some additional quantity maximized or minimized. Thus, in this section, we consider trimming a rotor subject to finding an extremum for some functional, the optimum value being G_0.

$$G_0 = \frac{1}{T}\int_0^T g_0(x_j, \lambda_m, \theta_k)dt + h_0(x_i, \lambda_m, \theta_k) \tag{32}$$

The functional is taken in the same form as the trim constraints in Eq. (11) where the average of g_0 is to be optimized (such as power, climb rate, attitude, etc.); or h_0 is any point value to be optimized at $t = 0$ (although this latter case would not be common).

4.2. Formulation

The formulation of the problem is based on the calculus of variations. Although this is not optimal control in the usual sense [41] (since the "controls" are not functions of time), the formulation takes on the flavor of optimal control. To begin, we adjoin the differential equations, Eq. (1), the multi-body constraints, Eq. (2), the quasi-periodicity conditions, Eq. (9), and the trim constraints, Eq. (11), to the functional in Eq. (32). This forms the optimality functional, W;

$$W = \int_0^T (f_i - \dot{x}_i)y_i(t)dt + Y_i[x_i(T) - P_{ij}x_j(0) - z_i] + \int_0^T F_m \mu_m(t)dt$$
$$+ r_L\left[\frac{1}{T}\int_0^T g_l\,dt + h_l - G_l\right] \quad i,j = 1,N; \quad m = 1,J; \quad l = 0,L \tag{33}$$

where $y_i(t)$ are introduced Lagrange multipliers (called co-states), Y_i are constant co-periodicity multipliers, the $\mu_m(t)$ are co-multipliers corresponding to the λ_m, σ_l are co-controls, and $\sigma_0 \equiv 1$.

Next, we take the variation of Eq. (33) and set it to zero. Collection of terms with like variations gives the optimality conditions. The $\delta y_i(t)$ terms and $\delta\mu_m(t)$ terms recover the original differential equations and multi-body constraints, Eqs. (1) and (2). The δY_i terms recover the periodicity conditions, Eq. (9); and the $\delta\sigma_l$ conditions recover the trim constraints, Eq. (11). Thus, these are the same as in the conventional trim. The added equations for optimal trim arise from the δx_i, $\delta x_i(0)$, $\delta x_i(T)$, $\delta\theta_k$, and $\delta\lambda_m$ terms. To do this, the $\delta\dot{x}_i$ terms in Eq. (33) must be integrated by parts as follows:

$$\int_0^T - \delta \dot{x}_i y_i dt = \delta x_i(0) y_i(0) - \delta x_i(T) y_i(T) + \int_0^T \delta x_i \dot{y}_i dt \tag{34}$$

When this is done, we can collect the appropriate terms.

First, we look at the δx_i equations. From Eqs. (33) and (34), these are

$$\dot{y}_i = -\frac{\partial f_j}{\partial x_i} y_j - \mu_m \frac{\partial F_m}{\partial x_i} - \sigma_l \frac{\partial g_l}{\partial x_i} \cdot \frac{1}{T}$$

$$i = 1, N; \qquad l = 0, L; \qquad m = 1, J \tag{35}$$

These are the co-state differential equations. Next, we consider the $\delta \lambda_j$ equations with the same repeated-index sums as in Eq. (35).

$$y_i \frac{\partial f_i}{\partial \lambda_j} + \mu_m \frac{\partial F_m}{\partial \lambda_j} + \sigma_l \frac{1}{T} \frac{\partial g_l}{\partial \lambda_j} = 0 \tag{36}$$

These are the co-multiplier equations. Third, we take the $\delta x_i\,(T)$ and $\delta x_i\,(0)$ equations

$$\delta x_i\,(T)\colon Y_i - y_i\,(T) = 0, \; Y_i = y_i\,(T) \tag{37}$$

$$\delta x_i\,(0)\colon -y_j\,(T) P_{ji} + y_i\,(0) + \sigma_l \frac{\partial h_l}{\partial x_i(0)} = 0 \tag{38}$$

However, since $P^T = P^{-1}$, we can rewrite Eq. (38) in the same form as the x_i equations.

$$y_i\,(T) = P_{ij}\, y_j\,(0) + P_{ij}\, \sigma_l \frac{\partial h_l}{\partial x_j\,(0)} \tag{39}$$

Equation (39) then becomes a periodicity constraint on $y_i(t)$ just as Eq. (9) is for $x_i\,(t)$.

Last, we have the $\delta \theta_k$ equations. These involve some terms arising from the fact that T can depend on θ_k (e.g., when Ω is one of the controls).

$$\frac{1}{T} \int_0^T (T y_i\,[\partial f_i / \partial \theta_k] + T \mu_m\,[\partial F_m / \partial \theta_k]$$

$$-\sigma_l g_l [\partial T / \partial \theta_k] / T + \sigma_l\,[\partial g_l / \partial \theta_k])\, dt + [\partial z_i / \partial \theta_k]\,[P_{ij}]\,[y_j(0) + \sigma_l\,(\partial h_l / \partial x_j(0))]$$

$$+ \sigma_l g_l\,(T)[\partial T / \partial \theta_k] / T = 0 \tag{40}$$

4.3. Interpretation

Equations (35), (36), (39), and (40) form the additional equations that are the optimal trim conditions. Fortunately, these equations are an exact analog to Eqs. (1), (2), (9), and (11) such that optimal trim can be treated as a conventional trim but with $2N$ states (x_i and y_i), $2\,J$ multipliers (λ_m and μ_m), and $K + L$ controls (θ_k and σ_l). The added equations are linear in the co-variables, and the co-variables do not affect the original equations. This can lead to numerical advantages.

In Eq. (35) we see the co-state equations. The right-hand side is the equivalent of f_i. Note that the matrix multiplying y_j is the negative transpose of the corresponding matrix in the δx_i equations, Eq. (13). Thus, the eigenvalues of the co-states are the negatives of those of the states. Thus, stable x_i imply unstable y_i (and *vice versa*); and y_i will have the same number of zero roots at the origin as does x_i. (This implies it can be advantageous to time march from T to 0 when solving for $y_i(t)$). Equation (36), the co-multiplier equations, is exactly in the form of the F_j.

For periodicity, we have already seen that Eq. (39) is in the form of Eq. (9) with the last term (a function of σ_l) being the analog of $z_i(\theta_k)$, quasi-periodicity. Last, Eq. (40) is seen as the sum of integral and point constraints like the g_l and h_l of Eq. (11).

Since optimum trim cam be put in the same form as conventional trim (even for multi-blade trim), the same solvability and trimmability conditions can be applied. Also, since there are $K + L$ controls and $K + L$ trim constraints, the trimmability matrix is always square. When it is invertible, it implies that the controls are capable of enforcing the trim constraints and of finding an extremum of the function to be optimized.

Of course, the practicality of optimum trim is dependent upon the ability to take the derivatives required. Because many solution techniques will compute Jacobians anyway, these derivatives may be available numerically without too much extra work. Otherwise, explicit analytic derivatives or finite-difference derivatives would be required.

5. APPLICATIONS

5.1. Classes of Solution Methods

The natural division of the trim equations into four categories implies that solution techniques can be organized on the basis of how they approach each condition. That is what we will do in this section. Since optimal trim can be written in the same form as conventional trim, we consider conventional trim with no loss of generality. The four types of equations are: 1) differential equations, Eq. (1); implicit equations, Eq. (2); quasi-periodicity conditions, Eq. (9); and trim constraints, Eq. (11). Every general trim algorithm must address each of these.

In essence, any solution method for the differential equations is some version of the method of weighted residuals [26]. That is, one takes trial functions as approximations to x_j and test functions y_{ji} and forms residual error functionals, e_i.

$$e_i = \int_0^T (f_j - \dot{x}_j) y_{ji}\, dt \tag{41}$$

By limiting the functional spaces of x_i and y_{ji}, one can set the e_i to zero. In conventional time marching, the test and trial function are of the order of the method (e.g., fourth-order Runge-Kutta) and are defined over short intervals. The same is true of h-version time finite elements. In some methods, Eq. (41) may be integrated by parts to make x_i weak or strong [28] or to obtain an action or a Hamiltonian. In some methods, the test and trial functions are defined over the entire range, such as in p-version time finite elements. Reference [28] shows that the Fourier series method is a special case of a strong p-version finite element.

In addition to the choice of functions, methods based on Eq. (41) must have an algorithm for iteration. In essence, one needs a Jacobian of the derivatives of e_i with respect to the unknown coefficients of the trial functions. This can be done either numerically (by finite differences), analytically (when possible), or approximately by segregation of some linear terms to form a matrix which is an approximation to the Jacobian. All methods (whether Fourier series, predictor-corrector, time finite elements, etc.) use some form of exact or approximate Jacobian in this way. Thus, all equation solvers are defined by the answers to the questions: 1) what are the trial functions, 2) what are the test functions, and 3) how is the Jacobian formed? When equations are implicit or involve multi-body constraints, there are trial and test functions for λ_m and in Eq. (2) just as in Eq. (41).

Next, we consider solution methods for satisfaction of periodicity. Although many methods exists, they can all be categorized as either enforced periodicity (i.e., periodicity by assembly) or transfer-matrix methods (i.e., periodicity by iteration on initial conditions). In the former method, the class of trial functions is restricted to those which are periodic. For example, all Fourier test functions are periodic; and time finite elements assemble the $t = 0$ and $t = T$ elements to enforce periodicity. Methods weak in displacement add the periodicity weakly to Eq. (41) [28].

In transition matrix approaches (such as periodic shooting or transfer-matrix time marching), periodicity is also weakly enforced; and the new guess on initial conditions is based on the inversion of a Jacobian that is the identity matrix (I) minus the transition matrix $\Phi(T)$. The transfer matrix approach has an advantage over assembly in that the Floquet transition matrix is automatically obtained from the trim iteration. Although there is also a Jacobian for assembly methods (usually done approximately by segregation of linearized terms), the periodicity constraint eliminates the Floquet information.

The computational trade-offs involve the solution of smaller dense matrices (p-version and Fourier) versus larger sparse matrices (h-version with enforcement by assembly), or versus very small transfer matrices but with multiple solutions due to iteration (shooting). The "best" method depends on the size of the problem, nonlinearity of the problem, and codes available.

The solution methods for the trim constraints also must involve some sort of Jacobian. Trim values are either integrated or point values; thus, no weighted residuals are required. Instead, one needs the Jacobian of the error in each G_l due to changes in each θ_k. However, since θ_k affects the solution x_j (which also appears in g_l or h_l), the solution of the trim equations must be coupled with the solution of the differential equation. As with the other parts of the problem, solution strategies can be classified as to how they compute this Jacobian.

5.2. Sequence of Analyses

Methods may also be categorized by the sequence of the various analyses. This concept has already been introduced above in the analysis of periodicity conditions. Assembly methods are parallel solutions of the differential equations and periodicity. Thus, initial conditions are found simultaneously with the solution. Transition-matrix methods, on the other hand, solve the equations and initial conditions sequentially. For systems with a zero eigenvalue (i.e., an eigenvalue of $\Phi(t)$ equal to unity), neither method can converge to a periodic solution (when used in sequence with trim constraints) due to the singularity of the Jacobian, Eq. (27). Thus, it would seem that general methods must place initial conditions and controls in parallel (although the differential equations may be solved in series). Based on Eq. (27), it might be possible to separate only the zero-eigenvalue modes into the parallel algorithm, leaving all other initial conditions sequential. For example, one could do a singular value decomposition of the initial-condition Jacobian and move the zero-frequency initial conditions to the control iteration. This idea is an example of how the theory here offers a theoretical framework for designing trim approaches.

The presence of zero-frequency modes also impacts methods that use assembly, since such modes may be only quasi-periodic and not truly periodic. Thus, the assembly of t = 0 and $t = T$ for such systems must include an offset, z_i. In Fourier methods, this implies addition of an extra term in the series

$$x_i = a_0 + b_0(t/T - 1/2) + \sum a_n \cos(n\Omega t) + b_n \sin(n\Omega t) \tag{42}$$

where $b_0 = z_i$ unless the offset is removed by a suitable change of variable.

The method of time marching until all transients decay is a method of iteration on initial conditions with a very approximate Jacobian. (The values of $x_i(T)$ are used as the next guess for $x_i(0)$). It also diverges for zero eigenvalues or when any eigenvalue has a negative real part. However, for stable systems without instabilities it can be very efficient. For numerical comparison of some of these methods, see [42].

5.3. Auto-Pilot Trim

The method of auto-pilot trim is to take some or all of the controls and allow them to be functions of time. Thus, they cease to be control parameters and become states. Then, the corresponding trim constraints are removed and replaced with differential equations for the new states (i.e., new f_i's). These state equations must be such as to eliminate the errors in the lost trim conditions. Thus, one simple auto-pilot model is

$$\tau\ddot{\theta}_k + \dot{\theta}_k = a[A_{lk}]^{-1}\{G_l - g_l\} \tag{43}$$

where τ is an optional time constant (usually of the order of T) and "a" is a gain (usually of the order of $1/T$, [32]. Since auto-pilots require the inverse of A_{lk}, the trimmability matrix is an important aspect of auto-pilots. This suggests that a combination of auto-pilot and periodic shooting could be made in which the Jacobian from a periodic-shooting cycle could then be used for the auto-pilot.

When the system contains zero eigenvalues, the auto-pilot in Eq. (43) will not converge because it is basically an integral controller. Proportional control must be added for a zero root, rate control for a once-repeated zero root (no spring or damper), etc.

$$\tau\ddot{\theta}_k + \dot{\theta}_k = [A_{lk}]^{-1}\{aG_l - ag_l - b\dot{g}_l - c\ddot{g}_l\} \tag{44}$$

However, the b and c terms need only be added to g_l that are either damped rigid-body translations or undamped rigid-body translations, respectively.

It should be noted, however, that the auto-pilot method of trim changes the flight-mechanics dynamics. Thus, it cannot be used if the flight mechanics modes are desired. However, the auto-pilot could be used to get close to exact trim and then followed by periodic shooting with the auto-pilot removed to finalize the trim. This offers the possibility of a combined approach in which periodic shooting and auto-pilot are used iteratively to adaptively change A_{kl} (through the Jacobian) and then complete trim with the auto-pilot off.

6. NUMERICAL EXAMPLES

6.1. Model

In order to demonstrate the various aspects of the trim theory and to test various trim strategies, we have set up a trim test bed program. A schematic of the model is shown in Fig. 1. The example helicopter consists of a four-bladed main rotor, four-bladed tail rotor, fuselage, engine/drive-train system, and a horizontal stabilizer. The ratio of tail rotor Ω to main-rotor Ω is 5. The elasticity of rotor blades is represented at the root by a set of

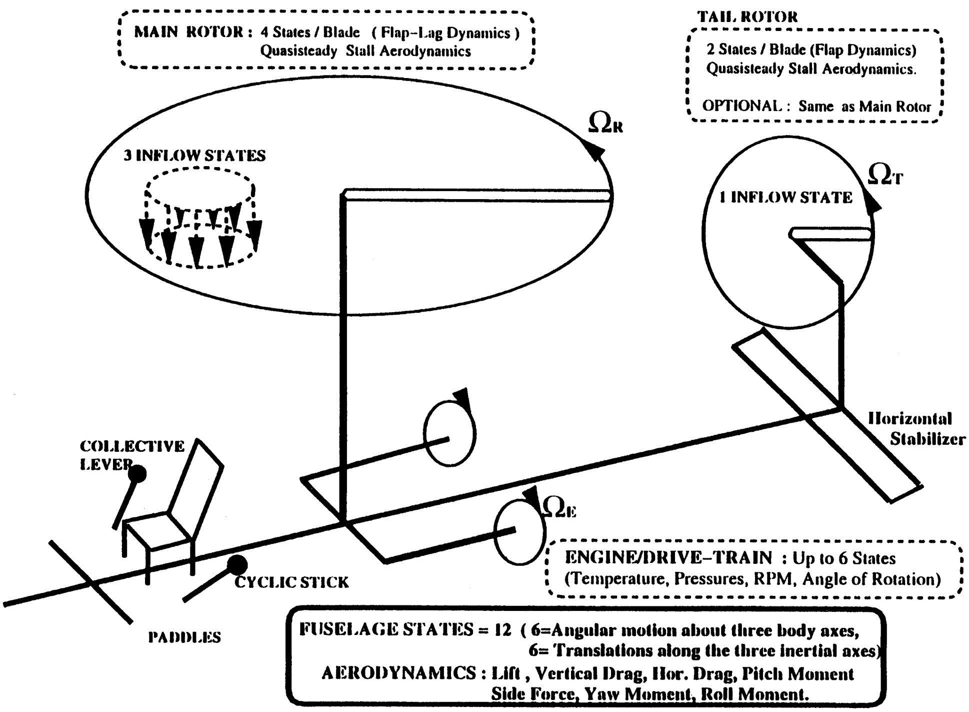


Figure 1 Schematic of rotor-body trim model.

flap and lag springs, and the fuselage is considered rigid. The main-rotor blades have flap and lag degrees of freedom, but the tail-rotor blades have only flap degrees of freedom. The fuselage has three large-angle degrees of freedom for orientations (Λ, Θ, Φ) and three degrees of freedom for translations.

The drive train has either 0 states (fixed Ω) or 2 states (angle Φ and rotation rate Ω of the hub), and the engine has from 0 to 4 states depending on the model. A zero-state engine has only torque as input or output. Higher-order models can include temperatures, pressures, and compressor speeds as states [43], [44]. There are 3 dynamic inflow states for the main rotor and 1 uniform dynamic inflow state for the tail rotor. Thus, the total count of state variables is:

Main Rotor	16
Tail rotor	8
Inflow	4
Fuselage	12
Drive Train	0–2
Engine	0–4
TOTAL	40–46

The rotor blades have NACA 0012 airfoils with quasi-steady aerodynamics including stall. The fuselage has full 3-D aerodynamics including inflow impingement from the main rotor. Other aerodynamic interferences are neglected. The numerical values chosen for the examples resemble the example helicopters of Ref. [40]. They are typical of existing helicopters and are given in Table 1. Finally, the effects of altitude on air density have been included in the computations.

6.2. Trim Conditions

The control parameters are collective pitch (θ_0), cyclic pitch (θ_s, θ_c), tail rotor pitch, (θ_T), throttle position (θ_e), and rotor period (T). With no engine-drive-train dynamics, the throttle position is eliminated and there is constant RPM; but, with these dynamics, there is an unsteady RPM and there are four possible combinations of the engine variables, throttle and RPM:

1. T and θ_e both specified (untrimmable)
2. T specified, θ_e unknown (trimmable)
3. T unknown, θ_e specified (trimmable)
4. T and θ_e unspecified (overtrimmable)

Condition 1 is not feasible. Condition 4 can be made into an optimal trim problem, for example, by minimization of power.

For the trimmable cases (2 and 3 above), the entire rotorcraft system has 5 controls (θ_0, θ_s, θ_c, θ_T, and θ_e or T). These would be the usual controls for a conventional helicopter. It follows from trimmability that there must be 5 trim constraints. Furthermore, since this

Table I Basic Parameters in Trim Calculations of Example Helicopter

Ambient Air Density at MSL, *slugs/ft^3*		=	0.002378
Vehicle Gross Weight, *lbs*		=	20000
Vehicle C.G. from Fuselag C.G., *ft*		=	{0.205, 0.007, 0.597}
Desired Linear speeds, *knots*		=	{115, 0, 0}
Description		**Rotor-1**	**Rotor-2**
Number of blades		4	4
Radial stations		8	8
Nominal rotor speed, *rpm*		210.000	1050.000
Blade radius, *ft*		30.000	6.000
Aerodynamic root-cutout, *ft*		4.500	0.900
Effective flap-lag hinge location, *ft*		3.000	0.800
Blade chord, *ft*		2.000	0.810
Blade inertia about hinge, *slugs-ft^2*		2870.000	6.250
Spring stiffness (flap), *ft-lbf/rad*		100000.000	15800.000
Spring stiffness (lag), *ft-lbf/rad*		450000.000	152500.000
Lag-damping coeff., *ft-lbf/rad-sec*		1500.000	40.000
Linear twist, *deg*		-10.000	-5.000
Airfoil static stall angle, *deg*		12.000	12.000
Airfoil lift-curve slope, *1/rad*		5.730	5.730
Airfoil minimum drag coeff.		0.008	0.008
Distance of rotor center from fus. C.G, *ft*		{0.5,0,7.5}	{37.5,1.5,6}
Rotor weight, *lbf*		1521.218	89.312
Fuselage Data			
Characteristic aerodynamic drag area, *ft^2*		=	{20, 120, 100}
Characteristic aero. volume {roll, pitch, yaw}, *ft^2*		=	{230, 1800, 810}
Characteristic area for lift in *z*-direction, *ft^2*		=	75.00
Incidence at zero lift, *deg*		=	5.000
Characteristic area for side force, *ft^2*		=	300.0
Aspect ratio of hor. stabilizer		=	4.000
Aero. area of hor. stabilizer, *ft^2*		=	20.000
Lift-curve slope for hor. stab., *1/rad*		=	5.730
Min. drag coeff. for hor. stab.		=	0.008
Distance of H. S. from fus. C.G., *ft*		=	{35,0,-1.5}
Fuselage weight, *lbf*		=	18389.470
Engine/drive-train inertia, *slugs-ft^2*		=	1500.000
Engine/drive-train damping factor η		=	0.667
Fuselage inertia matrix, *slugs-ft^2* =	4300	0	0
	0	37900	0
	0	0	33600
{.., .., ..} = {-Y, X, Z}			

is a free-flight model, the system has 4 roots at the origin (3 body translations and the relative rotation between rotor and body). Therefore, 4 of the 5 trim constraints must involve these 4 quantities. The most natural choice for these 4 trim constraints would consequently be the time-average of the X, Y, Z fuselage positions (set to zero) and a point constraint that $\psi(0) = 0$. This leaves one more necessary trim constraint for trimmability.

For this, one might choose the time average of either roll, pitch, or yaw (although, in hover, yaw is the only choice). In forward flight, one might rather choose some function to be optimized, such as the Euclidean norm of roll-pitch-yaw or the power required. Here, we investigate all of these possibilities.

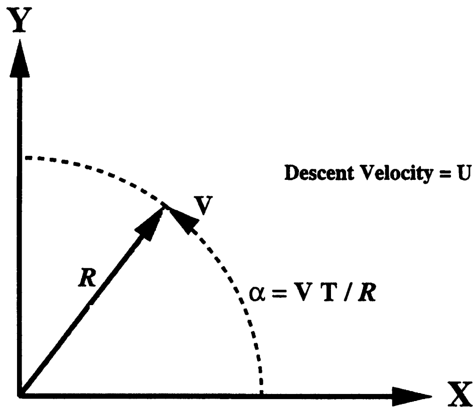
The periodicity conditions are of the simple variety, Eq. (6), except for one of the fuselage degrees of freedom. For level flight, the Y component (forward speed) would have $Y(T) - Y(0) = VT$ as quasi-periodicity. A constant-acceleration case (such as a steady descending turn) would probably best be handled in two steps. First, use point constraints rather than time averages for the Z, Y, Z constraints (i.e., use initial conditions). Then, use the forward speed, V, the descent speed, U, and the radius of the turn, $\bar{R}$, to define the quasi-periodicity conditions as shown in Fig. 2. The periodicity conditions and the fuselage velocity states ($\dot{X}$, $\dot{Y}$, $\dot{Z}$, $\dot{\Lambda}$, $\dot{\Theta}$, $\dot{\Phi}$) are of the simple kind, the state at $t = 0$ equals the state at $t = T$. As in level flight, the additional constraint could be placed on roll, pitch, or yaw; or one could choose to minimize power. For a coordinated turn, the extra constraint would be placed on roll angle, Fig. 2. It should also be noted that, for some definitions of Euler angles (in a turning trim), the periodicity constraints on Λ, Θ, Φ might become nonlinear so that $z_i(\theta_k)$ would become $z_i(\theta_k, x_j)$. Although we did not include this case in the preceding proofs, it is easily incorporated. The same effect could occur in multi-body formulations with a single global frame [45].

6.3. Test Cases

All of the numerical results to follow are from the method of periodic shooting. These involve 40–42 states and 4–6 controls for regular trim and 80–84 states and 7–11 controls (including co-controls) for optimal trim. Thus, the shooting Jacobian is of order 44 to 95 depending on the case. Figures 3 and 4 show the results of a level-flight trim with a prescribed time-averaged yaw angle, Λ. Plotted in Fig. 3 are the resultant controls (there are no engine dynamics); and plotted in Fig. 4 are the resultant pitch angle, roll angle, and two angle norms.

Notice that there are two different zero-pitch solutions, -30° and $+28^\circ$. If Λ is continued to $\pm 180^\circ$, there are actually four such solutions. Therefore, as a test, we attempted to trim with the condition $\Theta = 0$ rather than with a specified yaw angle. As expected, the periodic shooting could settle on any one of the four possible solutions, depending on the first guess. This is in agreement with [33] which notes the existence of multiple trim points. Also, a case was run in which roll angle was constrained to be zero (a probable pilot's choice). Two possible solutions can be found. One is $\Lambda = 2.0^\circ$, as seen in Fig. 4, and the other is a rearward flight case. Notice that Figs. 3 and 4 are not symmetric with Λ due to the tail-rotor inflow dynamics. We have also successfully trimmed for the case of lost tail rotor in which θ_T and tail rotor thrust are removed along with the trim constraint on pitch, roll, or yaw.

Next, we ran optimal trim cases for this same configuration with several different functionals, G_0. For functionals, we chose, $\sqrt{\Theta^2 + \Phi^2}$, $\sqrt{\Lambda^2 + \Theta^2 + \Phi^2}$, and power required. In precise agreement with Fig. 4, the optimal trim algorithm found minima for the first two functionals at $\Lambda = 2.0^\circ$ and $\Lambda = 0.5^\circ$, respectively. When power required was taken as the functional, the optimal trim provided a minima at $\Lambda = +0.5^\circ$ in confirmation of Fig. 5, which shows power versus yaw angle Λ.



$$X(T) - X(0) = -(1 - \cos\alpha) R$$

$$Y(T) - Y(0) = \sin\alpha\ R$$

$$Z(T) - Z(0) = -U T$$

$$\Lambda(T) - \Lambda(0) = \alpha$$

$$\Theta(T) - \Theta(0) = 0$$

$$\Phi(T) - \Phi(0) = 0$$

$$X(0) = R \ ; \ Y(0) = 0 \ ; \ Z(0) = 0$$

$$(1/T) \int_0^T \Phi \, g \, dt = V^2 / R$$

Figure 2 Quasi-periodicity for steady descending turn.

The final cases to be presented are with two-state drive-train dynamics of the form

$$\dot{\psi} = \Omega \tag{45}$$

$$\dot{\Omega} = -\eta\Omega + \theta_e - Q_e \tag{46}$$

Trim problems have been solved for T known and θ_e unknown; as well as for θ_e and T both unknown for a specified power; and optimal trim problems have been solved with neither known. Figures 6–8 show results for trimming to a prescribed average value of Ω (i.e., T known). The average yaw angle is constrained to be zero. As RPM decreases, θ_0 increases; and the blades begin to stall, Fig. 6. Thus, below 175 RPM the system is untrimmable. Figure 7 shows that pitch and roll angles also begin to diverge as 175 RPM is approached.

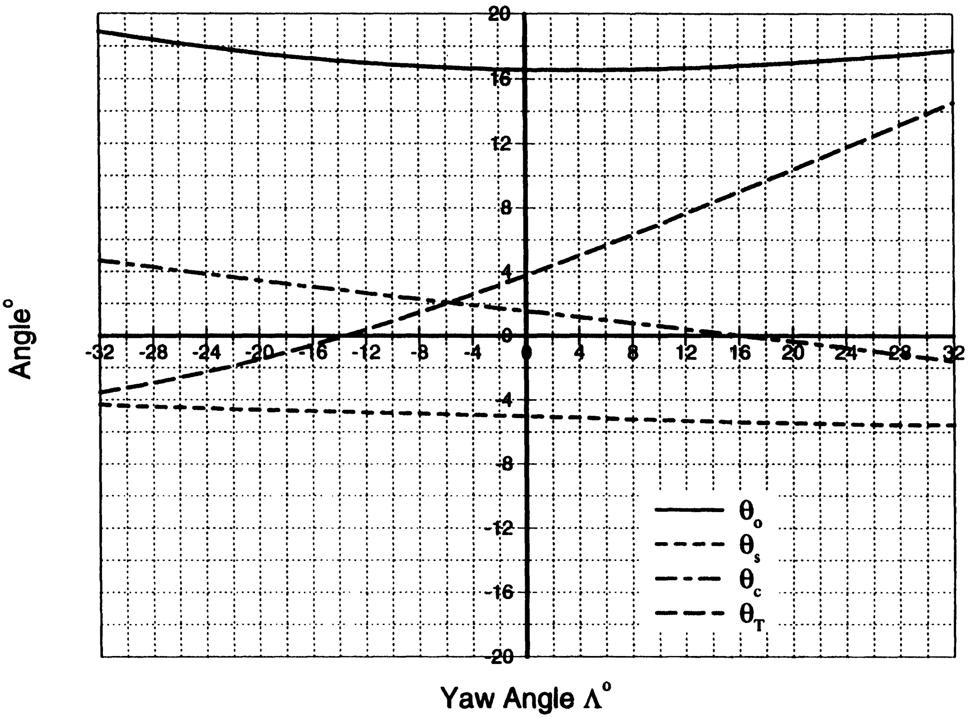


Figure 3 Trimmed pilot controls versus fuselage yaw angle at nominal rotor RPM.

Figure 8 shows the power required from the engine at each case. The power also becomes large as Ω approaches 175 RPM due to the increase in drag associated with stall. The same results would be found for θ_e given, Ω unknown. Thus, if one were to trim to minimum power (or to a fixed power) with Ω as the unknown, these low RPM values would never be reached.

Figure 8 shows that power also increases at high RPM due to the fact that θ_0 decreases as Ω increases. In other words, induced power remains fairly constant with Ω, while profile power increases as Ω^3. Thus, there is an optimum RPM. To test the optimal trim theory, we ran an optimal trim case with power as the objective function. The minimum point at 200 RPM is reached when periodic shooting is applied to the optimal trim equations. When $\Lambda = 0$ is removed as a constraint, the optimal trim equations find the combination of Λ and Ω that gives minimum power.

6.4. Numerical Issues

All of the computations were performed on a SGI-INDY workstation in double-precision. For numerical integration of dynamic equations, a predictor-corrector algorithm (with variable order and variable step-size) was employed. The CPU time requirement for one iteration of the Newton-Raphson algorithm, is 20 minutes for non-optimal trim and 360

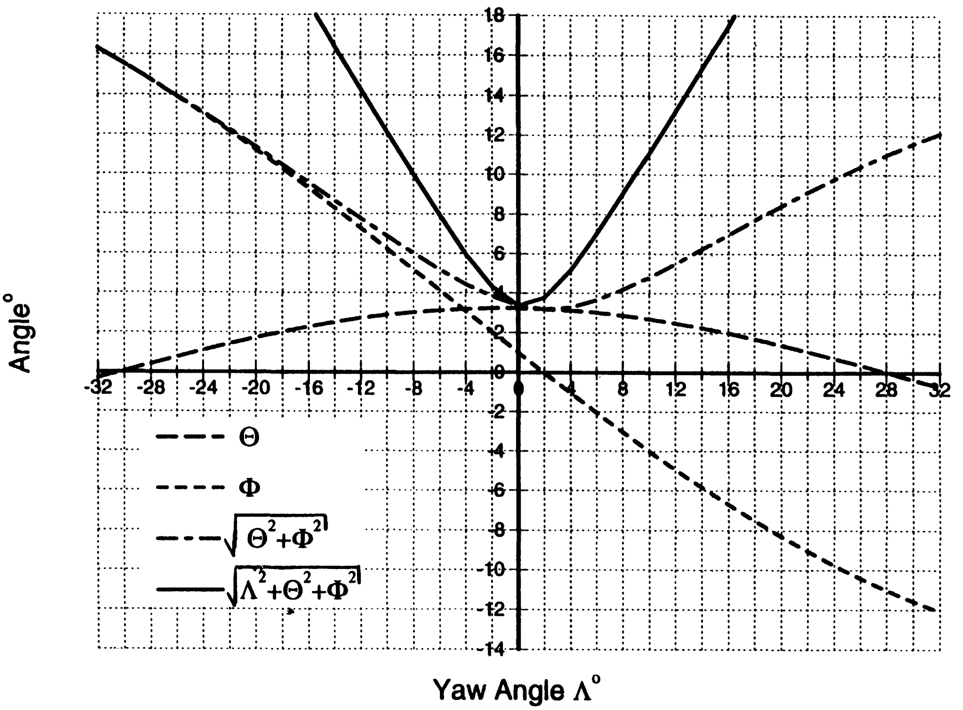


Figure 4 Trimmed fuselage angles versus fuselage yaw angle at nominal rotor RPM.

minutes for optimal trim. These CPU requirements are governed by the accuracy of the sought solution. The number of iterations required to reach the final solution depends on the initial guess and the trim-definition. For example, trimming to zero fuselage pitch angle is much more difficult with an arbitrary initial guess than is trimming to zero yaw angle. Similarly, optimal trim is much more difficult with an arbitrary initial guess. However, with a judicious initial guess based on previously found trim-solutions of zero yaw angle, trimming takes about 5–7 iterations. In our applications, we non-dimensionalize time which simplifies the optimum trim. Considering the fact that we did not take advantage of multi-blade trim (which would give another factor of 4 savings in CPU), and also considering the complexity of trim solutions we are trying to achieve, the computational requirements are reasonable. Moreover, the trimming algorithm is readily amenable to parallel computing, which could bring in tremendous reduction in CPU time as demonstrated in [46] with respect to Floquet stability analysis.

We also used Floquet theory (on the partition of the periodic-shooting Jacobian that is the transition matrix) to find system eigenvalues. These included the four zero-frequency modes (three translations and engine angle) as well as the other 3 flight mechanics modes, some of which are unstable. (Actually, the vertical translation mode is not exactly at the origin due to the dependency of air density on altitude.) These, however, were no problem to trim as long as we used the parallel approach. This is similar to the results of [33] in which a parallel trim method (harmonic balance with augmented trim equations) allowed

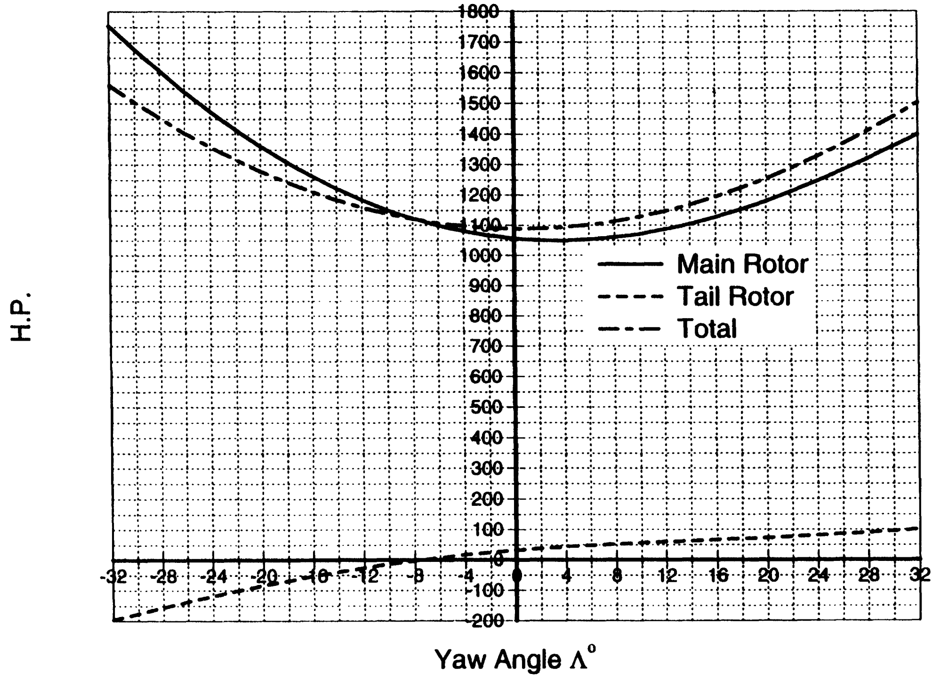


Figure 5 Trimmed aerodynamic power versus fuselage yaw angle at nominal rotor RPM.

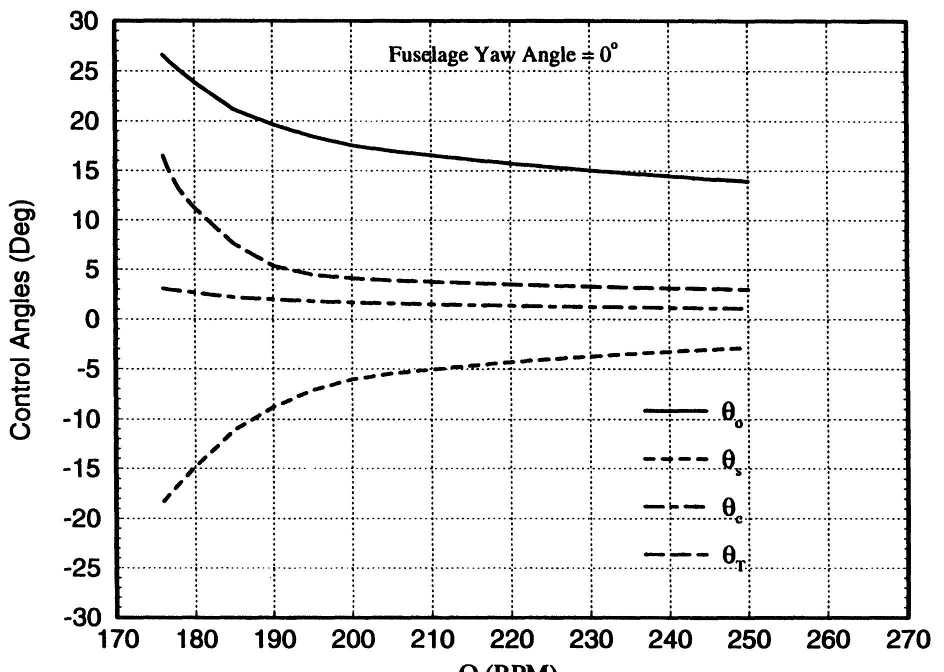


Figure 6 Pilot control angles versus main rotor RPM with drive-train dynamics.

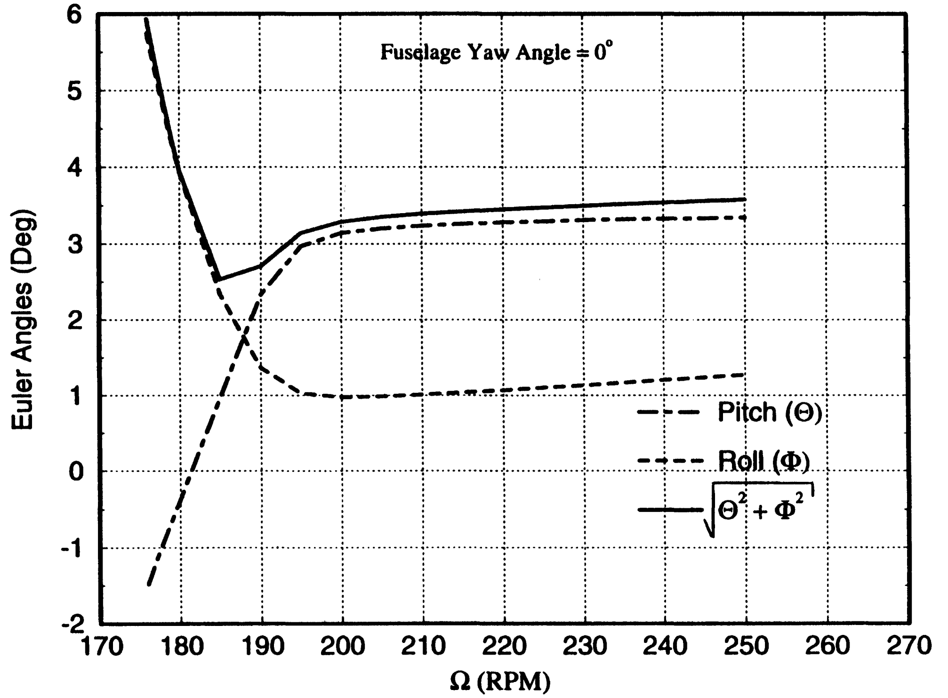


Figure 7 Trimmed fuselage angles versus main rotor RPM with drive-train dynamics.

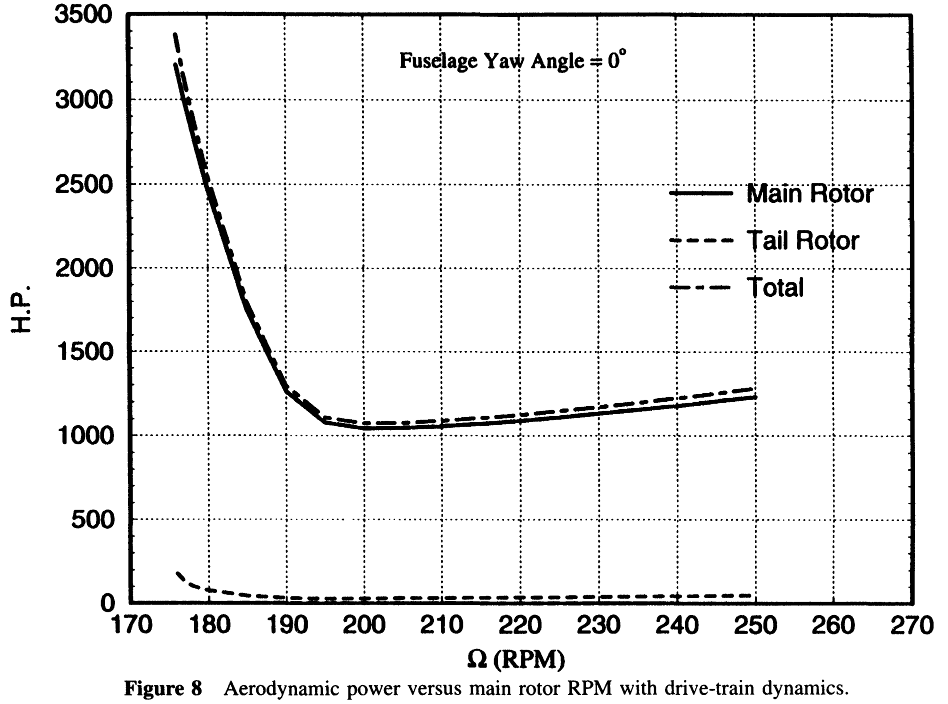


Figure 8 Aerodynamic power versus main rotor RPM with drive-train dynamics.

a unified handling-qualities and aeroelasticity analysis. Also, because flapping states are highly damped, the flapping co-states are highly unstable. Thus, we found it useful (although not necessary) to time march co-states from T to 0 rather than from 0 to T to suppress the instabilities.

7. SUMMARY AND CONCLUSIONS

The problem of rotorcraft trim has been set in a general mathematical formulation in terms of explicit equations, implicit equations, quasi-periodic conditions, and trim constraints, Eqs. (1), (2), (9), and (11). In order for trim to be achievable, the solvability matrix, the trimmability matrix, and the secular term matrices must be square and non-singular, Eqs. (12), (25), and (31). This implies that free-flight conditions cannot be trimmed by sequential strategies in which periodicity and controls are nested. A parallel trim strategy is required. Optimal trim, in which a parameter is minimized or maximized, can be placed in a form identical to that of conventional trim but with twice as many states. The above hold true independent of the types of equations or of the particular solution strategies.

All trim algorithms can be classified by how they treat each of the aspects of trim. For the solution to the differential equations and implicit equations, all methods are variations on the method of weighted residuals. For periodicity, all methods employ either constrained periodicity or transfer-matrix iteration. For trimmability, all are weighted-residual methods either in parallel or in sequence. Auto-pilot methods are a means of changing controls to state variables after which the above theorems still apply. Numerical results with periodic shooting demonstrate some of these conclusions.

One important aspect of the trim theory presented here is that it allows multi-blade trim for single- or multi-rotor aircraft, and this can result in a significant savings in computer time. Another important point is that, although this trim theory is very general, it does not mandate that every trim methodology need be completely general. Special cases that allow simplifications should certainly be utilized, and the theory here does not preclude such simplifications.

8. ACKNOWLEDGEMENTS

The basic trim theory and numerical examples were done under the Georgia Tech/ Washington University Center of Excellence for Rotorcraft Technology, jointly sponsored by the Army Research Office, Bell Helicopter Textron, and United Technologies-Sikorsky, Contract DAAL03-93-G-0002. The extensions to optimal trim and unsteady rotor speeds were done under ARO grant No. DAAHO4-94-6-0351, Gary Anderson Technical Monitor.

9. NOMENCLATURE

a, b, c	auto-pilot gains
a_n, b_n	Fourier coefficients
A_{ij}	trimmability matrix
B_{ij}	secular-term matrix

C^n_{ij}	Fourier coefficients
D_{ij}	solvability matrix
e_i	error vector
f_i	explicit state functions
F_i	implicit state functions, multi-body constraints
g	acceleration of gravity, ft/sec^2
g_l	constraint functions
G_l	desired values of constraints
h_l	point constraints (t = 0)
H_{li}	zero-pole constraint matrix
i	$\sqrt{-1}$ except when used as subscript
I	identity matrix
i, j, k, l, m	indices
J	number of Lagrange multipliers
K	number of controls
L	number of trim constraints
M	number of harmonics
M_i, N_i	smallest factors
n	harmonic number
N	number of states
p_{ij}	permutation matrix
P	total power, h.p.
P_{ij}	multiblade matrix
Q_e	rotor torque/engine inertia, 1/sec^2
Q_i	number of symmetric sectors, *ith rotor*
r_i	transformed, perturbed states
R	number of rotating interfaces
$\bar{R}$	radius of turn, ft
S	number of zero eigenvalues with discrete eigenvectors
t	time, sec
T	period, sec
U	rate of descent, ft/sec
V	horizontal velocity, ft/sec
W	adjoined functional
X, Y, Z	body translations, ft
x_i	states
y_i	co-states
z_i	quasi-periodicity movements
α	angle of turn, rad
$\delta(t)$	impulse function
δ_{ij}	Kronecker delta
η	drive-train damping
η_i	characteristic exponents
θ_k	control variables
Θ	pitch angle, positive nose down, rad
θ_0	collective pitch, deg
θ_s, θ_c	cyclic pitch, deg

θ_T	tail rotor pitch, deg
θ_e	engine control (throttle)
λ_m	Lagrange multipliers
Λ	yaw angle, positive counter-clockwise, rad
Λ_j	eigenvalues of Floquet Transition matrix
μ_m	co-multipliers
σ_k	co-controls
τ	time constant, sec
ϕ_{ij}	right-hand eigenvectors
Φ	roll angle, positive advancing side up, rad
$\Phi(T)$	Floquet transition matrix
$\psi(t)$	hub rotation angle, rad
ψ_{ji}	left-hand eigenvectors
Ω, $\bar{\Omega}$	average system rotation rate, rad/sec
Ω_i	average speed of ith rotor
$\bar{\Omega}_i$	multiblade Ω_i;$\bar{\Omega}_i = Q_i\Omega_i$
$(\hat{\ })$	( ) with λ_j removed explicitly
$\delta(\ )$	perturbation of ( )
$(\dot{\ })$	$d(\ )/dt$

References

1. *Webster's Ninth New Collegiate Dictionary*, Merriam-Webster, Springfield, Mass., 1991.
2. Peters, D. A., Flap-lag stability of helicopter rotor blades in forward flight, *Journal of the American Helicopter Society*, **20**(*4), October (1975).*
3. Johnson, W., *Helicopter theory,* Dover Publications, New York, 1994, pp. 132, 182, 243–249, 758–759, 772–774.
4. Ormiston, R. A., Rutkowski, M. J., Ruzicka, G. C., Saberi, H., and Jung, Y., Comprehensive aeromechanics analysis of complex rotorcraft, *Proceedings of the NASA/AHS Aeromechanics Specialists Conference*, San Francisco, January 19–21, 1994, Paper PS. 1.
5. Yen, J. G,, Corrigan, J. J., Schillings, J. J., and Hseieh, P. Y., Comprehensive analysis methodology at Bell helicopter: conference, *Proceeding of the Aeromechanics Specialists Conference,* San Francisco, January 19–21, 1994, Paper PS. 2.
6. Johnson, W., Technology drivers in the development of CAMRAD II, *Proceedings of the Aeromechanics Specialists Conference*, San Francisco, January 19–21, 1994, Paper PS. 3.
7. Schultz, L. A., Panda, B., Tarzanin, F. J., Derham, R. C., B. Oh, K., and Dadone, L., Interdisciplinary analysis for advanced rotors—Approach, capabilities and states, *Proceedings of the Aeromechanics Specialists Conference*, San Francisco, January 19–21, 1994, Paper PS. 4.
8. Bir, G., and Chopra, I., Status of University of Maryland advanced rotorcraft code (UMARC), *Proceedings of the Aeromechanics Specialists Conference*, San Francisco, January 19–21, 1994, Paper PS. 5.
9. Hsu, T.-K., and Peters, D. A., Coupled rotor/airframe vibration analysis by a combined harmonic-balance, impedance matching method, *Journal of the American Helicopter Society*, **27**(1), No. 1, (1982), 25–34.
10. Fuh, J.-S., Panda, B., and Peters, D., Assembly of finite-element helicopter subsystems with large relative rotations, *Journal of the American Helicopter Society*, **35**(2), (1990).
11. Peters, D. A., Trim techniques Workshop on Unified Equations for Rotorcraft System Modelling, NASA AMES, February 6–8, 1979, Paper No. 18.
12. Wheatley, J. B., An Aerodynamic analysis of the autogiro rotor with a comparison between calculated and experimental results, NACA Rept. 487, 1934.
13. Bailey, F. J. J., A simplified theoretical method of determining the characteristics of a lifting rotor in forward flight, NACA Rept. 716, 1941.
14. Peters, D. A., and Ormiston, R. A., Flapping response characteristics of hingeless rotor blades by a generalized harmonic balance method, NASA TN D-7856, February 1975.
15. Eipe, A., *Effect of blade flexibility, structural parameters and trim conditions on rotor loads*, Doctor of Science Thesis, Washington University, St. Louis, December 1979.

16. Wei, F.-S. and Peters, D. A., Lag damping in autorotation by a perturbation method, *Proceedings of the 34th Annual National Forum of the American Helicopter Society*, May 1978.
17. Stephens, W. B., and Peters, D. A., Rotor body coupling revisited, *Journal of the American Helicopter Society*, **32**(*1*), *(1987).*
18. Ryan, T. K., *Effects of the fuselage on helicopter trim by an iterative harmonic balance,* Master of Science Thesis, Washington University, St. Louis, December 1983.
19. Peters, D. A., Trim restraint springs and dampers, Final Report, 2GCHAS Project Office, July 29, 1989.
20. Peters, D. A., and Izadpanah, A. P., Helicopter trim by periodic shooting with Newton-Raphson iteration, *Proceedings of the 37th Annual Forum of the American Helicopter Society*, New Orleans, May 1981, Paper 81-23.
21. Achar, N. S., and Gaonkar, G. H., Computational aspects of helicopter trim analysis and damping levels from Floquet theory, AHS International Technical Specialists' Meeting on Rotorcraft Basic Research, Georgia Tech, March 15–27, 1991, Paper No. 20.
22. Berman, A., Chen, S-Y., Gustavson, B. and Hurst, P., Dynamic system coupler program, Disco 4-0, Vol. 1, Theoretical Manual Final Report DAAK51-83-C-0041, Report No. R-1790, 15 September 1985.
23. Berman, A., A generalized coupling technique for the dynamic analysis of structural systems, *Journal of the American Helicopter Society*, **25(3)**, (1980).
24. Friedmann, P. P., and Kottapalli, S. B. R., Coupled flap-lag-torsional dynamics of hingeless rotor blades in forward flight, *Journal of the American Helicopter Society*, **27(4)**, (1982).
25. Schrage, D. P., and Peters, D. A., Effect of structural coupling parameters on the flap-lag forced response of a rotor blade in forward flight using Floquet theory, *Vertica*, **3(5)**, (1979), 77–79.
26. Peters, D. A., and Izadpanah, A. P., hp-version finite elements for the space-time domain, *Computational Mechanics*, ***3***, (1988), 73–88.
27. Borri, M, Ghiringhelli, G. L., Lanz, M., Mantegazza, P. and Merlini, T., Dynamic response of mechanical systems by a weak Hamiltonian formulation, *Computers and Structures* ***20***, (1985), 495–508.
28. Hou, L.-J. and Peters, D. A., Periodic trim solutions with hp-version finite elements in time, *Mathematical and Computer Modelling*, **17(3)**, (1993), 29–46.
29. Panda, B., and Chopra, I., Flap-lag-torsion stability in forward flight, *Journal of the American Helicopter Society*, **30(4)**, (1985), 30–39.
30. Johnson, W., Development of a comprehensive analysis for rotorcraft—II. Aircraft Model, Solution Procedure and Applications, *Vertica*, ***5***, (1981), 185–216.
31. Peters, D. A., Kim, B. S., and Chen, H. S., Calculation of trim settings for a helicopter rotor by an optimized Automatic controller, *Journal of Guidance, Control, and Dynamics*, **7(1)**, (1984), 85–91.
32. Peters, D. A., Chouchane, M., and Fulton, M., Helicopter trim with flap-lag-torsion and stall by an optimized controller, *Journal of Guidance, Control, and Dynamics*, **13(3)**, (1990).
33. Celi, R., Aeroelasticity of helicopter in maneuvering flight, Aeroelasticity and fluid structure interaction problems, ASME Winter Annual Meeting, Chicago, November 6–11, 1994.
34. McVair, J. S. G., and Bradley, R., A generic tilt-rotor simulation model with parallel implementation and partial periodic trim algorithm, *Proceedings of the Eighteenth European Rotorcraft Forum*, Avignon, France, September 15–18, 1992, Paper No. 138.
35. Peters, D. A., Fast Floquet theory and trim for multi-bladed rotorcraft, *Journal of the American Helicopter Society*, **39(4)**, (1994).
36. Tran, C. T., and Petot, D., Semi-empirical model for the dynamic stall of airfoils in view of the application to the calculation of responses of a helicopter blade in forward flight, *Vertica*, **5(1)**, (1981), 35–53.
37. Johnson, W., General time-domain unsteady aerodynamics of wings, *Proceedings of the 25th Aircraft Symposium of Japan Society for Aeronautical and Space Sciences*, Tokyo, December 14–16, 1987.
38. Peters, D. A., and He, C. J., Comparison of measured induced velocities with results from a closed-form finite state wake model in forward flight, *Proceedings of the 45th Annual National Forum of the American Helicopter Society*, Boston, MA, May 22–24, 1989.
39. Romanowski, M. C. and Dowell, E. H., Using eigenmodes to form an efficient Euler based unsteady aerodynamics analysis, *Aeroelasticity and Fluid Structure Interaction Problems*, ASME Winter Annual Meeting, Chicago, November 6–11, 1994, pp. 147–160.
40. Prouty, R. W., *Helicopter Performance, Stability, and Control*, PWS Engineering, Boston, 1986, pp. 683–701.
41. Garnot-Dao, M. A., Contribution to performance assessment of helicopter rotor speed control by numerical optimization, *Proceedings of the Eighteenth European Rotorcraft Forum*, Vol. II, Avignon, France, September 15–18, 1992, Paper No. 150.
42. O'Malley, J., Izadpanah, A. P., and Peters, D. A., Comparisons of three numerical trim methods for rotor airloads, Ninth European Rotorcraft Forum, Stresa, Italy, September 13–15, 1983, Paper No. 58.
43. Duyar, A. and Gu, Z., Identification of the open loop dynamics of the T700 turboshaft engine, *Proceedings of the 48th Annual National Forum of the American Helicopter Society*, Washington, D. C., June 3–5, 1992, pp. 399–416.

44. Rock, S. and Neighbori, W. K., Integrated flight/propulsion control for helicopters, *Proceedings of the 49th Annual National Forum of the American Helicopter Society*, May 19–21, 1993, pp. 1433–1444.
45. Bauchau, O. A., and Kang, N. K., A multibody formulation for helicopter structural dynamic analysis, *Journal of the American Helicopter Society*, **38(2)**, (1993), 3–14.
46. Gaonkar, G. H., Nakadi, R. M., Subramanian, S., and Nagabhushanam, J., Parallel computing concepts and methods toward large-scale Floquet analysis of helicopter trim and stability, *Proceedings of the AHS Aeromechanics Specialists Conference,* San Francisco, January 19–21, 1994, Paper No. 5.6.

APPENDIX A: DETERMINATION OF PERIOD

The determination of the period of motion is rather trivial for simplified analyses in which there is a body attached to a constant-velocity rotor, $T = 2\pi/\Omega$. However, when one considers general systems with several rotating substructures and elastic engine-drive-train dynamics, the concept of the period of motion demands special attention and the introduction of general concepts. The first concept is that of average angular velocity. Because we are looking for a periodic motion, every interface between rotating substructures will have an average angular velocity equal to $2\pi/T$, where T is the time it takes for the relative rotation between the two structures to make a complete revolution.

The second concept is that, in order for periodicity to exist, all rotating interfaces must have average angular rates that are commensurate. For rotors driven by engines through transmissions, this will always be the case because one can count the relative number of gear teeth [30]. Thus, for a system with R rotating interfaces, the relative Ω's at each interface can always be written as

$$\frac{\bar{N}_1}{D_1}\Omega_1 = \frac{\bar{N}_2}{D_2}\Omega_2 = \frac{\bar{N}_3}{D_3}\Omega_3 = \cdot\cdot\cdot = \frac{\bar{N}_R}{D_R}\Omega_R \tag{47}$$

where $\bar{N}_i$ and D_i are integers. The highest common Ω (lowest common T) can be found by the following steps:

a) Divide Eq. (47) by the smallest number evenly divisible by every N_i (i.e., the lowest common numerator).

b) Multiply the resultant equation by the largest common factor of the resulting denominators.

The resultant equation gives the largest common Ω.

$$\Omega \equiv \frac{\Omega_1}{M_1} = \frac{\Omega_2}{M_2} = \cdot\cdot\cdot = \frac{\Omega_R}{M_R} \tag{48}$$

where the M_i are the resulting denominators having no common divisor. The smallest common period is then $T = 2\pi/\Omega$, and M_i is the number of revolutions of the ith interface for $0 \leq t < T$.

As pointed out in [30], the resultant period can be so large as to be impractical for computations. However, there are methods that can greatly reduce the minimum T. First, there is the method of multi-blade trim [35]. This concept can be applied if there is one non-rotating substructure to which the other rotating substructures are attached. In that case, each Ω_i in Eq. (48) becomes an absolute angular velocity of a substructure. Reference [35] shows that, for such systems, the trim and Floquet stability can be determined from solutions over a much shorter time period, T.

Consider that the i-th substructure has Q_i symmetric sectors. For example, a 6-bladed rotor with identical blades would have Q = 6; a 4-bladed x-rotor [40] with two non-orthogonal teetering blades would have Q = 2; or a multi-bladed rotor with one differing blade would have Q = 1. Thus, a sector is not necessarily a blade, it is simply a geometric partitioning of the rotating structure. Now, $\bar{\Omega}_i \equiv Q_i\Omega_i$ is the equivalent rotation rate for which $T_i = 2\pi/\bar{\Omega}_i$ gives a periodic position for the i-th rotor. Thus, we can rewrite Eq. (48) as

$$\frac{\bar{\Omega}_1}{Q_1 M_1} = \frac{\bar{\Omega}_2}{Q_2 M_2} = \cdots = \frac{\bar{\Omega}_R}{Q_R M_R} \tag{49}$$

When Eq. (49) is multiplied by the largest common factor of the denominators, Q_iM_i, the result is the largest common angular velocity of the multi-blade system, $\bar{\Omega}$:

$$\bar{\Omega} \equiv \frac{\bar{\Omega}_1}{N_1} = \frac{\bar{\Omega}_2}{N_2} = \cdots = \frac{\bar{\Omega}_R}{N_R} \tag{50}$$

where N_i are the resulting denominators having no common factor. The smallest common multi-blade period is $T = 2\pi/\bar{\Omega}$, and N_i is the number of sectors of the i-th substructure that rotate past a given point during one period, T. If the denominators of Eq. (49) have any common factors, then T can be reduced by those factors.

For example, consider the Aerospatiale AS332L1 [40]. There are two rotating substructures, the main rotor and the tail rotor. The ratio $\Omega_2/\Omega_1 = 4.80$ so that the largest Ω is

$$\Omega = \frac{\Omega_1}{5} = \frac{\Omega_2}{24} \tag{51}$$

Thus, the shortest period is 5 main rotor revolutions (or 24 tail-rotor revolutions). However, this helicopter has $Q_1 = 4$, $Q_2 = 5$. Thus, for multi-blade trim

$$\frac{\bar{\Omega}_1}{20} = \frac{\bar{\Omega}_2}{120}, \qquad \frac{\bar{\Omega}_1}{1} = \frac{\bar{\Omega}_2}{6} = \bar{\Omega} \tag{52}$$

Equation (52) implies one can do a multi-blade trim with a period of only one main-rotor blade passage (one-fourth revolution) which is 6 tail-rotor blade passages (five-fourths of a revolution). Thus, the multi-blade trim gives a savings of a factor of 20 in the trim and Floquet solution.

A second method of reducing the common period is to approximate the exact ratios ($\bar{N}_i/D_i$) in Eq. (47) with ratios of smaller integers. When coupled with the multi-blade method discussed above, this can result in very reasonable periods with negligible error in the tail-rotor position at $t = T$. Table II demonstrates this technique for 11 helicopters given in [40]. The first four columns give the helicopter designation, main rotor sectors, tail rotor sectors, and approximate integer ratio of tail rotor to main rotor RPM. (Note that the 4-bladed AH64 tail x-rotor has only 2 repeating sectors.) The column "error" is the relative error in Ω_2/Ω_1 between the "exact" RPM ratios and the integer ratios given in the table, (exact-approximate)/exact. The sixth column is the ratio of $\overline{\Omega}$'s which gives the smallest multi-blade period. The last four columns give the number of main-rotor revolutions both for normal trim and for multi-blade trim, the number of tail-rotor revolutions for multi-blade trim, and the error (in degrees) of the final tail-rotor position due to the approximate Ω_2/Ω_1 ratio. Note that this error can be kept to 1% to 2% of a rotor revolution (4° to 7°). For example, the 206 L-3 helicopter can be trimmed in 2 main rotor revolutions if one can accept a 14.6° tail-rotor error at the end of the 2 main-rotor rotations. However, if more accuracy is desired, the tail-rotor error can be reduced to 4° by utilizing 13 main-rotor revolutions.

In summary, the common period can be found by simple analysis of gear ratios; and this period can be greatly reduced by use of approximate gear ratios and multi-blade trim. Rotating components with full symmetry (such as engine rotors and compressors) do not need to be considered in this computation since Q is effectively infinity, making any T acceptable. Thus, there is no barrier to trimming multi-engine rotorcraft with unequal engine RPM (e.g., engines geared to the rotors by a differential).

APPENDIX B: MULTI-BLADE PERMUTATION MATRIX

Let the state variables of Q identical sectors of a rotating substructure be partitioned by sector number, where sectors are numbered in the direction of rotation.

Table II Determination of common period for typical rotors

Helicopter	Q_1	Q_2	Ω_2/Ω_1	*Error in RPM*	$Q_2\Omega_2/Q_1\Omega_1$	*Main Rotor Full Revs*	*Main Rotor Multi-Blade Revs*	*Tail Rotor Revs*	*Tail Rotor Azimuth Error at T*
AS 332L1	4	5	24/5	0.0004	6/1	5	0.25	1.20	0.3°
AH 64	4	2*	34/7	0.0020	17/7	7	1.75	8.50	5.1°
UH 60A	4	4	23/5	0.0020	23/5	5	1.25	4.75	1.0°
Westland 30	4	4	5/1	0.0080	5/1	1	0.25	1.25	3.5°
S-76A	4	4	11/2	0.0010	11/2	2	0.50	2.75	1.4°
AH-1S	2	2	41/8	0.0006	41/8	8	4.00	20.50	4.2°
SA 365N	4	13⁺	69/5	0.0002	899/20	5	5.00	69.00	4.6°
AS 350B	3	2	102/19	0.0002	68/19	19	6.33	34.00	2.6°
A 109	4	2	38/7	0.0020	19/7	7	1.75	9.50	7.0°
			65/12	0.0001	65/24	12	6.00	32.50	1.6°
206 L-3	2	2	13/2	0.0060	13/2	2	1.00	6.50	14.6°
			84/13	0.0002	84/13	13	6.50	42.50	4.1°
BO-105	4	2	21/4	0.0020	21/8	4	2.00	10.50	6.5°

*4-Bladed x-rotor.

⁺Fenestron.

$$\{x_j\} = \left\{ \begin{matrix} \{x_i\}_1 \\ \{x_i\}_2 \\ \cdot \\ \cdot \\ \cdot \\ \{x_i\}_Q \end{matrix} \right\} \quad \begin{matrix} i = 1, N \\ j = 1, Q N \end{matrix} \tag{53}$$

The permutation matrix for these states is defined by

$$[p] = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & I_N \\ 0 & I_N & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_N & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & I_N & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & I_N & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I_N & 0 \end{bmatrix} \tag{54}$$

where I_N are identity matrices of order $N \times N$. For systems expressed in multiblade coordinates, the permutation matrix is I_N except for the even-bladed differential coordinate, a_d, for which the permutation matrix is $- I_N$.